close dependence of the operations of subtraction and multiplication of numbers on the same operations for lines, it would seem that there must be some connection.

It is evident that the Greeks were even farther from a continuum of numbers than they were from a continuum of their line symbols. For no number in their system existed which expressed their incommensurable lines, not to mention the countless kinds of incommensurability of which they knew nothing.

These considerations indicate that Greek mathematics rested on a very narrow basis so long as it clung to its line notation. The sense of rigor, as shown by postulating the existence of a product of two factors certainly would not allow them to assume a continuous system, as less careful mathematicians have done. This line notation did not admit of sufficient expansion to allow them to establish on *that* such a system. Thus, until the foundation of their mathematical science was utterly changed, an advance to algebra and calculus was impossible.

YALE UNIVERSITY,
April, 1898.

MAXIMA AND MINIMA OF FUNCTIONS OF SEVERAL VARIABLES.

BY PROFESSOR JAMES PIERPONT.

IN treating the theory of maxima and minima in my lectures this year I have been astonished to find that the presentation of this theory in all English and American textbooks on the calculus which I could consult was false. That the older editions of such standard treatises as Todhunter, Williamson, and Byerly should be wrong in this particular was not astonishing since it was only in 1884 that Peano in his critical notes to the Calcolo Differenziale of Genocchi called attention to the point in question. Since then L. Scheeffers,* O. Stolz,† and von Dantscher‡ have devoted memoirs to this interesting but difficult subject and their results have found a place in the new edition of the Cours d'Analyse of C. Jordan and the Grundzüge of O. Stolz.

* *Mathematische Annalen*, vol. 35, p. 541.
† *Sitzungsberichte Vienna Academy*, 1890 (June).
‡ *Annalen*, vol. 42, p. 89.

Our English and American authors seem however to be totally ignorant of these facts. In the latest editions of the treatises just mentioned* as well as in the countless new works on the calculus that are constantly appearing we find the same incorrect reasoning repeated with an innocent ignorance that must be highly amusing to continental mathematicians. Yet this deplorable state of affairs would not have called forth the present note if the first calculus† written in the English language with the avowed purpose of presenting the subject in accordance with modern standards of rigor, did not employ the traditional incorrect treatment. That the author has not been led to give an incorrect criterion is not due to his method but because he gives no criterion at all for the critical case, for which he contents himself by remarking that *the expansion must be continued further.*‡ That this manner of reasoning is not confined to England and America and may be adopted by an otherwise careful writer is illustrated by the excellent work of Demartres, Cours d'Analyse,§ written in thorough harmony with the modern spirit of rigor. Here the reasoning is carried to its logical consequence for the critical case just alluded to and a false criterion is arrived at.

In the light of these facts I think it worth while to call the attention of American mathematicians to this matter once more. The point in question is very simple. To determine whether $f(x_1, x_2, \cdots, x_n)$ has an extreme at the point $P = (a_1, a_2, \cdots, a_n)$ we develop it into Taylor's series, this being possible, and get

$$\Delta = f(x_1, \cdots, x_n) - f(a_1, \cdots, a_n) = H_1 + H_2 + H_3 + \cdots$$

If P is an extreme Δ must have one sign for all points in its vicinity.|| To deduce a criterion, the assertion is now made in the works we are criticising that the sign of Δ depends upon the sign of H_m, that being the first derivative which does not vanish identically, while for the particular points for which H_m may vanish the sign of Δ depends upon the sign of the first derivative which does not vanish for

* For example, the 8th edition of Williamson's Differential Calculus, dated 1895.

† Infinitesimal Calculus by Horace Lamb, Cambridge University Press, 1897.

‡ Loc. cit., pp. 596–597.

§ Hermann, Paris, 1892–96 ; cf. vol. 1, p. 72–73.

|| The case when Δ may be zero for certain points is not considered in our text-books.

these points. This, of course, is generally wrong. The reason why so many mathematicians have fallen into this error seems to be this: For small values of the increments $h_1, h_2, \cdots, h_n$, H_m is an infinitesimal of the mth order, while all following terms being of higher order would be generally infinitely small in comparison with H_m. This is also the probable reason why some writers set

$$\varDelta = H_m + H_{m+1}$$

all other terms being simply neglected. For a single variable this is quite true, and without further thought it has been assumed for n variables. The fact that H_m can vanish for all points of a right line in any domain however small about P while the higher terms may not, a fact which at once marks a difference between functions of one and of several variables, has not made mathematicians more cautious in making the above assumption.

