THEOREM. If $F(\xi)$ be an $IQ\,[p^s, p^n]$ belonging to the class λ (not the principal), $F(\xi^{p^n} - \xi)$ decomposes into p^n $IQ\,[p^s, p^n]$ of the class $\lambda + 1$; but if the former belong to the principal class, the latter is simply an $IQ\,[p^{s+1}, p^n]$ of the first class.

PARIS, *April* 15, 1897.

ON A SOLUTION OF THE BIQUADRATIC WHICH COMBINES THE METHODS OF DESCARTES AND EULER.

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THE product of

$$x^2 - v^{\frac{1}{2}} x + \tfrac{1}{2}(p + v + qv^{-\frac{1}{2}}) = 0 \qquad (1)$$

and

$$x^2 + v^{\frac{1}{2}} x + \tfrac{1}{2}(p + v - qv^{-\frac{1}{2}}) = 0 \qquad (2)$$

is

$$x^4 + px^2 + qx + \tfrac{1}{4}\left[(p+v)^2 - q^2v^{-1}\right] = 0. \qquad (3)$$

All of this except one term coincides with the short form of the general biquadratic,

$$x^4 + px^2 + qx + r = 0. \qquad (4)$$

Since v is at our disposal we may treat (3) and (4) as equivalent, term by term, so that we have, after clearing ing of fractions,

$$4rv = v\,(p + v^2) - q^2,$$

$$v^3 + 2pv^2 + (p^2 - 4r)\,v - q^2 = 0.* \qquad (5)$$

* Up to this point this solution is precisely that of Descartes, except that the indeterminate quantity is here introduced in the form of a square root. It seems remarkable that the extreme facility with which the method of Descartes, which consists in separating the biquadratic into quadratic factors, may be combined with that of Euler, which consists in exhibiting the roots of the biquadratic as sums of square roots of the three roots of a cubic, should not heretofore have been observed. If the combination should, contrary to the writer's expectation, be found lacking in novelty, it may nevertheless be held that it has not attracted the attention which it deserves.

The three roots of this cubic being v_1, v_2, v_3, we have $v_1 + v_2 + v_3 = -2p$, $v_1 v_2 v_3 = q^2$, $v_1^{\frac{1}{2}} v_2^{\frac{1}{2}} v_3^{\frac{1}{2}} = \pm q$. As each of the square roots $v_1^{\frac{1}{2}}$, $v_2^{\frac{1}{2}}$, $v_3^{\frac{1}{2}}$, has two values of opposite signs, we may affix such signs as we please to $v_1^{\frac{1}{2}}$ and $v_2^{\frac{1}{2}}$, provided we adjust that of $v_3^{\frac{1}{2}}$ to agree with the requirement $v_1^{\frac{1}{2}} v_2^{\frac{1}{2}} v_3^{\frac{1}{2}} = \pm q$. We may in fact confine our attention to the case $v_1^{\frac{1}{2}} v_2^{\frac{1}{2}} v_3^{\frac{1}{2}} = -q$, and give positive values to $v_1^{\frac{1}{2}}$ and $v_2^{\frac{1}{2}}$, affixing to $v_3^{\frac{1}{2}}$ the sign opposite to the intrinsic sign of q. (It will be found that any other permissible adjustment of the signs will produce the same results in the end.)

Solving the quadratics (1) and (2) by using the value of $v_1^{\frac{1}{2}}$ obtained from the cubic (5), we have

$$
\begin{aligned}
x &= \tfrac{1}{2} v_1^{\frac{1}{2}} \pm \tfrac{1}{2} \sqrt{(-2p - v_1 - 2qv_1^{-\frac{1}{2}})} \\
&= \tfrac{1}{2} v_1^{\frac{1}{2}} \pm \tfrac{1}{2} \sqrt{(v_2 + v_3 + 2v_2^{\frac{1}{2}} v_3^{\frac{1}{2}})} \\
&= \tfrac{1}{2} v_1^{\frac{1}{2}} \pm \tfrac{1}{2} (v_2^{\frac{1}{2}} + v_3^{\frac{1}{2}}), \qquad (6) \\
x &= -\tfrac{1}{2} v_1^{\frac{1}{2}} \pm \tfrac{1}{2} (v_2^{\frac{1}{2}} - v_3^{\frac{1}{2}}). \qquad (7)
\end{aligned}
$$

As an illustration let us take that employed by Euler and many succeeding writers,*

$$x^4 - 25x^2 + 60x - 36 = 0.$$

Here $\quad p = -25,\ q = 60,\ r = -36$, and by (5),

$$v^3 - 50v^2 + 769v - 3600 = 0,$$

a cubic of which the roots are 9, 16, 25, so that we have $v_1^{\frac{1}{2}} = 3$, $v_2^{\frac{1}{2}} = 4$, $v_3^{\frac{1}{2}} = -5$. Then, from (6) and (7),

$$x = \tfrac{1}{2} \cdot 3 \pm \tfrac{1}{2} (4 - 5),$$

$$x = -\tfrac{1}{2} \cdot 3 \pm \tfrac{1}{2} (4 + 5)$$

the four values of x being 1, 2, 3, -6.

* See, for example, the article "Algebra" in the ninth edition of the *Encyclopædia Brittanica*. This illustration has been attacked because Euler himself recognized the "irreducibility" of the cubic as an obstacle which he evaded by recourse to trigonometrical functions. The cubic has rational roots, however, and the belief that an equation having a rational root can be rejected as irreducible has long been obsolete.