If C be the contact transformation whose defining functions are the above X_i, P_i, Z; Q an arbitrary point transformation; and L the transformation of Legendre as generalized by Lie it may be shown analytically and geometrically that

$$C = LQL.$$

In case the contact transformations degenerate into point transformations, Q must be projective. Among the results of the note are complete generalizations of those of a memoir of G. Vivanti, *Rend. del circ. mat. di Palermo*, vol. 5 (1891). F. N. COLE.

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CONCERNING A LINEAR HOMOGENEOUS GROUP IN $C_{m,q}$ VARIABLES ISOMORPHIC TO THE GENERAL LINEAR HOMOGENEOUS GROUP IN m VARIABLES.

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1. While the present paper is concerned chiefly with continuous groups, its results may be readily utilized for discontinuous groups.* Indeed, the finite form of the general transformation of the group is known *ab initio*. Further, the method is applicable to the construction of a linear $C_{m,q}$-ary group isomorphic to an arbitrary m-ary linear group.

2. The formula of composition of m-ary linear homogeneous substitutions

$$(a_{ij}): \qquad \xi_i' = \sum_{j=1}^{m} a_{ij}\xi_j \qquad (j = 1, \cdots, m)$$

is as follows, where the matrix (a_{ij}') operates first:

$$(a_{ij}'') = (a_{ij})(a_{ij}'),$$

where $$a_{ij}'' = \sum_{k=1}^{m} a_{ik}a_{kj}' \qquad (i, j = 1, \cdots, m).$$

* An analogous isomorphism between certain linear groups in the Galois field of order p^n has heen discussed by the writer in an article presented to the London Mathematical Society.

Using Sylvester's *umbral* notation, consider the qth minors of the determinant $| a_{ij} |$

$$\begin{vmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \cdots & a_{i_1 j_q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_q j_1} & a_{i_q j_2} & \cdots & a_{i_q j_q} \end{vmatrix} \equiv \begin{vmatrix} i_1 & i_2 \cdots & i_q \\ j_1 & j_2 \cdots & j_q \end{vmatrix}_a .$$

The formula* expressing the qth minors of $|a_{ij}''|$ in terms of the qth minors of $|a_{ij}|$ and of $|a_{ij}'$ is as follows:

$$(1) \qquad \begin{vmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{vmatrix}_{a''} = \sum_{l_1, \cdots, l_q} \begin{vmatrix} i_1 \cdots i_q \\ l_1 \cdots l_q \end{vmatrix}_a \cdot \begin{vmatrix} l_1 \cdots l_q \\ j_1 \cdots j_q \end{vmatrix}_{a'}$$

the summation extending over the $C_{m,q}$ combinations $l_1, l_2, \cdots, l_q$ of the m integers $1, 2, \cdots, m$ taken q at a time.

3. Consider the $C_{m,q}$-ary linear substitution

$$[a]: \qquad Y'_{i_1 i_2 \cdots i_q} = \sum_{l_1, \cdots, l_q} \begin{vmatrix} i_1 & i_2 \cdots & i_q \\ l_1 & l_2 \cdots & l_q \end{vmatrix}_a Y_{l_1 l_2 \cdots l_q},$$

where the sets $(i_1, \cdots, i_q)$ and $(l_1, \cdots, l_q)$ take successively the $C_{m,q}$ combinations of the integers $1, 2, \cdots, m$ taken q together and where further

$$i_1 < i_2 < \cdots < i_q; \quad l_1 < l_2 < \cdots < l_q.$$

Its determinant has been called the qth compound of the determinant

$$\begin{vmatrix} 1 & 2 \cdots m \\ 1 & 2 \cdots m \end{vmatrix}_a$$

and equals† the latter raised to the power $C_{m-1, q-1}$.

In virtue of (1) we have the composition formula:

$$[a] \cdot [a'] = [a''].$$

Hence, if the substitutions (a) form a group, so do also the substitutions $[a]$. We will speak of the latter group as the "qth compound of the m-ary group." Hence the theorem:

An arbitrary linear group is isomorphic to each of its compounds.

4. Consider the more general substitution

$$[a]_\epsilon: \quad X'_{i_1 \cdots i_q} = \sum_{l_1, \cdots, l_q} \varepsilon_{l_1 \cdots l_q}^{i_1 \cdots i_q} \begin{vmatrix} i_1 \cdots i_q \\ l_1 \cdots l_q \end{vmatrix}_a X_{l_1 \cdots l_q},$$

* Scott, Theory of Determinants, p. 53.

† Muir, Theory of Determinants, § 174.

where the ε's denote ± 1. The product $[\alpha]_\epsilon \cdot [\alpha']_\epsilon$ equals

$$X'_{i_1 \cdots i_q} = \sum_{j_1, \cdots, j_q} \left\{ \varepsilon^{i_1 \cdots i_q}_{l_1 \cdots l_q} \varepsilon^{l_1 \cdots l_q}_{j_1 \cdots j_q} \left| {}^{i_1 \cdots i_q}_{l_1 \cdots l_q} \right|_\alpha \left| {}^{l_1 \cdots l_q}_{j_1 \cdots j_q} \right|_{\alpha'} X_{j_1 \cdots j_q} \right\}.$$

Hence if we define the ε's such that

$$\varepsilon^{i_1 \cdots i_q}_{l_1 \cdots l_q} \cdot \varepsilon^{l_1 \cdots l_q}_{j_1 \cdots j_q} = \varepsilon^{i_1 \cdots i_q}_{j_1 \cdots j_q} \tag{2}$$

we have the formula of composition

$$[\alpha]_\epsilon \cdot [\alpha']_\epsilon = [\alpha'']_\epsilon.$$

But $[\alpha]_\epsilon = 1$ will correspond to $(\alpha) = 1$ if and only if

$$\varepsilon^{i_1 \cdots i_q}_{i_1 \cdots i_q} = +1 \qquad \left({}^{i_1, \cdots, i_q = 1, \cdots, m}_{i_1 < \cdots < i_q}\right). \tag{3}$$

From (2) and (3) it follows that

$$\varepsilon^{i_1 \cdots q}_{1 \cdots q} = \varepsilon^{1_i \cdots i_q}_{i_1 \cdots i_q}. \tag{4}$$

Hence if we set

$$Y_{i_1 \cdots i_q} \equiv \varepsilon^{i_1 \cdots i_q}_{1 \cdots q} X_{i_1 \cdots i_q},$$

it follows from (2) and (4) that $[\alpha]_\epsilon$ takes the form $[\alpha]$ of § 3. Since $[\alpha]$ is the transformed of $[\alpha]_\epsilon$ by a linear substitution, their determinants are equal.

