CERTAIN CLASSES OF POINT TRANSFORMATIONS IN THE PLANE.

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A POINT transformation is an operation by which a point is carried into the position of some point. As far as the general definition is concerned the path described by the point and the time consumed in the change of position are immaterial, accordingly the coördinates of the final position of the point are functions only of the coördinates of its initial position, and a point transformation of the xy-plane into itself is represented analytically by two equations of the form.

$$x_1 = X(x, y), \quad y_1 = Y(x, y), \qquad (1)$$

where the functions X and Y are independent analytic functions in the Weierstrassian sense.

By such a transformation point is transformed into point, lineal element* into lineal element, curve into curve, intersecting curves into intersecting curves, curves in contact into curves in contact. By imposing geometrical conditions on the transformation, there result corresponding analytical conditions for the determination of the forms of the functions X and Y and thus particular categories of point transformations arise.

For example, if the transformation (1) is to change straight line into straight line, or in other words, to leave the ordinary differential equation of the second order

$$y'' = 0$$

invariant, the functions X and Y are found to have the forms

$$X \equiv \frac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3}, \quad Y \equiv \frac{a_2x + b_2y + c_2}{a_3x + b_3y + c_3}, \qquad (2)$$

which define the *general projective transformation* of the xy-plane. If, further, the point transformation is to transform parabola into parabola, or what amounts to the same

* The term lineal element is here used in the sense introduced by Lie, namely, to designate the ensemble of a point and a straight line through the point.

thing, to leave invariant the ordinary differential equation of the fourth order

$$5y'''^2 - 3y''y^{\text{iv}} = 0,$$

the functions X and Y have the forms

$$X \equiv a_1x + b_1y + c_1, \quad Y \equiv a_2x + b_2y + c_2, \tag{3}$$

which define the so-called *general linear transformation* which leaves the line at infinity invariant. If the point transformation is to leave the areas of all figures in the plane invariant, X and Y have the forms (3) with the additional condition that the determinant

$$\Delta \equiv a_1b_2 - a_2b_1 = 1. \tag{4}$$

If the circular points at infinity are to be invariant by the transformation, X and Y are of the form

$$X \equiv \rho\,(x \cos \alpha - y \sin \alpha + a), \quad Y \equiv \rho\,(x \sin \alpha + y \cos \alpha + b)\ ; \tag{5}$$

arbitrary ρ gives a so-called *similitudinous transformation* which preserves the forms of figures ; ρ equal to unity gives a *Euclidian motion* in the plane. If the product of the radii vectores of the original point and the transformed point is to be constant, say unity, we have the *transformation by inversion* whose X and Y are defined by the equations

$$X \equiv \frac{x}{x^2 + y^2}, \quad Y \equiv \frac{y}{x^2 + y^2}. \tag{6}$$

Examples might be multiplied further, but it is not to the purpose here. It may be added, however, that some of the most interesting cases are those where X and Y are transcendental functions, notably the *logarithmic* and *exponential* transformations.

It is proposed in this note to determine the forms of X and Y and present a few of the properties of the point transformations of the xy-plane respectively defined by the following characteristic properties : 1° Cartesian subtangent of the transformed curve is to be $\frac{m}{n}$ times the Cartesian subtangent of the original curve ; 2° Cartesian subnormal $\frac{m}{n}$ times Cartesian subnormal ; 3° Cartesian subtangent $\frac{m}{n}$ Cartesian subnormal; 4° Cartesian subnormal $\frac{m}{n}$ Cartesian subtangent; 5° polar subtangent $\frac{m}{n}$ polar subtangent; 6° polar subnormal $\frac{m}{n}$ polar subnormal; 7° polar subtangent $\frac{m}{n}$ polar subnormal; 8° polar subnormal $\frac{m}{n}$ polar subtangent.

1°. The point transformations of this category are to be of such a nature that point is changed into point and curve transformed into curve in such a manner that the Cartesian subtangent of the transformed curve is equal to $\frac{m}{n}$ times the Cartesian subtangent of the original curve.

