SYSTEMS OF CONTINUOUS AND DISCONTINUOUS SIMPLE GROUPS.

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§ 1.

Known systems of discontinuous simple groups.

The following list gives the orders of the systems of discontinuous simple groups which have thus far been determined. Here $p =$ prime, m and $n =$ integers.

(1) $$p$$

(2) $$\tfrac{1}{2}\, n\,! \qquad (n > 4)$$

(3) $$\frac{(p^{nm}-1)\,(p^{nm}-p^n)\cdots(p^{nm}-p^{nm-n})}{d(p^n-1)}$$

$(m, n, p) \neq (2, 1, 2)$ or $(2, 1, 3)$; d is the greatest common divisor of m and p^n-1.

(4) $$\tfrac{1}{2}\,(p^{2nm}-1)\,p^{n(2m-1)}\,(p^{n(2m-2)}-1)\,p^{n(2m-3)}\cdots(p^{2n}-1)\,p^n \qquad (\text{for } p > 2)$$

$$(2^{2nm}-1)\,2^{n(2m-1)}\cdots\cdots(2^{2n}-1)\,2^n \qquad (m > 2).$$

(5) $$(2^m-1)\cdot(2^{2m-2}-1)\,2^{2m-2}\,(2^{2m-4}-1)\,2^{2m-4}\cdots(2^2-1)2^2 \qquad (m > 2).$$

(6) $$(2^m+1)\cdot(2^{2m-2}-1)2^{2m-2}\cdots(2^2-1)2^2 \qquad (m > 2).$$

The two triply-infinite systems (3) and (4) were obtained* by generalizing to the Galois Field of order p^n the two doubly-infinite systems set up by Jordan.† The system (3) is obtained in the decomposition of the general linear homogeneous group on m indices, in whose substitutions both indices and coefficients are "marks" of the Galois Field of order p^n. Concretely, (3) is the group of

*3. L. E. DICKSON. "The analytic representation of substitutions on a power of a prime number of letters, with a discussion of the linear group.—*Annals of Mathematics*, 1897.

4. L. E. DICKSON. A triply-infinite system of simple groups.—*Quarterly Journal of Mathematics*, vol. 29, 1897.

These results were announced in a paper read before the AMERICAN MATHEMATICAL SOCIETY at Buffalo, August 31, 1896.

† Traité des Substitutions, p. 106 and pp. 176, 178.

linear fractional substitutions of determinant unity on $m-1$ indices. The system (4) is obtained in the decomposition of the Abelian group on $2m$ indices; viz, that subgroup of the general linear homogeneous group on $2m$ indices (in which both coefficients and indices are marks of the Galois Field of order p^n), every substitution of which if operating simultaneously on two such sets of $2m$ indices

$$\xi_i, \eta_i\,;\ \rho_i, \sigma_i \qquad (i = 1 \cdots m)$$

multiplies the function

$$\sum_{i=1}^{m} (\xi_i \sigma_i - \eta_i \rho_i)$$

by a constant factor.

The systems (3) and (4) have in common the doubly-infinite system (viz, (3) for $m=2$ and (4) for $m=1$) set up first by Professor E. H. Moore* and a few months later by Professor W. Burnside† as a generalization of the well known modular group of order

$$\tfrac{1}{2} p (p^2 - 1), \qquad (p > 3).$$

The system (5) is due to Jordan, ibid, p. 205. It is obtained in the decomposition of the First-Hypoabelian group. For the sake of comparison I have changed the form of the expression giving its order, viz:

$$(P_m - 1)\, 2^{2m-2} (P_{m-1} - 1)\, 2^{2m-4} \cdots (P_2 - 1)\, 2^2 \quad (m > 2)$$

where $$P_s = 2^{2s-1} + 2^{s-1}.$$

The system (6) is obtained in the decomposition of the Second-Hypoabelian group. The order of the latter is given incorrectly in Jordan, ibid, p. 207, as equal to the order of the First-Hypoabelian group. The number of solutions of the congruence (42), p. 207, is P_n and not $2^{2n} - P_n$; the reference in § 282 should be to § 259 and not to § 260. I have not seen the simple groups (6) given explicitly anywhere.

