duced one of the heroes of mathematics; but there are now among us a dozen universities in each of which something, be it much or little, is being added to that splendid monument of human thought which bears the record of conquests made by so many of the intellectual giants of our race.

Among these giants Sylvester has without question the right to be reckoned. In the history of mathematics, his place will not be with the very greatest; but his work, brilliant and memorable as it was, affords no true measure of his intellectual greatness. Those who came within the sphere of his personality, could not but feel that, through the force of circumstances combined with the peculiarities of his poetic temperament, his performance, splendid as it was, has not adequately reflected his magnificent powers. Those of us who were connected with him cherish his memory as that of a sympathetic friend and generous critic. And in this university, as long as it shall exist, he will be remembered as the man whose genius illuminated its early years, and whose devotion and ardor furnished the most inspiring of all the elements which went to make those years so memorable and so fruitful.

HYPERBOLEA AND THE SOLUTION OF EQUATIONS.

BY MR. C. H. HINTON.

In the following pages, after a few remarks on the system of mathematics in vogue in Hyperbolea, I wish to show that a consideration of the methods of the Hyperboleans leads to a graphical representation of quantities by which, given an appropriate train of mechanism, not only the real, but also the imaginary roots of an equation can be mechanically found.

Hyperbolea is a land in which distance is measured by the function $\sqrt{x^2 - y^2}$. This, with its attendant consequences, sufficiently defines the locality.

Let AB be a straight line. Numbers give the ordination of positions on it. The length between any two positions is a physical notion. If p is a material rod the intervals AC and BD are said to be equal if the rod p occupies at one time the interval AC, at another time BD without observable distortion in the transference. Taking a two dimensional number system we have besides the system ABC another system of positions $AA'A''$, $BB'B''$ and so on. The positions A, A' are said to be at the same distance asA, B if the same material rod will occupy the interval A, A' as occupied AB, it being presumed that no distortion takes place.

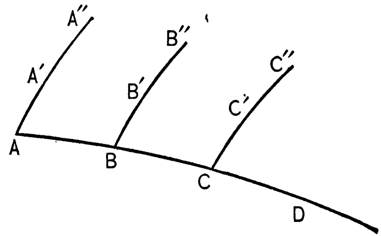

FIG. 1.

The operation of transferring a rod which occupies the interval AB to assume the position AA' is termed rotation. Where in the twofold number system the extremity which is not fixed at A will be found during the process of transference is not given *a priori*. We will suppose a number system so arranged in a material universe that the intervals satisfy the following conditions: The intervals AB, BC, CD can all in turn be occupied by the same rod, also AA', $A'A''$ and each set, such as BB', $B'B''$, and finally each set, such as $A'B'$, $B'C'$, etc.

So much being settled we are in accordance to a certain extent with our ordinary system of coördinates. But the po-

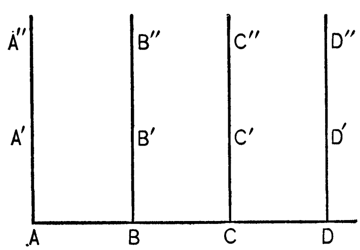

FIG. 2.

sitions which the extremity of the rod will occupy in its rotation have still to be defined. It has in no way been precised by the assumption already made. The shape of the " circle " is as yet indeterminate. Let x and y be the numbers denoting the position of the moveable extremity of the rod in the specified two-fold number system. We find from observation that in our world the law of rotation is this. During rotation $\sqrt{x^2 + y^2}$ has the same value that x alone has before rotation commenced. Now in Hyperbolea $\sqrt{x^2 - y^2}$ has the same value that x alone has before rotation commenced. From the particular law of length which we assume springs the definition of angle, and all subsequent geometrical relations. Hence the system of rectangular coördinates above drawn is very false if assumed to be a faithful representation of the way of measuring the plane in hyperbolea. But we are familiar with it. It is most practical. In fact it is the only practical way to use our own length and angles in our drawings. Then the physical laws of Hyperbolea will be given as a distortion of our physical laws. There is no help for this in our physical conditions, and we must remember that our physical laws of motion and rotation would seem just as distorted and curious to the Hyperboleans as theirs do to us.

