Spheres cutting 4 given spheres s_1, s_2, s_3, s_4 at given angles φ_1, φ_2, φ_3, φ_4.

H contains the determinant formed by the discriminant of φ, bordered with the coefficients of the planes as a factor. This expression is of the same degree as H, both in $a_{i,k}$ and the coefficients of φ, hence the remaining factor is numerical. By comparing corresponding terms in the two expressions, this is seen to be ± 1. The sign to be prefixed depends upon h in both cases, but the H method has two advantages, viz.:

H is of order $n - 1$; the bordered discriminant is of order $2n$.

H appears in many other connections; the bordered discriminant would have to be calculated for this purpose.

CORNELL UNIVERSITY,
July 19, 1897.

FLUID MOTION.

Hydrodynamics. By HORACE LAMB, F. R. S. Cambridge University Press. 1895. 8vo., pp. 604.

The appearance of a new treatise on any branch of higher mathematics rarely calls for anything else than congratulations to the author, and the volume before us is no exception to the rule. The problems of hydrodynamics present so many difficulties and the opportunities for students to obtain a connected view of them are so rare that any additional help is valuable. Professor Lamb, however, has gone much further than merely producing a continuous account of the subject as it stands at the present time. He has given us a treatise which will easily rank first amongst those in the English and perhaps in any language. The only other English treatise of the same scope, that by Basset published in 1888, although an advance on those which had previously appeared, rather suffers by comparison, both in its plan and the manner in which it is carried out.

In looking over the list of authors which Professor Lamb gives in an index, we are struck by the frequency with which four or five names occur, and a closer examination of the references attached to other names reveals the fact that the mathematical development of hydrodynamics has been almost entirely due to these four or five writers. It must be concluded from this, either that some cause has prevented all but a very few mathematicians from seriously attacking it, or that, of those who have attacked it, few have been able to materially advance it. Whether this is mainly due to the inherent difficulties of the subject or to some other cause, it is not easy to say. Hydrodynamics certainly seems to present more possibilities of further development to pure and applied mathematicians than many other subjects which have been more frequently attacked. But one cannot help thinking that another factor has had far more influence in giving direction to the work of a student entering the field, namely, the manner in which the subject is presented to him for the first time.

It is customary to introduce almost every branch of physics either from the experimental side or by means or an elementary text-book setting forth the main ideas on which the development is made, the reading of the text-book going sometimes with and sometimes without experimental demonstrations. The mechanics of particles and rigid bodies, hydrostatics, sound, heat, geometrical and physical optics can all be approached through elementary text-books and from them it is not difficult for the student to obtain sufficient to enable him to deal intelligently with the experiments or, if he has received a mathematical training, to proceed to the higher treatises. In electricity and magnetism the number of such books is enormous. But in hydrodynamics, the case is quite different. Except where an individual instructor or professor may happen to have the ability and inclination to present it to the student in a form in which he can get the ideas without being obliged to devote his attention mainly to difficulties of analysis—and such men appear to be rare—there are only two courses open. One is to read it in the manuals, written for the practical engineer, in which the theory is scarcely touched, the other is to approach it from the mathematical side entirely, the applications coming only after the difficulties of the general theories have been mastered. For the latter course Lamb's and Basset's treatises are available ; Besant's hydromechanics contains only a short account. Thompson and Tait's Natural Philosophy also contains certain parts of the subject. In French and German books, fluid motion is generally treated as a part of theoretical dynamics with very few applications to problems.

That a beginner should be thus restricted in his efforts to obtain a grasp of this subject can hardly be right. When so much is done in other branches towards the explanation of fundamental theories and methods, it can scarcely be unreasonable to ask for a small or medium sized book in which the ideas underlying the modern treatment of fluid motion shall be dealt with more fully than is possible in a higher treatise. Such terms as flow, circulation, molecular rotation and the like, are more than mere names for mathematical formulæ and they could surely be treated without loss of rigour from the physical point of view. Indeed many hints of such treatment are to be obtained from the papers of Lord Kelvin, Stokes and others. Again, many of the applications which lie nearest to us, such as the tides and wave-motion when the squares of small quantities are neglected, should be capable of being explained by reducing the results to harmonic motions—a method already successfully used in other subjects. In the cases where the mathematical investigations are complicated or difficult no harm will be done by giving results along with a careful explanation of their limitations, *e. g.*, the lines of flow of a liquid past an obstacle, the motions of the particles in simple wave-forms, the general properties of a vortex ring, etc. May we not fairly ask one of our prominent mathematical physicists, in the interests of the subject, to supply this want. It is perhaps somewhat the fashion to decry attempts to popularize the higher regions of science, but there is little doubt that the stimulus created by the interest of a large number of students, has indirectly a beneficial effect on the subject by bringing to it more of those who are or hope to be engaged in research and especially those, at present devoted to pure mathematics, whose assistance might be of the greatest value.