Recently‡ I have succeeded in generalizing the system (5) to a doubly infinite system, the expression for the order being rather complicated. The Second-Hypoabelian

* Abstract in the BULLETIN OF THE MATHEMATICAL SOCIETY, December, 1893; complete in the Mathematical Papers of the Chicago Congress.

† "On a Class of Groups defined by Congruences," *Proceedings of the London Mathematical Society*, vol. 25, pp. 113-139, February, 1894.

‡ "The First-Hypoabelian group generalized," offered April 1st to the *Quarterly Journal of Mathematics*.

group presents serious difficulties to generalization, as the conditions defining it (Jordan, § 277) do not lead to a group at all in the $GF[p^n]$, $n > 1$.

§ 2.

*Systems of finite continuous transformation groups which are simple.**

Through the work of Killing†, which has been carried out with more rigor by Cartan,‡ we know that all continuous finite *simple* transformation groups, aside from five isolated ones, belong to one of the four systems set up by Sophus Lie, each of such fundamental importance in geometry and analysis:

(a) The groups with $l(l+2)$ parameters, isomorphic (= gleichzusammengesetzt) with the general projective group of R_l (= space of l dimensions).

(b_1) The groups with $l(2l+1)$ parameters, isomorphic with the general projective group of a non-degenerate surface of the second order in R_{2l} and hence also with the largest group of confotm transformations in R_{2l-1}.

(b_2) The groups with $l(2l-1)$ parameters, isomorphic with the general projective group of a non-degenerate surface of the second order in R_{2l-1}. (Here must $l > 2$.)

(c) The groups with $l(2l+1)$ parameters, isomorphic with the general projective group of a linear complex in R_{2l-1}.

The five isolated simple groups not falling in these four classes contain respectively 14, 52, 78, 133 and 248 parameters and exist as point-transformation groups in respectively 5, 15, 16, 27 and 57 variables (but no fewer); however, as Berührungstransformation groups in fewer variables.

§ 3.

Elementary deduction of the groups (c) and proof of their simplicity.

The following elementary proof had its origin in an attempt to carry over into Lie's continuous group theory the

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* Professor LIE has determined four classes of infinite transformation groups which are simple; *e. g.*, all transformations of space of n dimensions; again, all Berührungstransformationen.

† WILHELM KILLING, "Die Zusammensetzung der stetigen endlichen Transformationsgruppen, *Math. Ann.*, vols. 31, 33, 34, 36 (particularly vol. 33).

‡ ÉLIE CARTAN, "Ueber die einfachen Transformationsgruppen," *Leipziger Berichte*, pp. 393–420, 1893; "Sur la structure des groupes simples finis et continus," *Comptes Rendus*, vol. 116, p. 784 (17 Apr., 1893) and p. 962; also Thèse, Paris, Nony., 1894.

investigations of Jordan on Abelian groups (whose decomposition led to the system of simple groups (4)). It is remarkable how much simpler and more elegant the problem becomes for continuous groups. That the simple groups thus obtained are identical with the groups (c) is proved below.

Of the transformations of the general linear homogeneous group on $2n$ variables, we consider those which, when operating simultaneously on two independent sets of variables

$$x_i, y_i; \quad \xi_i, \eta_i \quad (i = 1, 2, \cdots n)$$

leave invariant the function

$$\varphi = \sum_{i=1}^{n} (x_i \eta_i - \xi_i y_i).$$

Let G denote the sub-group of the linear homogeneous group thus defined and S an arbitrary transformation of G; viz.:

$$\begin{aligned} x_i' &= \sum_{j=1}^{n} (a_j^{(i)} x_j + c_j^{(i)} y_j) \\ y_i' &= \sum_{j=1}^{n} (b_j^{(i)} x_j + d_j^{(i)} y_j) \end{aligned} \qquad (i = 1 \cdots n)$$

