on the structure of the substitution group. Similarly Lie makes the integration of a system of partial differential equations which admits of a finite continuous group of transformations depend upon the integration of a series of auxiliary systems, and the number of these systems, their nature and the way in which they are related to one another depends on the structure of the continuous group. The importance of the structure of finite continuous groups is further illustrated in Picard's theory of linear differential equations and more generally in the theory of differential equations which admit of a fundamental system of integrals.

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# A GEOMETRICAL LOCUS CONNECTED WITH A SYSTEM OF COAXIAL CIRCLES.

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SUPPOSE there be given a sheaf or pencil of rays whose centre is $P$ and a system of coaxial circles lying in the same plane. Two rays of the sheaf will be tangent to each circle of the system and two circles of the system will be touched by each ray of the sheaf. If we start with any one circle of the system $k_1$ and one of its tangent rays $m_1$, it is easy to determine the second circle $k_2$ to which this ray is tangent, then the second ray $m_2$ tangent to $k_2$, then the second circle $k_3$ to which $m_2$ is tangent and so on indefinitely, the circles and rays forming a continuous chain. Whether or not this chain of circles and rays will return into itself depends upon the location of the centre $P$ with respect to the system of coaxial circles and I undertake in the present paper *to find the locus of the point $P$ for which the chain of circles and rays will close with three circles and three rays.*

In other words, I undertake to find the locus of points through which three lines can be drawn tangent to three circles of a coaxial system in pairs.

Any two circles and a pair of common tangents form such a closed circuit containing two elements of each kind and the point $P$ may be any one of the six vertices of the complete quadrilateral formed by the four tangents common to the two chosen circles.

The problem before us resolves itself largely into one of elimination and in order to make the elimination as simple as possible I shall confine myself to the consideration of the case in which two of the circles are equal. This limitation upon the circles does not, however, particularize the locus, since a point which admits one such triplet of lines through it admits also an infinite number of such triplets. In order, too, that the rays may all be real in a system of threes we shall assume that the circles have real intersections.

Let the three circles of the closed circuit be represented by the equations:

$$(1) \qquad x^2 + y^2 - 2k_1 x - \delta^2 = 0,$$

$$(2) \qquad x^2 + y^2 + 2k_1 x - \delta^2 = 0,$$

$$(3) \qquad x^2 + y^2 - 2k_3 x - \delta^2 = 0;$$

and the three lines by

$$(1) \qquad y - m_1 x - c_1 = 0,$$

$$(2) \qquad y - m_2 x - c_2 = 0,$$

$$(3) \qquad y - m_3 x - c_3 = 0,$$

the coördinate axes being the radical axis of the system of circles and the line of centres.

The condition that line (1) may be tangent to circle (1) is

$$m_1^2 \delta^2 - 2m_1 c_1 k_1 - c_1^2 + k_1^2 + \delta^2 = 0,$$

that line (1) may be tangent to circle (2) is,

$$m_1^2 \delta^2 + 2m_1 c_1 k_1 - c_1^2 + k_1^2 + \delta^2 = 0,$$

that line (2) may be tangent to circle (2) is

$$m_2^2 \delta^2 + 2m_2 c_2 k_1 - c_2^2 + k_1^2 + \delta^2 = 0,$$

that line (2) may be tangent to circle (3) is

$$m_2^2 \delta^2 - 2m_2 c_2 k_3 - c_2^2 + k_3^2 + \delta^2 = 0,$$

that line (3) may be tangent to circle (3) is

$$m_3^2 \delta^2 - 2m_3 c_3 k_3 - c_3^2 + k_3^2 + \delta^2 = 0,$$

that line (3) may be tangent to circle (1) is

$$m_3^2 \delta^2 - 2m_3 c_3 k_1 - c_3^2 + k_1^2 + \delta^2 = 0.$$

If now we eliminate the $m$'s, $c$'s and $k$'s from the equations of the three lines and the six conditions of tangency we shall obtain the locus of the point of concurrence of the tangents. This elimination is clearly possible.

Proceeding with the elimination we may substitute the values of $c_1$, $c_2$ and $c_3$, derived from the equations of the tangents, into the six conditions of tangency and so obtain

$$(1)\quad m_1^2(\delta^2 + 2k_1 x - x^2) + 2y(x - k_1)m_1 + (\delta^2 + k_1^2 - y^2) = 0$$

$$(2)\quad m_1^2(\delta^2 - 2k_1 x - x^2) + 2y(x + k_1)m_1 + (\delta^2 + k_1^2 - y^2) = 0$$

$$(3)\quad m_2^2(\delta^2 - 2k_1 x - x^2) + 2y(x + k_1)m_2 + (\delta^2 + k_1^2 - y^2) = 0$$

$$(4)\quad m_2^2(\delta^2 + 2k_3 x - x^2) + 2y(x - k_3)m_2 + (\delta^2 + k_3^2 - y^2) = 0$$

$$(5)\quad m_3^2(\delta^2 + 2k_3 x - x^2) + 2y(x - k_3)m_3 + (\delta^2 + k_3^2 - y^2) = 0$$

$$(6)\quad m_3^2(\delta^2 + 2k_1 x - x^2) + 2y(x - k_1)m_3 + (\delta^2 + k_1^2 - y^2) = 0$$

six equations from which to eliminate the three $m$'s and the two $k$'s.