We confine our discussion to the group of the $[\alpha]$. Denote the general m-ary linear group by G_m and its qth compound by $C_{m,q}$.

INFINITESIMAL TRANSFORMATIONS OF $C_{m,q}$, §§ 5–7.

5. Consider first the case $m = 4$, $q = 2$. Setting

$$a_{jj} = 1 + a_{jj}\delta t, \quad a_{ij} = a_{ij}\delta t, \tag{5}$$

the general infinitesimal transformation of $C_{4,2}$ is seen to assign to the six variables $Y_{i_1 i_2}$ the following increments:

	$Y_{12}\delta t$	$Y_{13}\delta t$	$Y_{14}\delta t$	$Y_{23}\delta t$	$Y_{24}\delta t$	$Y_{34}\delta t$
δY_{12}	$a_{11}+a_{22}$	a_{23}	a_{24}	$-a_{13}$	$-a_{14}$	0
δY_{13}	a_{32}	$a_{11}+a_{33}$	a_{34}	a_{12}	0	$-a_{14}$
δY_{14}	a_{42}	a_{43}	$a_{11}+a_{44}$	0	a_{12}	a_{13}
δY_{23}	$-a_{31}$	a_{21}	0	$a_{22}+a_{33}$	a_{34}	$-a_{24}$
δY_{24}	$-a_{41}$	0	a_{21}	a_{43}	$a_{22}+a_{44}$	a_{23}
δY_{34}	0	$-a_{41}$	a_{31}	$-a_{42}$	a_{32}	$a_{33}+a_{44}$

Setting in turn one of the a_{ij} equal unity and the other 15 equal zero, we obtain 16 linearly independent infinitesimal transformations A_{ij}. These we exhibit (by detached coefficients) in sets of four each. We use the abbreviation

$$P_{ij} \equiv \frac{\partial f}{\partial Y_{ij}} \delta t.$$

Set (1)

	P_{12}	P_{13}	P_{14}
A_{11}	Y_{12}	Y_{13}	Y_{14}
A_{12}	0	Y_{23}	Y_{24}
A_{13}	$-Y_{23}$	0	Y_{34}
A_{14}	$-Y_{24}$	$-Y_{34}$	0

Set (2)

	P_{12}	P_{23}	P_{24}
A_{22}	Y_{12}	Y_{23}	Y_{24}
A_{21}	0	Y_{13}	Y_{14}
A_{23}	Y_{13}	0	Y_{34}
A_{24}	Y_{14}	$-Y_{34}$	0

Set (3)

	P_{13}	P_{23}	P_{34}
A_{33}	Y_{13}	Y_{23}	Y_{34}
A_{31}	0	$-Y_{12}$	Y_{14}
A_{32}	Y_{12}	0	Y_{24}
A_{34}	Y_{14}	Y_{24}	0

Set (4)

	P_{14}	P_{24}	P_{34}
A_{44}	Y_{14}	Y_{24}	Y_{34}
A_{41}	0	$-Y_{12}$	$-Y_{13}$
A_{42}	Y_{12}	0	$-Y_{23}$
A_{43}	Y_{13}	Y_{23}	0

The four transformations of each set generate a group of four parameters. Indeed A_{ii} is Euler's homogeneous operator for the variables of the ith set, which do not enter into the coefficients of the other three of that set, so that the latter are commutative. Thus, for set (1), we have the commutator relations

$$(A_{1j}A_{11}) = A_{1j}\ (j = 2, 3, 4); \quad (A_{1j}A_{1k}) = 0 \quad (j, k = 2, 3, 4).$$

Its invariants are found by expanding the four determinants of the third order, one of which is skew-symmetric and therefore zero. The other three give the function (Pfaffian)

$$F \equiv Y_{12}Y_{34} - Y_{13}Y_{24} + Y_{14}Y_{23}$$

multiplied by Y_{23}, Y_{24}, Y_{34} respectively.

A similar result holds for the other sets. A skew-symmetric determinant appears in set (2) if we change the signs in the first column, in set (3) if we change the signs in the first and second columns. It is seen that F is an invariant for the total group of 16 parameters. We obtain also the (here trivial) invariant system formed by the six variables $Y_{i_1i_2}$.

6. Consider the case of general m and q. Neglecting

terms having the factor δt^2, as will be proven allowable, we have at once

$$\left|\begin{matrix} i_1 i_2 \cdots i_q \\ i_1 i_2 \cdots i_q \end{matrix}\right|_a = 1 + (a_{i_1 i_1} + \cdots + a_{i_q i_q})\delta t\,;$$

$$\left|\begin{matrix} i_1 i_2 \cdots i_q \\ j_1 j_2 \cdots j_q \end{matrix}\right|_a = 0,$$

if two or more j's differ from every i.