To take a definite instance, let OA, fig. 3, be a rod of length 32 units. It will in Hyperbolea assume positions such that the coördinates of its extremity are $P(40, 24)$ $Q(68, 60)$ $R(130, 126)$. These points lie on the hyperbola $x^2 - y^2 = 32^2$.

Thus if we represent the locus of points in Hyperbolea equally distant from O, we represent it in our physical system as a hyperbola.

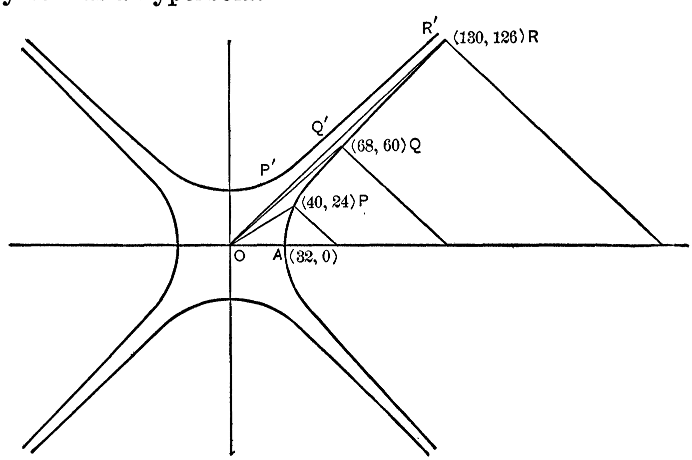

FIG. 3.

OP, OQ and OR are all equal in length (hyperbolic length).

If $P'Q'R'$ lie on the conjugate hyperbola and are conjugate radii it is evident that if any one such as OP' is to be a real length the coördinates of P' must be imaginary, thus $24\sqrt{-1}$, $40\sqrt{-1}$ will be the coördinates for P' and so on for the other points. Under this assumption $\sqrt{x^2 - y^2}$ will denote a real length. It is evident that a point on the asymptote is at a null distance from the origin. We have in fact in the asymptotes the lines of null length of the hyperboleans drawn extended. Thus we are able to show constructions which would only be matters of symbolism with them. Conversely they can put in extensive visible form constructions which we in our physical system know of merely as algebraical analogies. It is not only from O that the lines of null length start. From every point there are two directions in which the hyperbolean counts length as null. His plane, as ours, is granulated with lines of null length.

With regard to angle, the hyperbolean calls those angles equal which intercept equal arcs on his curve of equal distances. As equal distances are measured by the square root of the difference of the squares of the coördinates, $ds = \sqrt{dx^2 - dy^2}$ and $s = \sqrt{-1}\, a \log (x + y)$. Putting $x = a \cosh u$, $y = a \sinh u$, we get $ds = \sqrt{a^2 \sinh^2 u - a^2 \cosh^2 u}\, du$ or $s = \sqrt{-1}\, au$.

Hence for the hyperbolean the measure of an angle is $\frac{s}{a\sqrt{-1}}$ where s is the hyperbolic length of the arc subtended.

The geometric facts of Hyperbolea can be easily exhibited by means of a symbolism analogous with the Argand affix.

If we take a symbol j to denote a unit directed along the y axis then $a + bj$ will denote a vector having components, the one along the axis of numbers, a; and the other along the axis of y, b.

These vectors will compound and multiply like the vectors $a + b\sqrt{-1}$. The difference is that the modulus in this case is $\sqrt{a^2 - b^2}$.

Thus to take the instance of the hyperbola last taken $\rho = \left(\frac{5}{4} + \frac{3}{4}j\right)^n 32$ will give the three points P, Q, R for the values 1, 2, 3 of n.