To return, this position being once taken, the conclusion is correctly drawn that if H_m is a definite form the function has an extreme at P, while if H_m is indefinite no extreme exists. A peculiar mistake is now made by some authors by neglecting the fact that still a third case is possible, viz., when H_m is semi-definite. So Williamson, who in treating the case for two variables x, y states that an extreme exists if only the determinant

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \geqq 0$$

at P, which of course is wrong. In passing I wish to remark for the benefit of the non-professional mathematician that the article on the Infinitesimal Calculus which this author contributed to the last edition of the Encyclopedia Britannica, vol. 13, p. 24, contains the same mistake. To show how the case when H_m is a semi-definite form is treated we consider only two variables because a higher number is not usually given. Suppose then that

$$f(a + h, b + k) - f(a, b) = \varDelta = H_2 + H_3 + \cdots$$

and $$D = 0.$$

Then for all points on the right line L

$$h\frac{\partial^2 f}{\partial x^2} + k\frac{\partial^2 f}{\partial x \partial y} = 0$$

passing through P, $H_2 = 0$. This being so the reasoning already characterized as false shows that for all points on L we must have $H_3 = 0$ for an extreme, so that the sign of Δ for these points depends upon that of H_4. Hence the conclusion: $f(x, y)$ has an extreme at P, if H_4 for points on L has the sign of $\frac{\partial^2 f}{\partial x^2}$ at P.

To restate the criticism: the fundamental error is the assumption that when H_m does not vanish the value of the remaining terms

$$R = H_{m+1} + H_{m+2} + \cdots$$

is small in comparison with H_m so that the sign of

$$\Delta = H_m + R$$

depends upon that of H_m.

To illustrate the various points, consider

$$f(x, y) = x^2 - 6xy^2 + 8y^4$$

$$= H_2 + H_3 + H_4$$

for the point $P = (0, 0)$. This is a particular case of Peano's classic example. Here in the first place $D = 0$, so that according to Williamson we have an extreme and in fact a minimum. Let us apply now Demartres' criterion. The line L is here $h = 0$; for these points $H_3 = 0$ while the sign of H_4 being positive is the same as that of

$$\frac{\partial^2 f}{\partial x^2} = +2.$$

Hence we would have also here a minimum. The fact is that P is neither maximum or minimum, as is at once seen in letting x, y move along the parabola

$$h = mk^2.$$

Then $$\Delta = k^4 (m - 2)(m - 4),$$

which shows that for no domain however small about P, has Δ always one sign, since by varying m, that is for different parabolas, the sign of Δ is different.

This example illustrates very well the fundamental error stated above. According to this, for those points for which

$H_2 \neq 0$ the value of $R = H_3 + H_4$ is small compared with H_2. That this is not so is easily seen. For let x, y move along the parabola

$$k^2 = h.$$

Then $H_2 = h^2$ while $R = 2h^2$, *i. e.*, R is twice as large as H_2. Still the sign of Δ is that of H_2.

For the parabola

$$3k^2 = h$$

$H_2 = h^2$ while $R = -\frac{10}{9} h^2$, so that the sign of Δ is that of R and not that of H_2.

For the parabola

$$-10k^2 = h,$$

$$H_2 = h^2, \quad R = \frac{68}{100} h^2,$$

which is the only one of these three examples in which $R < H_2$.

In terminating this note I wish to remark that in the general case where no attempt is made to use Taylor's series, but where we suppose only the first derivatives to exist at the point in question, P, the oversight is frequently made that the extremes of $f(x_1, x_2, \cdots, x_n)$ are stated to be the points for which

$$f_{x_i}'(x_1, x_2, \cdots, x_n) \quad (i = 1, 2, \cdots, n)$$

vanish. This of course is not so in general, since the value of a partial derivative at a point may be quite different from the value obtained by taking $x_1, x_2, \cdots, x_n$ arbitrary, computing the general expression of $f_{x_i}'(x_1, x_2, \cdots, x_n)$ and then setting in this $x_1 = a_1$, $x_2 = a_2$, $\cdots$, $x_n = a_n$.

YALE UNIVERSITY,
June, 1898.