Consider the case in which $j_1, j_2, \cdots, j_{s-1}, j_{s+1}, \cdots, j_q$ form a permutation of $i_1, i_2, \cdots, i_{r-1}, i_{r+1}, \cdots, i_q$, while $j_s \neq i_r$. Since

$$i_k < i_{k+1}, \quad j_k < j_{k+1} \qquad (k = 1, \cdots, q-1),$$

the above permutation must be cyclic. According as $s < r$ or $s > r$, we readily see that

$$\left|\begin{matrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{matrix}\right|_a$$

must be of the respective forms:

$$\left|\begin{matrix} i_1 \cdots i_{s-1} & i_s & i_{s+1} \cdots i_{r-1} & i_r & i_{r+1} \cdots i_q \\ i_1 \cdots i_{s-1} & j_s & i_s \quad \cdots i_{r-2} & i_{r-1} & i_{r+1} \cdots i_q \end{matrix}\right|_a,$$

$$\left|\begin{matrix} i_1 \cdots i_{r-1} & i_r & i_{r+1} \cdots i_{s-1} & i_s & i_{s+1} \cdots i_q \\ i_1 \cdots i_{r-1} & i_{r+1} & i_{r+2} \cdots i_s & j_s & i_{s+1} \cdots i_q \end{matrix}\right|_a,$$

the cylic permutation being confined to the i's which run from i_s to i_r inclusive, and of the backward or forward type according as $s \gtrless r$. As the two cases are really not distinct, we consider only the first one, $r > s$.

Substituting for the a_{ij} their values in terms of the $a_{ij}\delta t$, the first determinant takes the following form (where for the moment a_{lk} is written for $a_{i_l i_k}\delta t$ and j for j_s):

$$\left|\begin{array}{cccccccccc}
1+a_{11} & a_{12} & \cdots & a_{1\,s-1} & a_{1j} & a_{1s} & a_{1\,s+1} & & a_{1\,r-1} & a_{1q} \\
a_{21} & 1+a_{22} & \cdots & a_{2\,s-1} & a_{2j} & a_{2s} & a_{2\,s+1} & & a_{2\,r-1} & a_{2q} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
a_{s-1\,1} & a_{s-1\,2} & \cdots & 1+a_{s-1\,s-1} & a_{s-1\,j} & a_{s-1\,s} & a_{s-1\,s+1} & & a_{s-1\,r-1} & a_{s-1\,q} \\
a_{s1} & a_{s2} & \cdots & a_{s\,s-1} & a_{sj} & 1+a_{ss} & a_{ss+1} & \cdot & a_{s\,r-1} & a_{sq} \\
a_{s+1\,1} & a_{s+1\,2} & \cdots & a_{s+1\,s-1} & a_{s+1\,j} & a_{s+1\,s} & 1+a_{s+1\,s+1} & & a_{s+1\,r-1} & a_{s+1\,q} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{r-1\,1} & a_{r-1\,2} & \cdots & a_{r-1\,s-1} & a_{r-1\,j} & a_{r-1\,s} & a_{r-1\,s+1} & & 1+a_{r-1\,r-1} & a_{r-1\,q} \\
a_{r1} & a_{r2} & \cdots & a_{r\,s-1} & a_{rj} & a_{rs} & a_{r\,s+1} & & a_{r\,r-1} & a_{rq} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{q1} & a_{q2} & \cdots & a_{q\,s-1} & a_{qj} & a_{qs} & a_{q\,s+1} & & a_{q\,r-1} & 1+a_{qq}
\end{array}\right|$$

In the expansion of this determinant, the only term of the first degree in the a's is seen to be a_{rj}. Hence the determinant equals

$$(-1)^{r-s} a_{i_r j_s} \delta t.$$

Similarly, the second determinant is found to have the same value.

The general infinitesimal transformation of the form $[a]$ is therefore as follows:

$$\delta Y_{i_1 \cdots i_q} \equiv Y'_{i_1 \cdots i_q} - Y_{i_1 \cdots i_q}$$

$$= \delta t \left\{ (a_{i_1 i_1} + \cdots + a_{i_q i_q}) Y_{i_1 \cdots i_q} \right.$$

$$\left. + \sum_{r,s}^{1 \cdots q} (-1)^{r+s} a_{i_r j_s} Y_{i_1 \cdots i_{s-1} j_s i_s \cdots i_{r-1} i_{r+1} \cdots i_q} \right\}$$

the summation also extending over all values of j_s from i_{s-1} to i_s exclusive. A simplification arises by introducing several coexistent notations for the same variable Y, viz:

$$Y_{i_1 \cdots i_{s-1} j_s i_s \cdots i_q} \equiv (-1)^{s-1} Y_{j_s i_1 \cdots i_{s-1} i_s \cdots i_q}.$$

Indeed, we may then perform the above summation with respect to s, and obtain for $\delta Y_{i_1 \cdots i_q}$ the simpler value

$$\delta t \left\{ (a_{i_1 i_1} + \cdots + a_{i_q i_q}) Y_{i_1 \cdots i_q} \right.$$

$$\left. + \sum_{r,j} (-1)^{r-1} a_{i_r j} Y_{j i_1 \cdots i_{r-1} i_{r+1} \cdots i_q} \right\},$$

$$(r = 1, \cdots, q; \quad j = 1, \cdots, m; \quad j \neq i_1, i_2, \cdots, i_q).$$

7. We may now readily obtain m^2 linearly independent infinitesimal transformations A_{lk} by setting in turn $a_{lk} = 1$ and the other a's equal zero.

For A_{ll}, $\delta Y_{i_1 \cdots i_q}$ is zero unless one of the i's equals l, while

$$\delta Y_{i_1 \cdots i_{r-1} l i_{r+1} \cdots i_q} = Y_{i_1 \cdots i_{r-1} l i_{r+1} \cdots i_q}.$$

Hence $$\delta Y_{l i_1 \cdots i_{r-1} i_{r+1} \cdots i_q} = Y_{l i_1 \cdots i_{r-1} i_{r+1} \cdots i_q}.$$

Hence A_{ll} has the form given below (for $k = l$).