We can also write as the equation of the hyperbola $x^2 - y^2 = 1$,

$$\rho = e^{ju} = 1 + ju + \frac{(ju)^2}{1.2} + \cdots = \cosh u + j \sinh u.$$

This equation gives the properties of the hyperbola. Thus

$$d\rho = je^{ju}\, du = (\, j \cosh u + \sinh u\,)\, du$$

showing that the tangent is parallel to the conjugate radius.

Now for real values of u we get only the points on one branch of the hyperbola. To get points on the other branch let $u = u + i\pi$ we get $\rho = e^{(u+i\pi)} = - e^{uj}$.

The points in the conjugate hyperbola are obtained by letting $u = u + \frac{i\pi}{2}$, whence

$$\rho \equiv e^{\left(u + \frac{i\pi}{2}\right)j} = ij\, e^{uj} = i(\cos\mathrm{h}uj + \sinh u)$$

the coördinates being imaginary. The hyperbolean can draw imaginary lines.

There is no period for real values of u. In Hyperbolea there is besides rotation a process which we may call Saltation, by which it is possible to pass from one vector to its conjugate. Saltation corresponds to an increase of u by $\frac{1}{2} i\pi$ and all dimaeters make this hyperbolic angle of $\frac{1}{2} i\pi$ with their conjugates.

Equal hyperbolic angles seem to us to grow smaller as their locality recedes from what we call the axis of the hyperbola. But the hyperbolean regards our equal angles as variable as increasing, and after a certain time as disappearing into a region of the plane which only exists as a matter of symbolism. The conception, too, of an axis midway between the asymptotes is unknown to him, all points on the hyperbola being at an equal distance, and there being an infinite rotation to the asymptotes, he has no means of distinguishing any one diameter from any other. Hence working with the symbolism $\rho = e^{ju}$ we can take any pair of conjugate diameters for primary axes and all the formulæ will hold, the coördinates being measured parallel to these conjugate ones.

If we project points on the real axis onto the circle $x^2 + y^2 = 1$ by means of lines parallel to the asymptote $x + iy = 0$ we get for the point corresponding to x' on the circle.

$$-\frac{1 + x'^2}{2x'},\ i\frac{1 - x'^2}{2x'}$$

which we cannot draw.

The hyperbolean projecting points on his axis of numbers onto the hyperbola $x^2 - y^2 = 1$ gets real points. It is to these points that I wish to call attention. We have an advantage over him in that we can draw these projecting lines, and so a mechanical process of solving equations which is symbolical to the hyperbolean can be actually carried out by us.

The point in the hyperbola for which $\rho = \cosh u + j \sinh u$ becomes, when projected on the axis of numbers by a parallel to the asymptote $x + y = 0$, a point at a distance from the origin equal to $\cosh u + \sinh u$. Conversely any point on the axis of numbers can be represented by a point on the hyperbola. The coördinates of the point on the hyperbola corresponding to a point a on the axis of numbers are given by $x + y = a$, $x^2 - y^2 = 1$ and are $\cosh u = \frac{a + \frac{1}{a}}{2}$ $\sinh u = \frac{a - \frac{1}{a}}{2}$

If now we draw from the origin vectors to such representative points on the hyperbola we find that:

1° the addition of distances from the origin corresponds to addition of the vectors by the parallelogram law.

2° the multiplication of the distances, considered as giving a point at a distance on the axis of numbers equal to the product of the distances, corresponds to the addition of the two hyperbolic angles of the representative vectors.

The first remark is justified by the fact that the distances on the axis of numbers are projected on the hyperbola by lines making an angle of 45° with the x axis. The second point follows from the fact that the vector of a point on a hyperbola can be written $\cosh u + j \sinh u$. The product of two such vectors having angles u and u' is evidently $\cosh(u + u') + j \sinh(u + u')$.

Putting $j = 1$ corresponds to finding the distance on the axis of numbers represented by the vector $\cosh u + j \sinh u$.

We have thus the means of carrying out the processes of addition and multiplication by vector addition and hyperbolic rotation.