The Cartesian subtangent of a curve $f(x,y) = 0$ at a point (x,y) is

$$y\frac{dx}{dy}, \quad \text{or } \frac{y}{y'}; \tag{7}$$

then the defining property of the point transformation sought gives the analytical condition

$$y_1\frac{dx_1}{dy_1} \equiv \frac{m}{n}\, y\frac{dx}{dy}. \tag{8}$$

This condition (8) is now to be turned to account to find the forms of X and Y in equation (1). Substituting in (8) the values of x_1 and y_1 from (1) there results

$$Y\frac{X_x + X_y y'}{Y_x + Y_y y'} \equiv \frac{m}{n}\frac{y}{y'},$$

or

$$nX_y y'^2 + (nYX_x - myY_y)\, y' - myY_1 \equiv 0. \tag{9}$$

This last equation must be identically true for all values of y', hence equating to zero the coefficients of the several powers of y'

$$X_y \equiv 0, \quad Y_x \equiv 0, \quad nYX_x - myY_y \equiv 0 \tag{10}$$

hence

$$X = X(x), \quad Y = Y(y), \quad X_x = m, \quad \frac{Y_y}{Y} = \frac{n}{y}\,* \tag{11}$$

* The last identity (10) breaks up into these two parts since X and Y are independent functions by hypothesis.

and finally

$$X = mx + a, \quad Y = by^n. \tag{12}$$

Then the transformation

$$x_1 = mx + a, \quad y_1 = by^n \tag{13}$$

is the most general point transformation which changes a curve c into a curve γ in such a manner that the subtangent of the point (x, y) of c is the $\frac{n}{m}$ th part of the subtangent of the corresponding point (x_1, y_1) of the transformed curve γ.

2°. The point transformation is to be found which changes curve into curve in such a manner that the Cartesian subnormal of the transformed curve is $\frac{m}{n}$ times the Cartesian subnormal of the original curve. The value of the Cartesian subnormal is

$$y\frac{dy}{dx}. \tag{14}$$

By virtue of the transformation there exists an identity of the form

$$y_1 y_1' \equiv \frac{m}{n} yy.' \tag{15}$$

Whence

$$myX_y y'^2 + (nYY_y - myX_x)y' - nYY_x \equiv 0. \tag{16}$$

This identity must obtain for all values of y', hence as in the preceding case

$$X = X(x), \quad Y = Y(y), \quad X_x = n, \quad YY_y = my, \tag{17}$$

therefore

$$x_1 \equiv X = nx + a, \quad y_1 \equiv Y = \sqrt{my^2 + b} \tag{18}$$

is the most general point transformation of the characteristic property 2°.

3°. By this third class of point transformations the Cartesian subtangent of the transformed curve is to be equal to the Cartesian subnormal of the original curve multiplied by the ratio $\frac{m}{n}$. This geometrical property expressed analytically becomes

$$\frac{y_1}{y_1'} \equiv \frac{m}{n} yy'. \tag{19}$$

Whence

$$myY_y y'^2 + (myY_x - nYX_y)y' - nYX_x \equiv 0; \tag{20}$$

hence

$$X = X(y), \quad Y = Y(x), \quad X_y = my, \quad Y_x = nY; \tag{21}$$

accordingly

$$x_1 = \frac{m}{2} y^2 + a, \quad y_1 = be^{nx} \tag{22}$$

are the equations of the most general point transformation possessing the assigned property.

4°. The defining property of this category of point transformation is that the Cartesian subnormal of the transformed curve shall be to the Cartesian subtangent of the original curve in the ratio m to n; hence the condition

$$y_1 y_1' \equiv \frac{m}{n} \frac{y}{y'} \tag{23}$$

which gives

$$nYY_y y'^2 + (nYY_x - myX_y)y' - myX_x \equiv 0. \tag{24}$$

Therefore

$$X = X(x), \quad Y = Y(y), \quad yX_y = n, \quad YY_x \equiv m; \tag{25}$$

and finally

$$x_1 = n \log y + a, \quad y_1 = \sqrt{2mx + b} \tag{26}$$

are the equations of the transformation sought.