For $A_{lk} (l \neq k)$, $\delta Y_{i_1 \cdots i_q}$ is zero unless some $i_r = l$, in which case

$$\delta Y_{i_1 \cdots i_{r-1} l i_{r+1} \cdots i_q} = \sum_j (-1)^{r-1} a_{lj} Y_{j i_1 \cdots i_{r-1} i_{r+1} \cdots i_q}.$$

Hence, since $a_{lj} = 0$ if $j \neq k$,

$$\delta Y_{l i_1 \cdots i_{r-1} i_{r+1} \cdots i_q} = Y_{k i_1 \cdots i_{r-1} i_{r+1} \cdots i_q}.$$

The m^2 independent transformations of the group $C_{m,q}$ are thus:

$$A_{lk} \equiv \sum_{i_1, \cdots, i_{r-1}, i_{r+1}, \cdots, i_q}^{1, \cdots, m} Y_{k i_1 \cdots i_{r-1} i_{r+1} \cdots i_q} P_{l i_1 \cdots i_{r-1} i_{r+1} \cdots i_q}.$$

$$(i_1 < i_2 < \cdots < i_q, \text{ and each } \neq l, \neq k).$$

Here $P_{l \cdots i_q}$ denotes

$$\frac{\partial f}{\partial Y_{l \cdots i_q}} \delta t.$$

Certain Properties of the Invariants of $C_{m,2}$, §§ 8–10.

8. For $q = 2$, we have the m^2 transformations of $C_{m,2}$

$$A_{lk} \equiv \sum_{i \neq k,\, l}^{i=1, \cdots, m} Y_{ki} P_{li} \qquad (l, k = 1, \cdots, m).$$

We may separate these m^2 transformations into m sets

$$[A_{l1}, A_{l2}, \cdots, A_{lm}] \qquad (l = 1, \cdots, m).$$

Those of the lth set involve only the $m - 1$ differential coefficients

$$P_{l1}, P_{l2}, \cdots, P_{l\,l-1}, P_{l\,l+1}, \cdots, P_{lm}.$$

For use below we exhibit them in a table (with detached coefficients). By our notation $Y_{ij} \equiv - Y_{ji}$.

	P_{l1}	P_{l2}	P_{l3}	$P_{l\,l-1}$	$P_{l\,l+1}$	P_{lm}
A_{ll}	Y_{l1}	Y_{l2}	Y_{l3}	$Y_{l\,l-1}$	$Y_{l\,l+1}$	Y_{lm}
A_{l1}	0	Y_{12}	Y_{13}	$Y_{1\,l-1}$	$Y_{1\,l+1}$	Y_{1m}
A_{l2}	Y_{21}	0	Y_{23}	$Y_{2\,l-1}$	$Y_{2\,l+1}$	Y_{2m}
A_{l3}	Y_{31}	Y_{32}	0	$Y_{3\,l-1}$	$Y_{3\,l+1}$	Y_{3m}
⋯	⋯	⋯	⋯	⋯	⋯	⋯
$A_{l\,l-1}$	$Y_{l-1\,1}$	$Y_{l-1\,2}$	$Y_{l-1\,3}$	0	$Y_{l-1\,l+1}$	$Y_{l-1\,m}$
$A_{l\,l+1}$	$Y_{l+1\,1}$	$Y_{l+1\,2}$	$Y_{l+1\,3}$	$Y_{l+1\,l-1}$	0	$Y_{l+1\,m}$
⋯	⋯	⋯	⋯	⋯	⋯	⋯
A_{lm}	Y_{m1}	Y_{m2}	Y_{m3}	$Y_{m\,l-1}$	$Y_{m\,l+1}$	0

It follows exactly as in § 5 that the m transformations of any set generate a group of m parameters.

Deleting the row A_{ll}, we obtain a skew-symmetric determinant of order $m-1$, which we denote by $D_{ll}^{(m-1)}$. Deleting the row A_{lk} and moving the column headed by P_{lk} into the place of the last column, we obtain a bordered skew-symmetric determinant $D_{lk}^{(m-1)}$, the first row and the last column forming its borders.

9. For m odd and $q=2$, we have*

$$D_{ll}^{(m-1)} = [1, 2, \cdots, l-1, l+1, \cdots, m]^2,$$

where the Pfaffian $[1, 2, \cdots, l-1, l+1, \cdots, m]$ includes the extreme cases $[1, 2, \cdots, m-1]$ and $[2, 3, \cdots, m]$. Further $D_{lk}^{(m-1)}$ factors into two Pfaffians of like order, which are seen to be

$$[l, i_1, i_2, \cdots, i_{m-2}], \ [i_1, i_2, \cdots, i_{m-2}, k],$$

where $i_1 < i_2 < \cdots < i_{m-2}$, and each $i \neq l, \neq k$.

Since the interchange of two indices merely changes the sign of the Pfaffian, it follows that all the determinants $D_{lk}^{(m-1)}$ vanish if and only if the Pfaffians

$$F_1 \equiv [2, 3, \cdots, m], \cdots, F_l \equiv [1, 2, \cdots, l-1, l+1, \cdots, m],$$
$$\cdots, F_m \equiv [1, 2, \cdots, m-1]$$

simultaneously vanish. It follows, therefore, from the general theory of Lie that every system of equations invariant under the group $C_{m,2}$, m odd, must include the m equations

$$F_k = 0 \qquad (k = 1, \cdots, m).$$

It follows readily from the properties of Pfaffians that the transformations A_{kj} have the following effect upon the Pfaffians F_k:

$$A_{kk}(F_k) = 0, \quad A_{ll}(F_k) = F_k\,\delta t \qquad (l = 1, \cdots, m; l \neq k),$$
$$A_{lj}(F_k) = 0, \quad A_{lk}(F_k) = (-1)^{l+k-1} F_l\,\delta t \quad \left({}^{j,\, l = 1, \cdots, m;}_{l \neq k,\; j \neq k\; l \neq j}\right).$$

For example,

$$A_{lk}[1, 2, \cdots, k-1, k+1, \cdots, m]$$
$$= [1, 2, \cdots, k-1, k+1, \cdots, l-1, k, l+1, \cdots, m]$$
$$= (-1)^{l-k-1}[1, 2, \cdots, k-1, k, k+1, \cdots, l-1, l+1, \cdots, m]$$
$$= (-1)^{l+k-1} F_l.$$

* Muir, Theory of Determinants, §§159, 163.