The conditions that S shall leave φ invariant are:

$$\sum_{\nu=1}^{n} (a_\mu^{(\nu)} d_\mu^{(\nu)} - b_\mu^{(\nu)} c_\mu^{(\nu)}) = 1, \quad \sum_{\nu=1}^{n} (a_\mu^{(\nu)} d_\lambda^{(\nu)} - b_\mu^{(\nu)} c_\lambda^{(\nu)}) = 0,$$

$$\sum_{\nu=1}^{n} (a_\mu^{(\nu)} b_\lambda^{(\nu)} - b_\mu^{(\nu)} a_\lambda^{(\nu)}) = 0, \quad \sum_{\nu=1}^{n} (c_\mu^{(\nu)} d_\lambda^{(\nu)} - d_\mu^{(\nu)} c_\lambda^{(\nu)}) = 0,$$

$$(\mu, \lambda = 1 \cdots n, \mu \neq \lambda),$$

altogether

$$n + n(n-1) + \frac{n(n-1)}{2} + \frac{n(n-1)}{2} = n(2n-1)$$

conditions.

Now S reduces to the identity if

$$a_i^{(i)} = 1, \ d_i^{(i)} = 1, \quad a_j^{(i)} = 0, \ d_j^{(i)} = 0 \quad (i, j = 1 \cdots n, \ i \neq j)$$

$$c_j^{(i)} = 0, \ b_j^{(i)} = 0 \quad (i, j = 1 \cdots n).$$

Hence we reach the general infinitesimal transformation of G by writing

$$a_i^{(i)} = 1 + \alpha_i^{(i)} \delta t, \ d_i^{(i)} = 1 + \delta_i^{(i)} \delta t, \ a_j^{(i)} = \alpha_j^{(i)} \delta t, \text{ etc.},$$

and subjecting the $\alpha_j^{(i)}$, $\beta_j^{(i)}$, $\gamma_j^{(i)}$, $\delta_j^{(i)}$ to the conditions following from the above $n\,(2n-1)$; viz,

$$\alpha_\mu^{(\mu)} + \delta_\mu^{(\mu)} = 0\,, \qquad \delta_\lambda^{(\mu)} + \alpha_\mu^{(\lambda)} = 0$$
$$\beta_\lambda^{(\mu)} - \beta_\mu^{(\lambda)} = 0\,, \qquad \gamma_\mu^{(\lambda)} - \gamma_\lambda^{(\mu)} = 0$$
$$(\mu, \lambda = 1 \cdots n,\ \mu \neq \lambda).$$

The most general infinitesimal transformation of the group G is thus

$$\delta x_i \equiv x_i' - x_i = \sum_{j=1}^{n} (\alpha_j^{(i)} x_j + \gamma_j^{(i)} y_j)$$
$$\delta y_i \equiv y_i' - y_i = \sum_{j=1}^{n} (\beta_j^{(i)} x_j - \alpha_i^{(j)} y_j)$$

where $\quad \beta_j^{(i)} = \beta_i^{(j)},\ \gamma_j^{(i)} = \gamma_i^{(j)} \quad (i, j = 1 \cdots n,\ i \neq j).$

Putting in turn each of the $2n^2 + n$ independent constants remaining equal 1 and all the others zero, we reach the $2n^2 + n$ linearly independent transformations of G:

$$Q_{ij} \equiv x_j \frac{\partial f}{\partial x_i} - y_i \frac{\partial f}{\partial y_j} \quad (i, j = 1 \cdots n).$$

$$L_i \equiv y_i \frac{\partial f}{\partial x_i}\,; \quad L_i' \equiv x_i \frac{\partial f}{\partial y_i} \quad (i = 1 \cdots n).$$

$$R_{ij} \equiv x_i \frac{\partial f}{\partial y_j} + x_j \frac{\partial f}{\partial y_i}\,; \quad N_{ij} \equiv y_i \frac{\partial f}{\partial x_j} + y_j \frac{\partial f}{\partial x_i} \quad (i, j = 1 \cdots n,\ i \neq j).$$

The notation Q_{ij}, L_i, etc., corresponds to that of Jordan, p. 174.