Take for instance the point 2 on the axis of numbers. To find the corresponding point on the hyperbola put $\cosh u + \sinh u = 2$. We get $\cosh u = \frac{2 + \frac{1}{2}}{2}$ $\sinh u = \frac{2 - \frac{1}{2}}{2}$ or $\cosh u = \frac{5}{4}$ $\sinh u = \frac{3}{4}$. The corresponding vector is $\rho = \frac{5}{4} + j\frac{3}{4}$. The point 4 is similarly $\rho_2 = \frac{34}{16} + j\frac{30}{16}$. These points are shown in fig. 3 if 32 is taken for unity; ρ_1 is then $\frac{40}{32} + j\,\frac{24}{32}$, ρ_2 is $\frac{68}{32} + j\,\frac{60}{32}$. The vector sum of these

$\frac{108}{32} + j\frac{84}{32}$ becomes 6 if projected back on the axis of x, *i. e.*, if $j = 1$. Thus, whether we add the vectors and project back or whether we simply add the numbers 4 and 2, we obtain the same result.

The product of these two vectors is $\frac{130}{32} + j\frac{126}{32}$ which projects back on the axis of numbers into 8 (put $j = 1$). And this product vector makes an angle with the axis of x equal to the sum of the factor vectors. Put $\rho_1 = \cosh u_1 + j \sinh u_1 = e^{ju_1}$,we get, if $j = 1$, $e^{u_1} = 2$, or $u_1 = \log 2$; in the same way $u_2 = \log 4$ and their product $e^{ju_1} \times e^{ju_2} = e^{j(u_1+u_2)}$. Thus the hyperbolic angles of the three vectors are respectively log 2, 2 log 2, 3 log 2, the sum of the first two equals the third.

In the following remarks I will not go into the question of the feasibility of constructing a chain of mechanism which shall move vectors along the hyperbola by equal, and multiples of equal, hyperbolic angles. Assuming such a mechanism and also one capable of summing vectors, I will show the process of finding the solution of an equation for real roots, and for mixed real and imaginary roots.

1. Take the quantic $x^2 - 3.5\,x + 3$ which has the roots 1.5 and 2.

$$\text{Put } x = e^{uj}; \quad 3.5 = e^{1.2528j}; \quad 3 = e^{1.0936j}.$$

This corresponds to the projection shown in the diagram, fig. 4, x being taken as unity and therefore u as null.

The vector ρ^2 representing x^2 lies along OA the vector representing $-3.5x$ lies along OB. OC represents the number 3 and OB the coefficient -3.5.

As x increases in value from 1 the moveable vector ρ^2 starts from OA moving by intervals of $2u$; the vector x or ρ starts from OB moving by intervals of u. The vector sum of ρ^2, ρ and OC gives the value of the quantic.

The directions of rotation and saltation are given by the letters s and m, s denoting a small and m a large hyperbolic angle; thus from s to m the hyperbolic angle increases, from m to s it diminishes.

The curve in fig. 5 gives the locus traced out by the extremity of the compound vector which represents the quantic. The numbers affixed show the values of u at the specified points.

It will be seen that the curve cuts the asymptote $x+y=0$ at the points .4 (written for .40554) and .69 written for .6931 these being log 1.5 and log 2.