The transformation A_{ll} therefore gives the following increments:

$$\delta F_l = 0, \quad \delta F_k = F_k \delta t \quad (k = 1, \cdots, m; \ k \neq l).$$

The transformation $A_{lk} (l \neq k)$ produces the increments

$$\delta F_k = (-1)^{l+k-1} F_l \delta t, \quad \delta F_j = 0 \quad (j = 1, \cdots, m; \ j \neq k).$$

It is readily seen that the m^2 linearly independent infinitesimal transformations in the m variables F_k,

$$(6) \quad A_{ll} = \sum_{k \neq l}^{k=1, \cdots, m} F_k \frac{\partial f}{\partial F_k} \delta t; \quad A_{lk} = (-1)^{l+k-1} F_l \frac{\partial f}{\partial F_k} \delta t,$$

generate a group whose finite transformations are:

$$(7) \qquad F_i' = \sum_{j=1}^{m} A_{ij} F_j \qquad (j = 1, \cdots, m)$$

where A_{ij} is the minor (without sign) complementary to a_{ij} in the determinant $|a_{ij}|$. Indeed if we apply formula (5) to the determinant

$$A_{ij} \equiv \begin{vmatrix} 1 \cdots j-1 \, j & j+1 \cdots & i-1 \, i+1 \cdots m \\ 1 \cdots j-1 \, j+1 \, j+2 \cdots & i & i+1 \cdots m \end{vmatrix}$$

we find as in §6, the results

$$(8) \qquad A_{ii} = 1 + \sum_{s \neq i}^{s=1, \cdots, m} a_{ss} \delta t, \ A_{ij} = (-1)^{i+j-1} a_{ji} \delta t$$

$$(i, j = 1, \cdots, m; \ i \neq j).$$

It follows that the general infinitesimal transformation of the form (7) gives the following increments:

$$\delta F_i = \left[\sum_{s \neq i}^{s=1, \cdots, m} a_{ss} F_i + \sum_{j \neq i}^{j=1, \cdots, m} (-1)^{i+j-1} a_{ji} F_j \right] \delta t$$

$$(i = 1, \cdots, m),$$

from which we readily obtain the m^2 linearly independent transformations (6). We may therefore enunciate the following theorem* concerning the individual finite transformations of the above groups.

* This theorem is capable of proof by determinants without having recourse to the infinitesimal transformations of the groups concerned.

For m odd, the second compound $C_{m,2}$ of the general m-ary linear homogeneous group G_m possesses a system of m invariant Pfaffians,

$$F_i \equiv [1, 2, \cdots, i-1, i+1, \cdots, m] \quad (i = 1, \cdots, m).$$

The transformation $[a]_2$ of $C_{m,2}$, corresponding to any given transformation (a_{ij}) of G_m, effects upon the F_i a linear transformation which is identical with that m-ary transformation $[a]_{m-1}$ of the $(m-1)^{st}$ compound of G which corresponds to (a_{ij}).

10. For m even and $q = 2$, the skew-symmetric determinant $D_{ll}^{(m-1)}$ vanishes identically. We readily find* that the bordered skew-symmetric determinants

$$D_{lk}^{(m-1)} = [l, i_1, i_2, \cdots, i_{m-2}, k]\,[i_1, i_2, \cdots, i_{m-2}]$$
$$(l, k = 1, \cdots, m;\ l \neq k)$$

if $i_1, i_2, \cdots, i_{m-2}, l, k$ form a permutation of $1, 2, \cdots, m$.

It is readily verified that the transformations A_{ii} leave unaltered the Pfaffian $[1, 2, \cdots, m]$, while the A_{ij} $(i \neq j)$ annul it. Hence $[1, 2, \cdots, m]$ is an invariant of $C_{m,2}$. Consider the $\frac{1}{2}m(m-1)$ Pfaffians

$$F_{i_1 i_2 \cdots i_{m-2}} \equiv [i_1, i_2, \cdots, i_{m-2}].$$

We find that the transformation A_{lk} gives the increments,

$$\delta F_{1\, i_2 \quad i_{m-2}} = 0 \qquad (\text{if every } i_s \neq l)\ ;$$

$$\delta F_{l\, i_2 \cdots i_{m-2}} = F_{k\, i_2 \cdots i_{m-2}}\delta t.$$

But these are the increments produced by the transformation A_{lk} of the group $C_{m,m-2}$ upon its variables $F_{i_1 i_2 \cdots i_{m-2}}$ [see § 7]. We have therefore proved the following theorem, capable of proof using only the finite transformations of the groups involved:

For m even, the secona compound $C_{m,2}$ of the general m-ary linear group G_m possesses as an isolated invariant the Pfaffian $[1, 2, \cdots, m]$ and as a system of invariants the set of $C_{m,2}$ Pfaffians.

$$[i_1, i_2, \cdots, i_{m-2}] \qquad \left(\begin{smallmatrix} i_1, i_2, \cdots, i_{m-2} = 1, \cdots, m \\ i_1 < i_2 < \cdots < i_{m-2} \end{smallmatrix}\right).$$

The transformation $[a]_2$ of $C_{m,2}$, corresponding to any given transformation (a_{ij}) of G_m, effects upon these Pfaffians a linear transformation identical with that $C_{m,m-2}$ -ary transformation $[a]_{m-2}$ of the $(m-2)^{nd}$ compound of G_m, which corresponds to the given (a_{ij}).