We verify the following *Klammerausdruck* relations:

$$(L_i\, Q_{ii}) = 2\, L_i\,;\ (L_i\, Q_{ij}) = 0\,;\ (L_i\ Q_{ji}) = N_{ij}.$$
$$(L_i L_i') = -\, Q_{ii}\,;\ (L_i R_{ij}) = -\, Q_{ij}\,;\ (L_i N_{ij}) = 0.$$
$$(L_i'\, Q_{ii}) = -2\, L_i'\,;\ (L_i'\, Q_{ij}) = -\, R_{ij}\,;\ (L_i'\, Q_{ji}) = 0.$$
$$(L_i'\, N_{ij}) = Q_{ji}\,;\ (L_i'\, R_{ij}) = 0.$$
$$(R_{ij}\, Q_{ii}) = -R_{ij}\,;\ (R_{ij}\, Q_j) = -2\, L_j'\,;\ (R_{ij} N_{ij}) = Q_{ii} + Q_{jj}.$$
$$(N_{ij},\, Q_{ii}) = N_{ij}\,;\ (N_{ij}\, Q_{ij}) = 2\, L_i\,;\ (N_{ij}\, Q_{kj}) = N_{ik}.$$

Lemma: If an invariant subgroup H of G contains the single infinitesimal transformation L_i (i=fixed), it contains all the transformations of G.

If H contains L_i, it contains N_{ij} $(j = 1 \cdots n,\ j \neq i)$, since $(L_i\, Q_{ji}) = N_{ij}$.

But $(N_{ij}\, Q_{ji}) = 2L_j$.

Hence H would contain every L_i and N_{ij}; hence also

$$(L_i R_{ij}) = -Q_{ij} \quad (j \neq i)\ ;\ (L_i'\, L_i) = Q_{ii}$$

$$(Q_{ii} L_i') = 2L_i'\ ;\quad (Q_{ij}\, L_i') = R_{ij}.$$

Theorem: The group G is simple.

Suppose, indeed, an invariant subgroup H of G exists. Let it contain the infinitesimal transformation (not the identity),

$$T = \sum_{ij}^{1\dots n} q_{ij}\, Q_{ij} + \sum_{i=1}^{n} l_i' L_i' + \sum_{i=1}^{n} l_i L_i + \sum_{i,j(i \neq j)}^{1\dots n} (r_{ij} R_{ij} + n_{ij} N_{ij}),$$

where, since $R_{ij} = R_{ji}$, $N_{ij} = N_{ji}$, we may take

$$r_{ij} = r_{ji},\ n_{ij} = n_{ji}.$$

Then must H contain

$$(L_k\, T) = 2q_{kk} L_k + \sum_{i=1\dots n}' q_{ik} N_{ik} - l_k' Q_{kk} - \sum_{i=1\dots n}' r_{ik} Q_{ki},$$

k being fixed and $\sum_i'$ indicating that $i \neq k$.

Similarly must H contain

$$(L_k'\, T) = -2q_{kk} L_k' - \sum_i' q_{ki} R_{ki} + l_k Q_{kk} + \sum_i' n_{ik} Q_{ik}.$$

Hence (for $s \neq k$) H contains

$$(L_k(L_k T)) = -2l_k' L_k;\ (L_k'(L_k' T)) = -2l_k L_k';$$

$$(L_s(L_k T)) = -r_{sk} N_{sk};\ (L_s'(L_k' T)) = -n_{sk} R_{sk};$$

$$(L_s'(L_k T)) = q_{sk} Q_{ks}$$

But $(Q_{sk} N_{sk}) = -2L_s;\ (N_{sk}(L_s R_{sk})) = -(N_{sk}\, Q_{sk}) = -2L_s.$

Since not every l_k', l_k, r_{sk}, n_{sk}, q_{sk}, q_{kk} is zero, T not being the identity, we find that H contains certainly one L_i and by the lemma the whole of G.

It remains to prove that the simple group G in $2n$ variables is the homogeneous form of the largest projective group in space of $2n - 1$ dimensions (with the coördinates $z, x_1, \cdots x_{n-1}, y_1, \cdots y_{n-1}$) leaving invariant the linear complex defined by the Pfaff's equation:

$$dz + \sum_{i=1}^{n-1} (x_i\, dy_i - y_i\, dx_i) = 0.$$

The $n\,(2n+1)$ infinitesimal transformations of the latter group are,* if $U \equiv \sum_{\nu=1}^{n-1} (x_\nu p_\nu + y_\nu q_\nu) + zr$:

$$p_i - y_i r,\ q_i + x_i r,\ r,\ zr + U,\ x_i q_k + x_k q_i,$$
$$x_i p_k - y_k q_i,\ y_i p_k + y_k p_i,\ zp_i - y_i U,\ zq_i + x_i U,\ zU.$$
$$(i, k = 1 \cdots n-1)$$