In making this appeal there is no desire to depreciate the text-books and treatises which have hitherto been published. Each has its own special object in view—the practical text-book for the engineer or the mathematical one for the student of pure science. What seems to be required is one which occupies a place intermediate between these two classes, a book which, while keeping in view the practical applications of problems hitherto attacked with some success, shall also fully explain the elements of the methods by which the problems are treated, special stress being laid on giving, wherever possible, a "picture view" of what is considered to actually take place when the fluid is in motion.

But we must return to our main topic. In his preface Professor Lamb states that the new volume may be regarded as a second edition of his "Treatise on the Mathematical Theory of the Motion of Fluids," published in 1879. The alterations and additions, however, have been so extensive that he has thought it right to change the title. In fact, the new work is more than twice as large as the older one.

Three new chapters have been added. One, on Problems in Three Dimensions, chiefly contains matter which has appeared in scientific journals since 1879, and it gives a fair idea of what progress has been made in this direction. The earlier chapter on Waves in Liquids, consisting of 28 pages has been expanded into two long chapters of about a hundred pages each. A new chapter has also been added on rotating bodies of fluid—a subject quite untouched in the earlier edition.

These additions do not by any means give a complete idea of the new volume, for so many improvements in details have been effected, especially in the matter of references, that we hardly recognize the ancestor. Professor Lamb's success in keeping the mathematics in a subordinate position, without loss of rigour, and in bringing forward physical interpretations of the formulae, has been well maintained. His theoretical treatment is, as might be expected, on the lines of the most recent methods, and he has given so many applications to every-day problems that we are almost tempted to lose sight of how much remains to be done in what has already been achieved. In fact, the main features of most of the recent as well as of the earlier work, with some few exceptions, have been included, and where omissions are intentionally made, full references are given.

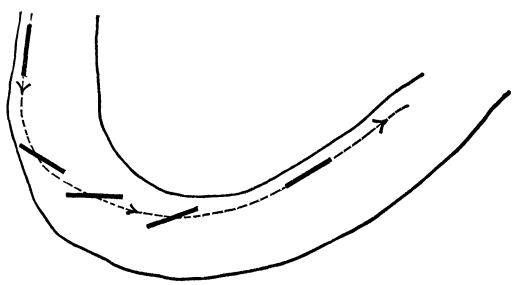
Coming to a more detailed examination, we find in the first chapter the development of the usual equations of motion, attention being also paid to expressions for the energy and the impulsive generation of a given state of motion. The importance of the latter in separating rotational from irrotational motion, insisted on by Lord Kelvin, is developed at the beginning of Chapter II., where the notion of a velocity potential is first introduced. It is here that Professor Lamb has made an innovation which, as he himself admits, is open to criticism, namely, the changing of the sign of the velocity potential. The question is similar to that of the universal introduction of the metric system, and the arguments in favor of a change seem much less strong. The point at issue is whether the trouble caused by making the change is fully compensated in the advantages gained. The rest of the chapter is mainly occupied with the integration of the equations in special cases, *e. g.*, those of steady motion and the efflux of liquid from a tank.

Chapter III., on Irrotational Motion, is mainly an essay on Green's theorem with Lord Kelvin's extension, some theorems on sources and sinks being added to the corresponding chapter in the earlier volume. In Chapter IV, on the motion of a liquid in two dimensions, full use is made of the properties of the complex variable and of conform representation. The proofs of these properties might well have been omitted. It is true that in 1879 no text-book in the English language on the Theory of Functions existed, but the same cannot be said now. The examples have been greatly improved by drawings of the stream-lines in the various cases. One of the most interesting of the latter is the case of a stream impinging on a flat rectangular board ; the tendency of the board to turn its flat side to the stream is fully explained. Several examples of this tendency in the motion of a boat, are given by Thompson and Tait, Art. 325. · The rules followed by coxswains of eights on the Cam and probably on other narrow winding rivers, appear to be due to the same cause.

The problem is this: What is the best method of steering an eight-oared boat when racing round a sharp bend in a narrow river? The rules usually given are, first, that the coxswain should approach the corner from the farther side of the river ; secondly, that he should start to turn as late as possible, and, thirdly (this is very frequently forgotten), if the bend be a sharp one, that he should start to turn almost suddenly and not gradually. If these rules be followed, it is found that only a very slight use of the rudder is necessary after the boat has once begun to turn. Should the coxswain take the corner badly, by starting to turn too gradually, the boat will tend to travel across to the outside of the bend and the rudder has to be put "hard on" the whole way round, thus stopping the "way" of the boat.

The explanation of these rules appears to be due to the couple tending to turn the boat broadside on to its direction of motion. The magnitude of this couple varies approximately as the angle between the boat's length and the direction of motion of the boat as a whole. Hence, in order that the couple may have as great an effect as possible throughout the turn, the angular velocity of the boat about a vertical line should be increased to its maximum as soon as possible. In other words, the rudder should be so used that the boat may start to turn round as quickly as possible; after attaining its maximum angular velocity, the couple will tend to assist it in turning and thus the necessity of afterwards using the rudder very much will be avoided. This is usually best attained by getting the tiller over rapidly but not so suddenly as to stop the way of the boat. Hence, in steering round a bend whose greatest curvature is at the centre of the bend (this is generally the case), the boat should be taken to the outer side of the river and the start to turn made rather late, so that the angular velocity may be got up quickly and the time occupied in turning made as small as possible.