* Muir, Theory of Determinants, § 163.

RECIPROCITY BETWEEN THE qTH AND THE $m-q$TH COMPOUNDS OF G_m, §§ 11–15.

11. We may express* the qth minors of the determinant $|A_{ij}|$ adjungate to $|a_{ij}|$ in terms of the $(m-q)$th minors of $|a_{ij}|$:

$$(9) \qquad \begin{vmatrix} i_1 & i_2 & \cdots & i_q \\ j_1 & j_2 & \cdots & j_q \end{vmatrix}_A$$

$$= D^{q-1} \left\| \begin{matrix} 1 & 2 & \cdots & i_1-1 & i_1+1 & \cdots & i_q-1 & i_q+1 & \cdots & m \\ 1 & 2 & \cdots & j_1-1 & j_1+1 & \cdots & j_q-1 & j_q+1 & \cdots & m \end{matrix} \right\|_a,$$

the double bars indicating that, in the two series of integers written in ascending order, j_1-1 does not necessarily fall under i_1-1, etc.

If therefore we write, for every $i_1 < i_2 < \cdots < i_q \leqq m$,

$$Y_{1\,2\,\cdots\,i_1-1\,i_1+1\,\cdots\,i_q-1\,i_q+1\,\cdots\,m} \equiv Z_{i_1 i_2 \cdots i_q}$$

the general substitution $[a]_{m-q}$ of the group $C_{m,m-q}$ becomes

$$(10) \qquad Z'_{i_1 i_2 \cdots i_q} = D^{1-q} \sum \begin{vmatrix} i_1 & i_2 & \cdots & i_q \\ j_1 & j_2 & \cdots & j_q \end{vmatrix}_A Z_{j_1 j_2 \cdots j_q},$$

the summation extending over every combination $j_1, j_2, \cdots, j_q$ of the integers $1, \cdots, m$ taken q together.

12. To obtain the general infinitesimal transformation (10) we proceed as in § 6, using formulæ (8). We find

$$\begin{vmatrix} i_1 & i_2 & \cdots & i_q \\ i_1 & i_2 & \cdots & i_q \end{vmatrix}_A = 1 + \left(q \sum_{s=1}^{m} a_{ss} - \sum_{s=1}^{q} a_{i_s i_s}\right) \delta t;$$

$$\begin{vmatrix} i_1 & i_2 & \cdots & i_q \\ j_1 & j_2 & \cdots & j_q \end{vmatrix}_A = 0 \qquad \text{(if two or more } j\text{'s differ from every } i\text{)};$$

$$\begin{vmatrix} i_1 & i_2 & \cdots & i_q \\ j_1 & j_2 & \cdots & j_q \end{vmatrix}_A = (-1)^{r+s-1} A_{i_r j_s}$$

$$= (-1)^{i_r+j_s+r+s-1} a_{j_s i_r} \delta t,$$

if j_s be the only j different from every i, and i_r be the only i different from every j. Further,

$$D^{-q+1} = 1 + (-q+1) \sum_{s=1}^{m} a_{ss} \delta t.$$

* Compare Muir, end of § 97.

Hence $\delta Z_{i_1 i \cdots i_q}$ equals δt times the expression

$$\Big(\sum_{s=1}^{m} a_{ss} - \sum_{s=1}^{q} a_{i_s i_s}\Big) Z_{i_1 i_2 \cdots i_q}$$
$$+ \sum_{r,s}^{1,\cdots,q} (-1)^{i_r + j_s + r + s - 1} a_{j_s i_r} Z_{i_1 \cdots i_{s-1} j_s i_{s+1} \cdots i_{r-1} i_{r+1} \cdots i_q}$$

summed also for $j_s = i_{s-1} + 1, \cdots, i_s - 1$. If we perform the summation with respect to s in the latter sum (see end of § 6), it becomes

$$\sum_{r,j} (-1)^{i_r + j + r} a_{j i_r} Z_{j i_1 \cdots i_{r-1} i_{r+1} \cdots i_q},$$

summed for

$$r = 1, \cdots, q; \quad j = 1, \cdots, m, \quad j \neq i_1, i_2, \cdots, \text{ or } i_q.$$

13. Setting $a_{lk} = 1$ and the other a's equal zero, we obtain m^2 linearly independent infinitesimal transformations A_{lk}'. Setting

$$Q = \frac{\partial f}{\partial Z} \delta t,$$

and proceeding as in § 7, we find

$$A_{lk}' = (-1)^{l+k-1} \sum_{i_1, \cdots, i_q}^{1 \cdots m} Z_{l i_1 \cdots i_{r-1} i_{r+1} \cdots i_q} Q_{k i_1 \cdots i_{r-1} i_{r+1} \cdots i_q}$$
$$(i_1 < i_2 \cdots < i_q, \text{ and each } i \neq l, \neq k).$$
$$A' - A_{kk}' = \sum_{i_1 \cdots i_q}^{1 \cdots m} Z_{k i_1 \cdots i_{r-1} i_{r+1} \cdots i_q} Q_{k i_1 \cdots i_{r-1} i_{r+1} \cdots i_q}$$
$$(i_1 < i_2 \cdots < i_q, \text{ and each } i \neq k),$$

where we denote by A' the following transformation

$$A' \equiv \sum_{i_1 \cdots i_q}^{1 \cdots m} Z_{i_1 i_2 \cdots i_q} Q_{i_1 i_2 \cdots i_q}$$

To prove that A' belongs to the group $C_{m, m-q}$ under consideration, we note that $A' - A_{kk}'$ contains $C_{m-1, q-1}$ terms, so that

$$mA' - \sum_{k=1}^{m} A_{kk}'$$

contains $mC_{m-1, q-1}$ terms which coincide in sets of q each, and among which every one of the $C_{m, q}$ terms of A' is represented. Hence, since $mC_{m-1, q-1} = qC_{m, q}$, it follows that

$$(m - q)A' = \sum_{k=1}^{m} A_{kk}'.$$

14. The set of m infinitesimal transformations of $C_{m,m-q}$,

$$A' - A_{kk}', \quad A_{lk}' \quad (l = 1, \cdots, m, l \neq k),$$

generate a group of m parameters which is identical with the group generated by the m transformations $A_{kl}(l=1, \cdots, m)$ of the group $C_{m,q}$. We thus see the exact manner in which the qth and $(m-q)$th compounds of the general m-ary linear group G_m are isomorphic.