It will be observed that the values of $m_1$ derived from equation (2) are the same as the values of $m_2$ derived from equation (3); that the values of $m_2$ derived from equation (4) are the same as the values of $m_3$ derived from equation (5); and that the values of $m_3$ derived from equation (6) are the same as the values of $m_1$ derived from equation (1). This is accounted for, of course, by the cyclic arrangement of the tangents.

From equations (1) and (2), $k_1$ being assumed different from zero and $y - m_1 x$ or $c_1$ also different from zero (since a zero value of $c_1$ is inadmissible for real intersections) we derive $m_1 = 0$ and hence $\delta^2 + k_1^2 - y^2 = 0$, as is also easily seen geometrically.

The three tangents being distinct $m_2$ and $m_3$ must be different from zero, since $m_1 = 0$, and hence:

From equation (3), $m_2 = -\dfrac{2y(x + k_1)}{\delta^2 - 2k_1 x - x^2}$

and from equation (6), $m_3 = -\dfrac{2y(x - k_1)}{\delta^2 + 2k_1 x - x^2}$.

These then must be the roots of equations (4) and (5). Therefore,

$$\frac{2y(x + k_1)}{\delta^2 - 2k_1 x - x^2} + \frac{2y(x - k_1)}{\delta^2 + 2k_1 x - x^2} = \frac{2y(x - k_3)}{\delta^2 + 2k_3 x - x^2}$$

and $\dfrac{2y(x+k_1)}{\delta^2-2k_1x-x^2}\cdot\dfrac{2y(x-k_1)}{\delta^2+2k_1x-x^2}=\dfrac{\delta^2+k_3^2-y^2}{\delta^2+2k_3x-x^2}$.

Making use of the relation $\delta^2+k_1^2-y^2=0$ previously obtained, these equations reduce to

$$\frac{2x(2y^2-\delta^2-x^2)}{(\delta^2+x^2)^2-4x^2y^2}=\frac{x-k_3}{\delta^2-2k_3x-x^2}\text{................ A},$$

and
$$\frac{4y^2(x^2-y^2+\delta^2)}{(\delta^2+x^2)^2-4x^2y^2}=\frac{\delta^2+k_3^2-y^2}{\delta^2-2k_3x-x^2}\text{.................. B.}$$

The result of eliminating $k_3$ from equations A and B is:

$$(x^2-y^2+\delta^2)\ (\delta^2+x^2+2xy)\ (\delta^2+x^2-2xy)\ (\delta^2+x^2-2\delta y)$$
$$(\delta^2+x^2+2\delta y)\ (\delta^4+2\delta^2x^2+x^4+4x^2y^2)=0,$$

the locus sought.

If, however, we eliminate $x$ and $y$ from

(1) either $x^2-y^2+\delta^2=0$ or $\delta^2+x^2\pm 2xy=0$,

(2) $\delta^2+k_1^2-y^2=0$,

and

(3) either equation A or equation B,
we find that these can only be satisfied in case $k_3=\pm k_1$, that is, in case the third circle coincides with either one or the other of the first two. Hence,

neither $x^2-y^2+\delta^2=0$

nor $\delta^2+x^2\pm 2xy=0$

constitutes a part of the proper locus. Moreover, the last factor in the complete locus vanishes only for imaginary values of $x$ and $y$, so that the proper locus of $P$ for which real constructions are possible is

$$\delta^2+x^2\pm 2\delta y=0,$$

namely, a pair of parabolas symmetrically situated with respect to both coördinate axes, each having a point of intersection of the circles for focus and the line of centres for directrix.

Also, eliminating $x$ and $y$ from

(1) $\delta^2+x^2\pm 2\,\delta y=0$,

(2) $\delta^2+k_1^2-y^2=0$,

and
(3) either equation A or equation B,
we find the following relation to exist among the circles when the circuit of threes is complete, viz:

$$k_1^4 k_3^4 - 4\delta^6 k_1^2 - 6\delta^4 k_1^2 k_3^2 - 4\delta^6 k_3^2 - 3\delta^8 = 0$$

an equation, in which two of the values of $k_3$ in terms of $k_1$ and $\delta$ are imaginary and the other two equal and of opposite signs.

The magnitude $k_3$, and so the centre of the third circle, can, however, be more easily obtained geometrically, as is seen from the following:

The equation of the proper locus is

$$x^2 + \delta^2 \pm 2\delta y = 0$$

while $y = \sqrt{\delta^2 + k_1^2} = r_1$, the radius of one of the equal circles, so that $x^2 = 2\delta r_1 - \delta^2$.

From equation A, $\dfrac{k^3}{x} = \dfrac{x^4 + 4\delta^2 y^2 - 3\delta^4 - 2\delta^2 x^2}{3x^4 + 2\delta^2 x^2 - \delta^4 - 4x^2 y^2}$

which, after substituting the above values for $x^2$ and $y^2$ in the right hand member, becomes

$$\frac{k_3}{x} = -\frac{\delta}{r - \delta}.$$

That is, *the straight line joining any chosen point of the proper locus with the corresponding point of intersection of the system of circles intersects the line of centres in the centre of the required third circle which, together with the two equal circles pertaining to the chosen point, completes the circuit.*

The locus of points, whose period in the proposed sequence of rays and circles is three, is thus seen to resolve itself into a pair of parabolas, while the invariance of the anharmonic ratio of four rays under projection yields immediately other such triplets of rays through a point of the locus. The projective extension of this result is sufficiently interesting to invite further investigation of a more purely geometric sort, and perhaps it is even more desirable to study and tabulate corresponding loci for periods four, five, etc., and to this subject I hope to return in a subsequent essay.

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