If we introduce† for the variables x_i, y_i, z respectively x_i/x_n, y_i/x_n, y_n/x_n, we reach the homogeneous group‡ (where the new transformations are written in the same order as the old above):

$$Q_{in},\ R_{ni},\ L_n',\ -Q_{nn},\ R_{ik} \text{ and } 2L_i',$$
$$Q_{ki},\ N_{ik} \text{ and } 2L_i,\ N_{in},\ -Q_{ni},\ -L_n$$
$$(i, k = 1 \cdots n-1;\ \ i \neq k \text{ in } N_{ik} \text{ and } R_{ik}).$$

This group is evidently identical with our group G. By the same transformation the above Pfaff's equation takes the homogeneous form (aside from the factor $1/x_n^2$):

$$\sum_{i=1}^{n} (x_i\, dy_i - y_i\, dx_i) = 0.$$

§4.

Semi-simple linear homogeneous groups whose defining function§ is the sum of n determinants of order $q > 2$.

Of the transformations of the general linear homogeneous group in $q\,n$ variables, we consider those which, when operating simultaneously on q independent sets of $n\ q$ variables, the j^{th} set of which may be exhibited thus

$$x_{i1}^{(j)}, x_{i2}^{(j)}, \cdots x_{iq}^{(j)}, \qquad (i = 1 \cdots n),$$

leave invariant the function

$$\varphi \equiv \sum_{i=1}^{n} D_i,$$

* Lie, Theorie der Transformationsgruppen, vol. II., p. 522.

† The formulæ for this change of variables are given in Lie, ibid, vol. I., p. 579.

‡ For $n = 2$, *i. e.*, for 4 variables, this homogeneous form is given in Lie, ibid, vol. II., p. 450.

§ Discontinuous linear groups with the same defining function are mentioned (but not studied) by Jordan, l. c., p. 219. The result reached in this paragraph, where $q > 2$, is wholly different from that of § 3, where $q = 2$.

where D_i denotes the determinant

$$D_i \equiv \begin{vmatrix} x_{i1}^{(1)}, & x_{i2}^{(1)}, & \cdots & x_{iq}^{(1)} \\ x_{i1}^{(2)}, & x_{i2}^{(2)}, & \cdots & x_{iq}^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ x_{i1}^{(q)}, & x_{i2}^{(q)}, & \cdots & x_{iq}^{(q)} \end{vmatrix} .$$

Denote by G the groups of transformations thus defined and let S be an arbitrary transformation of G; viz,

$$x_{ik}' = \sum_{j=1}^{n} (a_{j1}^{ik} x_{j1} + a_{j2}^{ik} x_{j2} + \cdots + a_{jq}^{ik} x_{jq})$$
$$(i = 1 \cdots n;\ k = 1 \cdots q).$$

The conditions that S shall leave φ invariant are:

$$(1) \qquad \sum_{i=1}^{n} \begin{vmatrix} a_{j1}^{i1}, & a_{j2}^{i1}, & \cdots & a_{jq}^{i1} \\ a_{j1}^{i2}, & a_{j2}^{i2}, & \cdots & a_{jq}^{i2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{j1}^{iq}, & a_{j2}^{iq}, & \cdots & a_{jq}^{iq} \end{vmatrix} = 1 \qquad (j=1, 2, \cdots n).$$

$$(2) \qquad \sum_{i=1}^{n'} \begin{vmatrix} a_{j_1k_1}^{i1} & \cdots\cdots & a_{j_qk_q}^{i1} \\ \vdots & & \vdots \\ a_{j_1k_1}^{iq} & \cdots\cdots & a_{j_qk_q}^{iq} \end{vmatrix} = 0$$

for $j_s = 1 \cdots n$; $k_s = 1 \cdots q$, provided not all the j_s's are equal and no pair $(j_\lambda, k_\lambda) = (j_\mu, k_\mu)$.