When we confine ourselves to the group of those transformations of G_m of determinant $D = 1$, the qth and the $(m - q)$th compounds are not merely isomorphic but identical. Indeed the $m^2 - 1$ transformations of the $C_{m,m-q}$,

$$A_{lk}' \ (l, k = 1, \cdots, m, l \neq k), \ A_{11}' - A_{kk}' \qquad (k = 2, \cdots, m)$$

are identical which the $m^2 - 1$ transformations

$$A_{kl} \ (k, l = 1, \cdots, m, k \neq l), \ A_{11} - A_{kk} \qquad (k = 2, \cdots, m)$$

of $C_{m,q}$, the corresponding transformations being given by the same pair of subscripts $(k\ l)$ or $(1\ k)$.

15. To illustrate the reciprocity between the groups $C_{m,q}$ and $C_{m,m-q}$, we take the example $m = 5$, $q = 2$. We write the table of §8 for the transformations of $C_{5,2}$ which belong to the set $l = 2$; viz.,

	P_{21}	P_{23}	P_{24}	P_{25}
A_{22}	Y_{21}	Y_{23}	Y_{24}	Y_{25}
A_{21}	0	Y_{13}	Y_{14}	Y_{15}
A_{23}	$-Y_{13}$	0	Y_{34}	Y_{35}
A_{24}	$-Y_{14}$	$-Y_{34}$	0	Y_{45}
A_{25}	$-Y_{15}$	$-Y_{35}$	$-Y_{45}$	0

By §6 we obtain the following transformations of $C_{5,3}$:

	$-P_{345}$	P_{145}	P_{135}	P_{134}
$A' - A_{22}'$	$-Y_{345}$	Y_{145}	Y_{135}	Y_{134}
$+ A_{12}'$	0	Y_{245}	Y_{235}	Y_{234}
$+ A_{32}'$	$-Y_{245}$	0	Y_{125}	Y_{124}
$- A_{42}'$	$-Y_{235}$	$-Y_{125}$	0	Y_{123}
$+ A_{52}'$	$-Y_{234}$	$-Y_{124}$	$-Y_{123}$	0

We thus observe that any term as $Y_{245}\,P_{145}$ of the latter table may be derived at once from the corresponding term $Y_{13}\,P_{23}$ of the former by taking as subscripts to the one those integers 1, 2, $\cdots$, 5 (in order), which do not occur among the subscripts to the other term. The rule which, if applied to the first table, gives the Pfaffian invariant $F_2 \equiv [1345]$ will, when applied to the second table, give

$$\overline{F}_2 \equiv -(Y_{213}Y_{245} - Y_{214}Y_{235} + Y_{215}Y_{234}),$$

which we will denote by $-_2[1345]$, the first subscript to the Y's being 2 throughout.

Forming the remaining four tables for the group $C_{5,2}$ and the corresponding tables for $C_{5,3}$, we obtain the following results:

$$F_1 \equiv [2345],\ \overline{F}_1 = {}_1[2345];\ F_3 \equiv [1245],\ \overline{F}_3 = {}_3[1245];$$
$$F_4 \equiv [1235],\ \overline{F}_4 = -{}_4[1235];\ F_5 \equiv [1234],\ \overline{F}_5 = {}_5[1234].$$

In general, if F_j or $F_{i_1 i_2 \cdots i_{m-2}}$ denote the Pfaffians formed from the tables of the transformations A_{kl} of $C_{m,2}$, we will denote by $\overline{F}_j$ or $\overline{F}_{i_1 i_2 \cdots i_{m-2}}$ the Pfaffians formed from the corresponding tables of the transformations A_{lk}', $A' - A_{kk}'$ of $C_{m,m-2}$.

GROUP INDUCED BY $C_{m,m-2}$ UPON ITS INVARIANTS, §§ 16–18.

16. For m odd and $q = 2$, the group $C_{m,m-q}$ has a system of m invariant Pfaffians $\overline{F}_j$ of degree $\frac{1}{2}(m-1)$. By §9, the transformation $A' - A_{kk}'$ effects upon the $\overline{F}_j$ the transformation

$$\sum_{j \neq k}^{j=1,\cdots,m} \overline{F}_j \frac{\partial f}{\partial \overline{F}_i}\delta t;$$

while $A_{lk}' \equiv (-1)^{k+l-1}A_{kl}$ produces the transformation

$$\overline{F}_k \frac{\partial f}{\partial \overline{F}_l}\delta t.$$

Since the Eulerian operator A' multiplies each $\overline{F}_j$ by $\frac{1}{2}(m-1)$, it follows that A_{kk}' produces the following transformation:

$$\frac{1}{2}(m-1)\sum_{j=1}^{m} \overline{F} \frac{\partial f}{\partial \overline{F}_j}\delta t - \sum_{j \neq k}^{j=1,\cdots,m} \overline{F}_j \frac{\partial f}{\partial \overline{F}_j}\delta t$$
$$\equiv \frac{1}{2}(m-3)\sum_{j \neq k}^{j=1,\cdots,m} \overline{F}_j \frac{\partial f}{\partial \overline{F}_j}\delta t + \frac{1}{2}(m-1)\ \overline{F}_k \frac{\partial f}{\partial \overline{F}_k}\delta t.$$