5°. The next four transformations are more readily studied in polar coördinates. Let the general point transformation of the $(r\theta)$ plane into itself be

$$r_1 = R(r, \theta), \quad \theta_1 = \Theta(r, \theta) \tag{27}$$

The transformation 5° is subject to the limitation that the polar subtangent of the transformed curve be to the polar subtangent of the original curve in the ratio m to n; this geometrical condition gives the following analytical condition for determining the forms of the functions R and Θ,

$$r_1^2 \frac{d\theta_1}{dr_1} \equiv \frac{m}{n} r^2 \frac{d\theta}{dr}, \tag{28}$$

or

$$mr^2 R_\theta \theta'^2 + (mr^2 R_r - nR^2 \Theta_\theta)\, \theta' - nR^2 \Theta_r \equiv 0\,; \tag{29}$$

hence

$$R = R(r), \quad \Theta = \Theta(\theta), \quad r^2 R_r = nR^2, \quad \Theta_\theta = m\,; \tag{30}$$

therefore

$$r_1 \quad R \equiv \frac{r}{ar + n} \qquad \theta_1 \equiv \Theta = m\theta + \beta \tag{31}$$

are the equations of the most general point transformation possessing the above-named property.

Similarly for the other cases we find:

6° Polar subnormal $\frac{m}{n}$ times polar subnormal,

if $$nR_\theta \theta'^2 + (nR_r - m\Theta_\theta)\, \theta' - m\Theta_r \equiv 0; \tag{32}$$

whence $$r_1 = mr + a, \quad \theta_1 = n\theta + \beta. \tag{33}$$

7°. Polar subtangent $\frac{m}{n}$ polar subnormal,

if $$nR^2 \Theta_\theta \theta'^2 + (nR^2 \Theta_r - mR_\theta)\theta' - mR_r \equiv 0; \tag{34}$$

whence $$r_1 = \frac{1}{a - n\theta}, \quad \theta_1 = mr + \beta. \tag{35}$$

8°. Polar subnormal $\frac{m}{n}$ solar subtangent,

if $$mr^2 \Theta_\theta \theta'^2 + (mr^2 \Theta_r - nR_\theta)\theta' - nR_r \equiv 0; \tag{36}$$

whence $$r_1 = m\theta + a, \quad \theta_1 = \beta - \frac{n}{r}.$$

9. A family of transformations is said to form a Lie group of transformations when the product of any two transformations of the family is equivalent to a transformation belonging to the family. By the product of any number of transformations is meant the transformation equivalent to their successive application. It is to be observed that these transformation products do not obey exactly the same laws as ordinary algebraic products; they always follow the associative law but do not of necessity obey the commutative law.

Consider the family of transformations 1°. Let S be the transformation of the family which changes the point (x, y) into the point (x_1, y_1) given by,

$$x_1 = mx + a, \quad y_1 = by^n; \qquad S \qquad (37)$$

let T be the transformation of the same family which transforms the point (x_1, y_1) into the point $x_2, y_2)$ given by the equations, say

$$x_2 = m_1 x_1 + a_1, \quad y_2 = b_1 y_1^{n_1}; \qquad T \qquad (38)$$

the product ST, that is, the transformation which changes the original point (x, y) directly into the point (x_2, y_2) is obtained by eliminating x_1 and y_1 from the equations (37) and (38); this elimination yields

$$\left.\begin{array}{l} x_2 = mm_1 x + m_1 a + a_1 = m_2 x + a_2, \\ y_2 = b^{n_1} b_1 y^{nn_1} = b_2 y^{n_2}. \end{array}\right\}; \quad ST \equiv V \qquad (39)$$

the equations (39) are of the same form as the equations (37) and (38) *i. e.*, the transformation V equivalent to the successive application of the transformations S and T of the family 1° has the same form as S and T and hence belongs to the family 1° ; this remarkable property is compressed into the statement that the family of point transformations 1° constitutes a continuous group of transformations.

Similarly it may be shown that the transformations 2° form a continuous group of transformations.