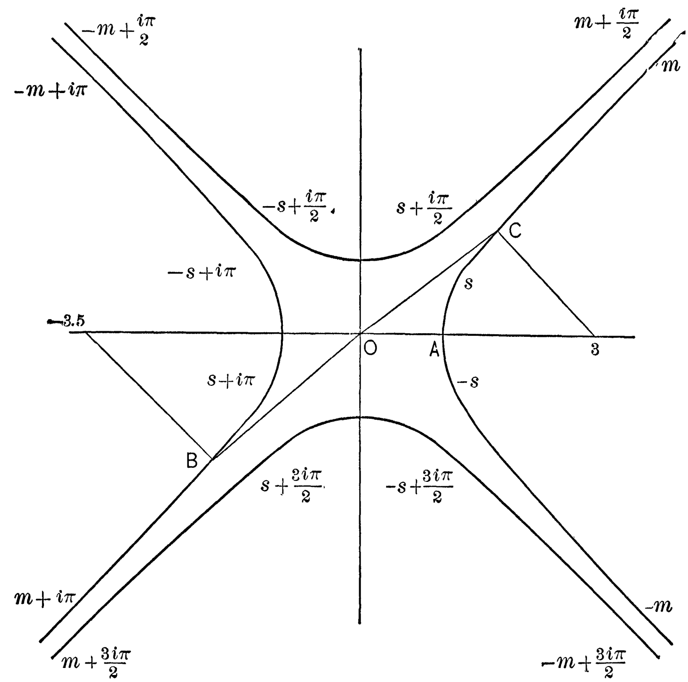
$-m+\frac{i\pi}{2}$
$m+\frac{i\pi}{2}$
$-m+i\pi$
m
$-s+\frac{i\pi}{2}$
$s+\frac{i\pi}{2}$
C
$-s+i\pi$
s
-3.5
O
A
3
$-s$
$s+i\pi$
B
$s+\frac{3i\pi}{2}$
$-s+\frac{3i\pi}{2}$
$m+i\pi$
$-m$
$m+\frac{3i\pi}{2}$
$-m+\frac{3i\pi}{2}$

FIG. 4.

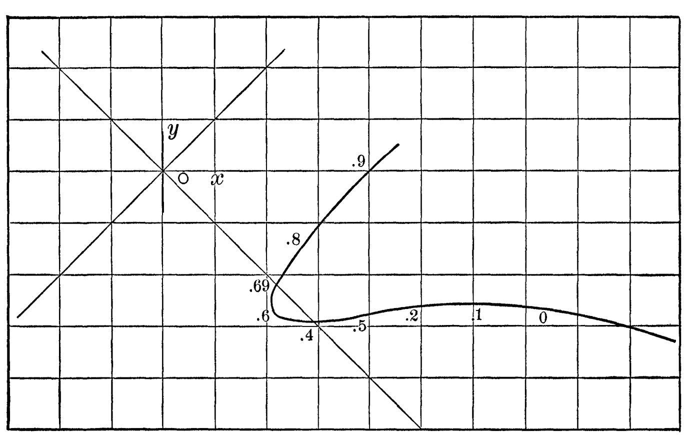
y
O
x
.9
.8
.69
.6
.4
.5
.2
.1
0

FIG. 5.

The projections of these points by a parallel to $x + y = 0$ give a null distance from the origin and hence $e^{\log 2}$ and $e^{\log 1.5}$ are solutions of the equation, namely, 2 and 1.5. Thus, granted the train of mechanism, the solution of an equation (as to its real roots) consists simply in determining the vectors corresponding to the coefficients and then making hyperbolic radii start from them as initial positions at the proper relative rates. The intersections of the curve traced out by the vector sum of the vectors with the asymptote $x + y = 0$ gives values of u for which e^u is a solution.

In default of the mechanism I give the calculation of a couple of the points:

X	$u = .2$	$u = .6931$
$\cosh 2u$	1.0803	2.125
$-\cosh 1.2528 + u$	-2.2534	-3.571
$+\cosh 1.0986$	1.6666	1.666
	$+.4935$	.220
Y		
$\sinh 2u$	.4105	1.375
$\sinh 1.2528 + u$	2.6201	-3.428
$\cosh 1.0986$	1.3333	1.333
	$-.2763$	$-.220$

To exhibit the process of finding both real and imaginary roots add a number to the previous expression so as to make the roots imaginary.

Take $x^2 - 3.5x + 4.625$, the roots of which are $x = 1.75 \pm 1.25\sqrt{-1}$.

As x is to be partly real, partly imaginary, put it equal to $e^{uj} + e^{\left(u' + \frac{i\pi}{2}\right)j}$

That is to say, represent it as a pair of radii the radius having the angle u together with the conjugate of that having the angle u'.