The finite transformations of the group induced upon the $\overline{F}_j$ by the group $C_{m,\,m-2}$ have therefore the form

$$(11) \qquad \overline{F}_i' = D^{\frac{m-3}{2}} \sum_{j=1}^{m} a_{ij}\,\overline{F}_k \qquad (i=1,\cdots,m).$$

17. For m even and $q=2$, the group $C_{m,\,m-q}$ has as an isolated invariant a Pfaffian of degree $m/2$ and as a system of invariants the C_m, Pfaffians $\overline{F}_{i_1 i_2 \cdots i_{m-2}}$ of degree $\frac{1}{2}(m-2)$. It follows from §§ 10, 13, 14 that the transformation $(-1)^{l+k-1}A_{lk}'\ (l \neq k)$ of $C_{m,\,m-2}$ gives rise to the following increments in the Pfaffian invariants:

$$(a) \qquad \begin{cases} \delta\overline{F}_{i_1 i_2 \cdots i_{m-2}} = 0 & \text{(if each } i \neq k) \\ \delta\overline{F}_{k i_2 \cdots i_{m-2}} = \overline{F}_{l i_2 \cdots i_{m-2}}\,\delta t\,; & \end{cases}$$

also that $A' - A_{kk}'$ produces the increments (a) (when l is replaced by k). Since A' multiplies each $\overline{F}_{i_1 \cdots i_{m-2}}$ by $\frac{1}{2}(m-2)$, it follows that A_{kk}' produces the increments

$$(b) \qquad \begin{cases} \delta\overline{F}_{i_1 i_2 \cdots i_{m-2}} = \frac{1}{2}(m-2)\,\overline{F}_{i_1 i_2 \cdots i_{m-2}}\,\delta t & \text{(if each } i \neq k), \\ \delta\overline{F}_{k i_2 \cdots i_{m-2}} = \frac{1}{2}(m-4)\,\overline{F}_{k i_2 \cdots i_{m-2}}\,\delta t. & \end{cases}$$

Having thus determined the infinitesimal transformations of the group induced by the group $C_{m,\,m-2}$ upon its system of invariants $\overline{F}_{i_1 \cdots i_{m-2}}$, we may readily show that the finite transformations of this group are

$$(12) \qquad \overline{F}'_{i_1 i_2 \cdots i_{m-2}} = D^{1-\frac{m}{2}} \sum_{j_1,\cdots,j_{m-2}}^{1\cdots m} \begin{vmatrix} i_1 \cdots i_{m-2} \\ j_1 \cdots j_{m-2} \end{vmatrix}_A \overline{F}_{j_1 j_2 \cdots j_{m-2}}.$$

Indeed, proceeding as in §§ 11–13, we find that the infinitesimal transformation gotten from (12) by setting $a_{kk}=1$ and the other a's $=0$ has precisely the increments (b), while that given by setting $a_{lk}=1$ and the other a's $=0$ has, when multiplied by $(-1)^{l+k-1}$, precisely the increments (a).

To give (12) another form, we set

$$\overline{F}_{i_1 i_2 \cdots i_{m-2}} \equiv W_{i_{m-1} i_m} \qquad \left(\begin{smallmatrix} i_2 < i_2 < \cdots < i_{m-2} \\ i_{m-1} < i_m \end{smallmatrix}\right),$$

when i_{m-1} is the first and i_m the second integer $< m$ which does not occur in the series $i_1, i_2, \cdots, i_{m-2}$.

Further, formula (9) of § 11 becomes for $q = m - 2$

$$\left|\begin{matrix} i_1 i_2 \cdots i_{m-2} \\ j_1 i_2 \cdots j_{m-2} \end{matrix}\right|_A = D^{m-3} \left|\begin{matrix} i_{m-1} i_m \\ j_{m-1} j_m \end{matrix}\right|_a .$$

Hence the transformation (12) takes the form*

$$(12_1) \qquad W'_{i_{m-1} i_m} = D^{\frac{m-4}{2}} \sum_{j_{m-1} j_m}^{1,\cdots,m} \left|\begin{matrix} i_{m-1} i_m \\ j_{m-1} j_m \end{matrix}\right|_a W_{j_{m-1} j_m} .$$

18. We may enunciate the results proven in §§ 16–17 for the individual transformations of the groups concerned:

To any given transformation (a_{ij}) *of determinant* D *of the general m-ary linear homogeneous group* G_m, *there corresponds a transformation* $[a]_{m-2}$ *of the* $(m-2)^d$ *compound* $C_{m, m-2}$ *which gives rise to a linear transformation upon its system of Pfaffian invariants, viz:*

1° : *for m odd, the m-ary transformation,*

$$\overline{F}_i' = D^{\frac{m-3}{2}} \sum_{j=1}^{m} a_{ij} \overline{F}_j \qquad (i = 1, \cdots, m),$$

which for $D = 1$, *is precisely the given transformation of* G_m.

2°. *for m even, the* $\frac{1}{2}m(m-1)$ *-ary transformation* (12) *or* (12_1), *where, for* $D = 1$, (12_1) *belongs to the second compound of* G_m, *and* (12) *to the* $(m-2)^d$ *compound of the* $(m-1)^{st}$ *compound of* G_m.

UNIVERSITY OF CALIFORNIA,
August 9, 1898.

A SECOND LOCUS CONNECTED WITH A SYSTEM OF COAXIAL CIRCLES.

BY PROFESSOR THOMAS F. HOLGATE.

(Read before the American Mathematical Society at its Fifth Summer Meeting, Boston, August 19, 1898.)

IN a paper read before this Society at its Toronto Meeting and published in the BULLETIN for November, 1897, I

* We may verify (12_1) directly, using the method of § 6 for $q = 2$. The presence of the factor $D^{\frac{m-4}{2}}$ influences only the transformations $A_{kk'}$. There occurs, however, some difficulty as to signs in passing from the W's to the F's. Likewise the results of §§ 11–14 could doubtless be proved by the method of § 6.