On the contrary, the transformations 3° do not constitute a group, as the following consideration shows. Let P be the transformation of the family 3° which changes the point (x, y) into the point (x_1, y_1) given by

$$x_1 = \frac{m}{2} y^2 + a, \quad y_1 = be^{nx}; \qquad P \qquad (40)$$

the point (x_1, y_1) is changed by a transformation Q of this same family into the point (x_2, y_2) for which, say

$$x_2 = \frac{m_1}{2} y_1^2 + a_1, \quad y_2 = b_1 e^{n_1 x}; \qquad Q \qquad (41)$$

the transformation W equivalent to the successive application of P and Q is found by eliminating x_1 and y_1 from the equations (40) and (41). This elimination gives the transformation which changes (x, y) directly into (x_2, y_2) and yields as the equations of the transformation

$$x_2 = \frac{m_1}{2} b^2 e^{2nx} + a_1, y_2 = b_1 e^{n_1 \frac{m}{2} y^2 + n_1 a}; PQ \equiv W \qquad (42)$$

but the equations (42) are not of the form (40), hence W, the product of P and Q, does not belong to the family containing P and Q, and the family of transformations 3° does not form a continuous group of transformations.

In the same manner it appears that the transformations 4° do not form a group.

It may also be verified readily that the families 5° and 6° possess the group property; and that the families 7° and 8° are not continuous groups.

10. An *infinitesimal* point transformation is one by which a point suffers an infinitesimal change of position. It differs from the identical transformation by an infinitesimal. If, by virtue of an infinitesimal point transformation, x and y receive the increments respectively

$$\delta x = \xi(x,y)\delta t, \quad \delta y = \eta(x,y)\delta t,$$

the infinitesimal transformation is represented symbolically by

$$Vf \equiv \xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y}.*$$

The equations of the group 1° give the identical transformation

$$x_1 = x, \quad y_1 = y$$

for the system of values

$$a = 0, \quad b = 1, \quad m = 1, \quad n = 1;$$

hence the system

$$a = \delta a, \quad b = 1 + \delta b, \quad m = 1 + \delta m, \quad n = 1 + \delta n$$

yields a transformation of the form 1° differing from the identical transformation by an infinitesimal, and the transformation corresponding to this system is an infinitesimal transformation of the group 1°; accordingly

$$x_1 = (1 + \delta m)x + \delta a,$$

$$y_1 = (1 + \delta b)y^{1+\delta n} = y(1 + \delta b)(1 + \frac{\delta n \log y}{1} + \cdots)$$

* For these details consult any one of Lie's published works.

are the equations of the infinitesimal transformation of the group 1°. Whence

$$x_1 - x \equiv \delta x = x\delta m + \delta a, \quad y_1 - y \equiv \delta y = y(\delta b + \delta n \log y)$$

to terms of the second order; therefore, if we make an obvious change in the designation of the infinitesimal constants δa, δb, $\cdots$, we have

$$\delta x = \xi(x, y)\delta t = (\alpha x + \beta)\delta t, \quad \delta y = \eta(x, y) = y(\lambda + \mu \log y)\,\delta t$$

and the symbol of the infinitesimal transformation of our group 1° is

$$Vf \equiv (\alpha x + \beta)\frac{\partial f}{\partial x} + y(\lambda + \mu \log y)\frac{\partial f}{\partial y}.$$

The system of values

$$a = 0, \quad b = 0, \quad m = 1, \quad n = 1$$

makes 2° an identical transformation, hence the system of values

$$a = \delta a, \quad b = \delta b, \quad m = 1 + \delta m, \quad n = 1 + \delta n$$

determines an infinitesimal transformation of the group 2°, namely

$$x_1 = (1 + \delta n)x + \delta a, \quad y_1 = \sqrt{(1 + \delta m)y^2 + \delta b}\,;$$

whence

$$x_1 - x \equiv \delta x = x\delta n + \delta a, \quad y_1 - y \equiv \delta y = \frac{\delta m}{2}y + \frac{\delta b}{y}$$

by expanding y_1 by the binomial formula and by neglecting terms of higher order. Then putting

$$\delta n = \lambda \delta t, \quad \delta a = \mu \delta t, \quad \delta m = \rho \delta t, \quad \delta b = \sigma \delta t,$$

the infinitesimal transformation of the group 2° has the symbolic form

$$Vf \equiv (\lambda x + \mu)\frac{\partial f}{\partial x} + (\rho y + \sigma y^{-1})\frac{\partial f}{\partial y}$$

Since the families 3° and 4° do not possess the identical transformation they have no infinitesimal transformations.*