These radii cannot be added together. The single vector sum of them must not be taken. They form an ununifiable couple representing a complex expression. But they can be multiplied together. They form a pair of lines representing the composite expression.

The new equation becomes, substituting the value of x in $x^2 - 3.5x + 4.625 = 0$

$$e^{2uj} + e^{2\left(u' + \frac{i\pi}{2}\right)j} - e^{(l\,3.5 + u)j} + e^{l\,4.625j} + 2e^{\left(u + u' + \frac{i\pi}{2}\right)j}$$
$$- e^{\left(l\,3.5 + u' + \frac{i\pi}{2}\right)j} = 0$$

Here the constant term $e^{i4.625j}$ or $e^{1.5315j}$ differs from the constant term of the preceding equation by $.75 + .87j$. Hence the axes being in the position shown in fig. 6, fig. 5, if its origin be placed at (.75, .87) would represent the value of the left hand side of the equation for real values of u. Fig. 6 is the graph of the new expression. Expanding the exponentials in the expression on the left hand side of the above equation we obtain $\rho = X + Yj + i\ (X' + Y'j)$. Values of $X\ Y$ etc. are given in the table below at the end of the paper. Taking the vector sum of the real vectors, and of the imaginary vectors we get two series of curves. In default again of the appropriate mechanism the calculation is given below. The mechanical process would differ from that indicated above simply in the circumstance that a means must be provided by which the angles traced out by each of the vectors, the real and the imaginary ones must be introduced so as to add to the rotation of the other in an appropriate way.

In fig. 6 the continuous lines denote the curves traced out by the real vector of the quantic as u and u' vary, the dotted lines denote the curves traced out by the imaginary vector of the quantic as u and u' vary.

Take for instance the curves for which $u = .4$. The real vector sum traces out the hyperbola marked $u = .4$. The second figure in the affixed designations denotes the value of u' for the specified points on the curve.

For the same value of u the imaginary vector sum traces out the hyperbola (dotted) $u = .4$. Corresponding positions on each are shown by the values of u' being identical, *i. e.*, points where both affixed figures are the same, are simultaneous positions of the extremities of the real and imaginary quantic vector.

If u' is constant and u varies the real vector describes hyperbolic sine cosine curves, the imaginary vector describes hyperbolas which all pass through the origin.

The extremity of either vector can move over the whole plane. Thus to any path of the one corresponds a definite path of the other. Hence the method of mechanically obtaining a solution of the equation would be to make the extremity of one vector move along the asymptote, the other then traces a curve which intersects the asymptote. The values of u and u' for these points of intersection give the real and imaginary parts of the root.

In the particular case of the quadratic expression since all the imaginary curves for which $u' =$ constant pass through the origin, the single point of the origin stands for a whole curve of the u series.

This value is in the figure $u' = \log 1.75 = .5597$ (written in the figure .56). Hence in the case of the quadratic the resultant imaginary vector has null components. For