When this angular velocity is obtained the bow will be slightly turned towards the inner bank owing to the momentum of the boat sideways (that is, in a sailing boat, the leeway) tending to carry it to the outer bank.



The diagram shows the course of the boat, the dotted lines representing the motion of some central point (the centre of mass, or "centre of reaction") of the boat, and the straight lines, the general angular direction, somewhat exaggerated, of the boat's length to its course. The arrows show the direction of motion.

The chief problems considered in chapter V., on the Irrotational Motion of a liquid in three dimensions, are the motion of one sphere, of two spheres and of an ellipsoid, in an infinite liquid. The next chapter deals mainly with the dynamical theory of the motion of solids through a liquid in the cases where a velocity potential may still be supposed to exist. The additions consist of an account of the method of using generalized coördinates, with some applications and several new problems.

In treating of Vortex Motion, Professor Lamb confines himself chiefly to the main propositions and theorems. After giving several proofs of the fact that vortex lines move with the fluid and pointing out the mathematical analogies with electromagnetics and with the conception of sources and sinks, a few articles are devoted to the impulse and energy of a vortex system. The chief interest of vortices lies in the vortex rings adopted by Lord Kelvin to formulate a theory of the constitution of matter. For analytical reasons, these rings have to be treated chiefly as having sections small in comparison with their diameter, and hence, for many purposes, the consideration of them is the same as that of rectilinear vortices. This part of the subject has now almost a literature of its own, one complete volume, that of Poincaré "*Théorie des Tourbillons*" having appeared last year. Professor Lamb has entirely omitted the investigations of Professor J. J. Thomson and has merely given one or two special examples. We cannot but think that this chapter might have been made more interesting without adding very much to its length. The strongest impression conveyed is that the vortex is a mere mathematical abstraction.

The two chapters on Tidal and Surface Waves respectively are in striking contrast to that just noticed. The happy combination of the results of theory and observation and the completeness of the treatment without tedious details, show Professor Lamb at his best. A glance at the topics selected will indicate the scope of the two chapters. In the former we have the Canal theory of the Tides, waves in a canal of variable section, waves of finite amplitude, propagation of waves in a sheet of waves, the tidal oscillations of a rotating sheet of water (which contains new matter) and a short appendix on tide-generating forces. The chapter on Surface Waves deals with waves on a sheet of water or a straight canal, trains of water, waves of finite amplitude, standing waves, the oscillations of a spherical mass of liquid, and capillary waves. One or two points call for remark. In several places the motions are characterized as being "infinitely small" when it is simply meant that small quantities of the second and higher orders are neglected; in one case (Art. 178), the small quantity is 140 feet, 10000 feet being considered a finite quantity. This misuse of the word "infinitely," is unfortunately only too common in physical text-books. The investigation in Art. 206 is not altogether clear. The arguments for the existence of free oscillations in a canal of any section as expressed by a Fourier's series for the part of the velocity potential which is independent of the time, appear to be somewhat vitiated by the neglect of the possibility of satisfying the surface conditions under the particular assumptions made.

Waves of Expansion which chiefly concern the theory of sound, and Viscosity which may be considered as a complement to Chapter VIII., follow. Finally, we are given a rather brief account of the equilibrium of rotating masses of liquid. Professor Darwin's figures of the various forms of

Jacobian ellipsoids are reproduced, but the investigations of Darwin and Poincaré on other possible forms receive little more than a passing mention.

The author is to be congratulated on the completion of a task which will earn him the gratitude of all those who are now or may in the future be interested in hydrodynamics. The manner in which his materials are put together and the fact that he never loses sight of the practical applications make the book unusually interesting; the large number of references will enable anyone to find out all that has been done in any branch. In fact, although the volume is a bulky one, we cannot but regret that it has not been divided into two and extended by including the investigations noted above as omitted, and by giving a much fuller index of subjects.

ERNEST W. BROWN.

NOTES.

THE Toronto Meeting of the British Association for the Advancement of Science, held August 18–25, like the Montreal meeting of 1884, proved a very gratifying success. The attendance was about 1,300 as against 1,700 at the Montreal Meeting. The papers presented were in number and value well up to the Association's standard. The meetings of the American Botanical Society, the American Society for the Promotion of Engineering Education, and the AMERICAN MATHEMATICAL SOCIETY were held in Toronto immediately preceding the Association Meeting, and many members of these Societies remained to attend the session of the Association. Many members of the American Association were also in attendance, a large number coming directly from Detroit after the adjournment of the American Association in the preceding week. In all about 250 Americans were present, to whom a cordial reception was extended by the Association throughout the proceedings. They participated freely in the general and sectional meetings; a considerable number were placed on important committees; and several were appointed vice-presidents.

The officers of Section A,—Mathematical and Physical Science—, were: President, Professor A. R. FORSYTH; Vice-Presidents, Professor W. E. AYRTON, Professor G. C. FOSTER, Professor O. HENRICI, Dr. G. W. HILL, Professor A. JOHN-