11. We have already found that the family of transformations 5° is a group, hence we shall find an infinitesimal transformation belonging to the family. In fact the following values of the arbitrary constants

$$\alpha = 0, \quad \beta = 0, \quad m = 1, \quad n = 1$$

reduce the transformation 5° to the identical transformation

$$r_1 = r, \quad \theta_1 = \theta.$$

Then the system of values

$$\alpha = \delta\alpha, \quad \beta = \delta\beta, \quad m = 1 + \delta m, \quad n = 1 + \delta n$$

determines the transformation of the group which differs from the identical transformation by an infinitesimal, that is, the infinitesimal transformation of the group, namely,

$$r_1 = \frac{r}{r\delta\beta + 1 + \delta n}, \qquad \theta_1 = (1 + \delta m)\,\theta + \delta\alpha$$

or to terms of the second order

$$r_1 = r\,(1 - \delta n - r\delta\beta), \quad \theta_1 = (1 + \delta m)\theta + \delta\alpha\,;$$

whence

$$\delta r \equiv r_1 - r = -r(x\delta\beta + \delta n), \quad \delta\theta \equiv \theta_1 - \theta = \theta\delta m + \delta\alpha;$$

or putting

$$\delta\beta = \lambda\delta t, \quad \delta n = \mu\delta t, \quad \delta m = \nu\delta t, \quad \delta\alpha = \rho\delta t,$$

the symbol of the infinitesimal transformation of the group 5° is

$$Vf \equiv \xi(r, \theta)\frac{\partial f}{\partial r} + \eta(r, \theta)\frac{\partial f}{\partial \theta} \equiv (\nu\theta + \rho)\frac{\partial f}{\partial \theta} - r(\lambda r + \mu)\frac{\partial f}{\partial r}.$$

In the same manner we find the infinitesimal transformation of the group 6° to have the symbol

$$Vf \equiv (fr + g)\frac{\partial f}{\partial r} + (h\theta + k)\frac{\partial f}{\partial \theta}.$$

The families 7° and 8° do not contain the identical transformation, hence they do not have infinitesimal transformations.*

* It is to be observed that it is not for the reason that these families are not groups that they do not contain infinitesimal transformations. For example, the family of ∞^1 transformations $x_1 = xt$, $y_1 = y + t - 1$, obviously does not form a group, and yet the family contains the identical transformation, as is seen by putting t equal to unity, and the infinitesimal transformation $x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$ found by putting $1 + \delta t$ for the parameter.

12. Apart from their properties as transformations, the above transformations are of interest because of certain applications to plane curves, notably to spirals which it is hoped to bring out in a subsequent note.

Since finishing this note the writer finds that the *finite* forms of the transformations discussed were given by Laisant in the *Nouvelles Annales de Mathématiques*, 2d series, vol. 7 (1868), p. 318, in the solution of a problem proposed by Haton de la Goupillière, *Nouvelles Annales*, vol. 6 (1867), problem No. 803. The wide divergence between the properties and the points of view of the present note and the solution referred to seem to warrant its presentation to the Society. The above-mentioned volumes of the *Nouvelles Annales* are to be had in the Library of Congress.

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CONTINUOUS GROUPS OF CIRCULAR TRANSFORMATIONS.*

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THE object of this paper is to present the outlines of a fairly complete theory of the continuous groups of linear fractional transformations of one variable. The method employed is quite different from the methods of Lie. Lie's classic theory is based upon the infinitesimal transformation; I shall make but little use of the infinitesimal transformation, but shall develop the subject from the consideration of the essential parameters of the transformation. The complex plane is chosen because it beautifully illustrates the methods. I have put together some old and some new facts and have sought to build up a general theory.

* Several terms have been proposed to designate the linear fractional transformations of the complex plane. Möbius introduced the term "Kreisverwandtschaft." Mathews' Theory of Numbers, page 107, translates this as "Möbius' Circular Relation." Professor Cole, in *Annals of Mathematics*, vol. 5, page 137, refers to "Orthomorphic Transformation," following Cayley; this seems too general for the special case here considered, since it is applicable to all conformal transformations. Darboux, in his Theorie des Surfaces, vol. 1, page 162, uses "transformation circulaire." It seems to me that "Circular Transformation" is the best yet proposed, for the fundamental property is expressed in the name.