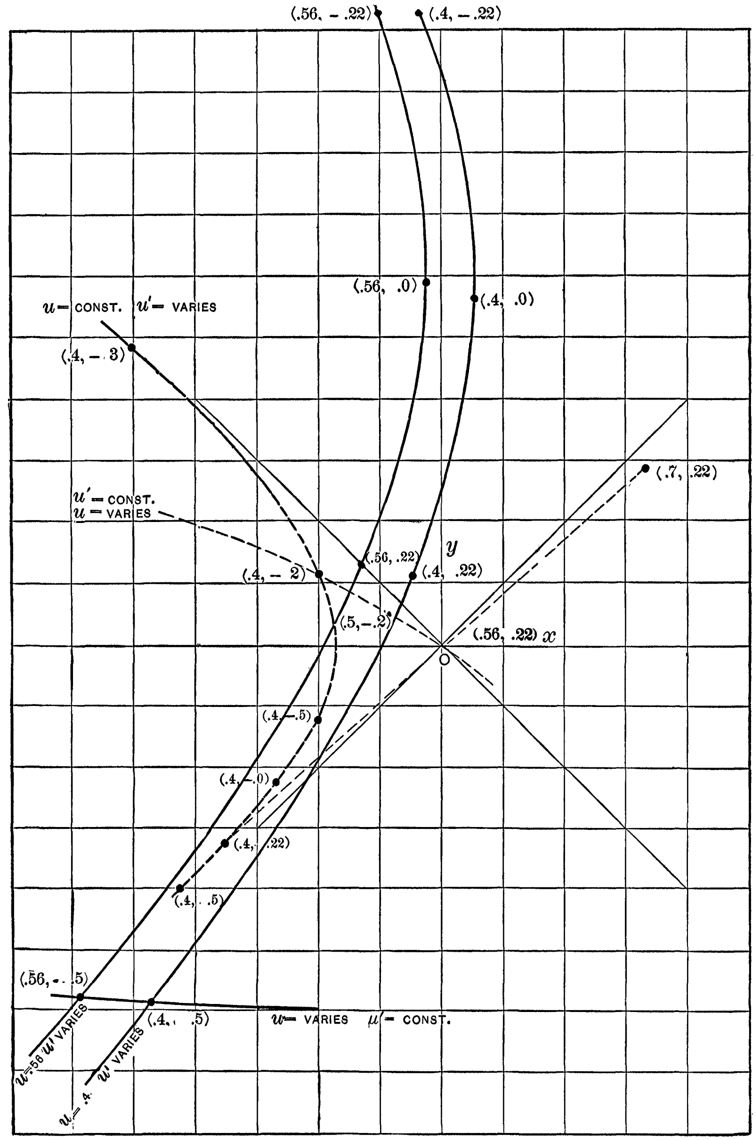

FIG. 6.

a solution the two imaginary vectors can be rotated round so as to form one diameter of the conjugate hyperbola. In general, however, the method would be to move one or the other summation vector along the asymptote. In the above figure, in which the dotted curve from (.4,.22) to (.7,.22) has its point (.56,.22) at the origin, looking at the corresponding points (.4,.22), we see that when the real vector passes from (.4,.22) on the curve $u=.4$ to (.56,.22) on the curve $u=.56$ the imaginary vector travels from (.4,.22) to the origin (.22 is written for .22314). Hence $e^{.5597j}+e^{.22314j+\frac{i\pi}{2}}$ is the root which gives, putting $j=1$, $e^{\log 1.75}+\sqrt{-1}e^{\log 1.25}$. Thus the root is found by a graphic process, and given an appropriate train of mechanism could be determined mechanically.

In an equation with real coefficients each complex root will have a conjugate. If the coefficients are imaginary it will be necessary to take the value $e^{u'+\frac{3i\pi}{2}}$. It is also necessary for negative real parts of the root to trace the curves for $u+i\pi$.

Considering the coördinates (x, y) $(x, \sqrt{-1}y)$ $(\sqrt{-1}x, y)$ $(\sqrt{-1}x, \sqrt{-1}y)$, we have access to one-quarter of the plane given by our symbolism. The hyperbolic being has also access to one-quarter of it, namely one-half of the region of (xy), one-half of the region of $(\sqrt{-1}x, \sqrt{-1}y)$.

The function taken to represent distance has a purely observational origin. Hence it seems probable that besides the parallel postulate there is another assumption in the Euclidean geometry. It is easily seen that the theory of parallels is the same in the hyperbolean geometry as in ours; and hence it follows that Euclid I.47 does not depend on the parallel postulate. We see that it cannot really be proved. Since a perfectly consistent system of geometry can be built on the assumption that the square on the hypotenuse is equal to the difference of the squares on the sides, the propositon that the square on the hypotenuse is equal to the sum of the squares on the sides of a right angled triangle cannot be considered to be proved. It would be interesting to trace where the assumption comes in. Probably some other property of the "circle" besides its being the locus of equidistant points is taken for granted.