The general infinitesimal transformation of G is found by substituting in S

$$a_{ik}^{ik} = 1 + \alpha_{ik}^{ik} \cdot \delta t, \ a_{rs}^{ik} = \alpha_{rs}^{ik} \cdot \delta t \left(\begin{matrix} i, r = 1 \cdots n;\ k, s = 1 \cdots q \\ (r, s) \neq (i, k) \end{matrix} \right)$$

The first set of conditions (1) on the a's give at once

$$\alpha_{j1}^{j1} + \alpha_{j2}^{j2} + \cdots + \alpha_{jq}^{jq} = 0 \qquad (j = 1 \cdots n).$$

To prove that the second set (2) gives simply (for $q > 2$)

$$\alpha_{j_1m}^{jl} = 0 \ \ (l, m = 1 \cdots q;\ j, j_1 = 1 \cdots n,\ j_1 \neq j),$$

consider first the conditions in which $j_2 = j_3 = \cdots = j_q = j$, say, and thus $j_1 \neq j$; viz,

$$\sum_{i=1}^{n} \begin{vmatrix} a_{j_1k_1}^{i1}, & a_{jk_2}^{i1}, & a_{jk_3}^{i1}, & \cdots & a_{jk_q}^{i1} \\ \vdots & & & & \\ a_{j_1k_1}^{iq}, & a_{jk_2}^{iq}, & a_{jk_3}^{iq}, & \cdots & a_{jk_q}^{iq} \end{vmatrix} = 0$$

For those determinants given by $i \neq j$, the elements of the last $q - 1$ columns all have the factor δt and hence the determinants have the factor δt^{q-1} and hence the factor δt^2. For the determinant $i = j$, the elements of the first column have the factor δt; likewise those of the l^{th}row, if l be the integer $\leqq q$ which is lacking among the numbers $k_2, k_3, \cdots k_q$, all different. The minor of $a^{jl}_{j_1 k_1}$ has the term

$$\pm a^{jk_2}_{jk_2} a^{jk_3}_{jk_3} \cdots a^{jk_q}_{jk_q} = \pm (1 + a^{jk_2}_{jk_2} \cdot \delta t) \cdots (1 + a^{jk_q}_{jk_q} \cdot \delta t)$$

and but one such. Hence on expanding the determinant according to the elements of the first column, we get the term $\pm a^{jl}_{j_1 k_1} \cdot \delta t$ together with terms of higher order in δt. Hence $a^{jl}_{j_1 k_1} = 0$.

The remaining ones of the conditions (2) are then satisfied identically. For unless $i = j_s$, the elements of the s^{th} column

$$a^{il}_{j_s k_s} = a^{il}_{j_s k_s} \cdot \delta t = 0 \quad (l = 1 \cdots q).$$

But i can not simultaneously equal $j_1, j_2, \cdots j_q$ for the conditions (2); hence the elements of at least one column of every determinant are all zero.

The group G thus contains $n\,(q^2 - 1)$ linearly independent infinitesimal transformations,

$$\delta x_{ik} \equiv x_{ik}' - x_{ik} = a^{ik}_{i1} x_{i1} + a^{ik}_{i2} x_{i2} + \cdots + a^{ik}_{iq} x_{iq}$$

where $$\sum_{k=1}^{q} a^{ik}_{ik} = 0 \qquad (i = 1 \cdots n).$$

The finite equations of the transformations of G are then evidently

$$x_{ik}' = a^{ik}_{i1} x_{i1} + \cdots + a^{ik}_{iq} x_{iq} \qquad (i = 1 \cdots n;\ k = 1 \cdots q),$$

where the determinants

$$| a^{ik}_{im} | = 1 \qquad (i = 1 \cdots n).$$

Thus the group G breaks up into n sub-groups, those transformations in which i has a fixed value forming the linear homogeneous group, leaving the determinant D_i invariant, *i. e.*, the special linear homogeneous group in q variables. Since G breaks up into n invariant *simple* sub-groups, and since all the transformations of any one are commutative with all of any other one, G is a so-called half-simple (halbeinfach) group.

LEIPZIG, *February* 6, 1897.

[The contents of § 3 and § 4 were presented February 19th before the Groups-Seminar of Professor Lie, who stated that the interesting result of § 4 was new and not what one would have expected.]