ON CERTAIN METHODS OF STURM AND THEIR APPLICATION TO THE ROOTS OF BESSEL'S FUNCTIONS.

BY PROFESSOR MAXIME BÔCHER.

Read at the February Meeting of the Society, 1897.

LAST November Mr. M. B. Porter, a graduate student at Harvard, submitted to me a proof that when $n > -\frac{1}{2}$ the well known theorem that between two successive positive roots of $J_n(x)$ lies at least one root of $J_{n+1}(x)$ can be extended to give the theorem that between two successive positive roots of $J_n(x)$ lies *just* one root of $J_{n+1}(x)$.* This proof consisted in applying to Bessel's equation the following proposition due to Sturm:†

If in a certain interval of the x-axis $\varphi_1(x) < \varphi_2(x)$ *then between two successive roots, lying in this interval, of a solution of the equation:*

$$\frac{d^2y}{dx^2} = \varphi_1(x) \cdot y,$$

there cannot lie more than one root of a solution of the equation:

$$\frac{d^2y}{dx^2} = \varphi_2(x) \cdot y.$$

The application to Bessel's functions is immediate when we let $y = \sqrt{x}\, J_n(x)$ for then y satisfies the differential equation:

$$\frac{d^2y}{dx^2} = \left(\frac{4\,n^2-1}{4\,x^2} - 1\right)y.$$

Not being aware at the time that the theorem above quoted is explicitly given by Sturm, Mr. Porter gave a proof of it which depends upon some better known theorems of the same mathematician. This proof, which is different from the one given by Sturm, is reproduced below (p. 210).

There has just appeared in the *American Journal*, (vol. xix, p. 75) another proof of the theorem concerning the

* For the sake of simplicity of statement we confine our attention to positive roots, the negative roots being numerically equal to them.

† *Liouville's Journal*, vol. i., p. 136. We will in future quote this article by merely mentioning Sturm's name.

roots of Bessel's functions by E. B. Van Vleck which also rests on a theorem of Sturm. This paper contains an extension of the theorems and methods in question to the subject of contiguous hypergeometric functions.

I propose to present now a third method for proving and extending the theorem, following out still other lines of thought marked out by Sturm (p. 160 seq.), and I will then compare the three methods showing wherein lie the peculiar advantages and possibilities of extension of each. My purpose has been to call attention to Sturm's methods rather than to elaborate the details of the theory of the roots of Bessel's functions. The very fact that the properties of the roots of Bessel's functions here considered are not generally known brings out in a striking way how little this fundamental paper of Sturm has really been read.

The relative position of the positive roots of $J_n(x)$ and $J_{n+1}(x)$ is evidently the same as that of the positive roots of $R_n(\xi)$ and $R_{n+1}(\xi)$ where $\xi = x^2/4$ and $R_n(\xi) = (2/x)^n J_n(x)$. The object of introducing $R_n(\xi)$ is that (cf. Gray and Mathews: *Treatise on Bessel Functions*, p. 46)

$$R_n'(\xi) = -R_{n+1}(\xi),$$

(the accent denoting differentiation) so that Rolle's theorem shows us that between two successive roots of $R_n(\xi)$ lies at least one root of $R_{n+1}(\xi)$. To show that there lies in this interval only one root of $R_{n+1}(\xi)$ we have merely to pass from Bessel's equation to the equation satisfied by $R_n(\xi)$:

$$\xi \frac{d^2y}{d\xi^2} + (n+1)\frac{dy}{d\xi} + y = 0.$$

A point at which $R_n'(\xi) = 0$ must be a maximum or a minimum of $R_n(\xi)$ as otherwise $R_n''(\xi)$ would also vanish at this point and this is seen from the differential equation to be impossible. If then between two successive vanishing points of $R_n(\xi)$ $R_n'(\xi)$ vanished more than once it would have to vanish at least three times and at one of these points $R_n(\xi)$ would have a minimum if positive, a maximum if negative. In either case $R_n(\xi)$ and $R_n''(\xi)$ would have the same sign at a point where $R_n'(\xi) = 0$, and this, as we see from the differential equation, is impossible. This completes the proof of the theorem that *between two successive positive roots of $J_n(x)$ lies one and only one root of $J_{n+1}(x)$.**

* Cf. Sturm, p. 162.

Precisely the methods just used enable us to obtain an analogous theorem, which the writer has never seen stated, concerning the roots of $J_{n+2}(x)$. We again consider two successive roots a and b $(0 < a < b)$ of $R_n(\xi)$. Between a and b lies, as we have seen, one and only one point c for which $R_n'(c) = 0$. From a to c, R_n and R_n' have the same sign so that we see from the differential equation that $R_n''(\xi)$ cannot vanish between a and c. $R_n''(\xi)$ must, however, vanish at least once in $a\,b$ (and, therefore, as we have just seen in $c\,b$) for $R_n(a) = R_n(b) = 0$, while $R_n'a$ and $R_n'(b)$ have opposite signs, so that it is clear from the differential equation that $R_n''(a)$ and $R_n''(b)$ have opposite signs. It remains to see whether R_n'' can vanish more than once in $c\,b$. It is easily seen that at every point where $R_n'' = 0$, R_n' has a maximum or a minimum so that if R_n'' vanishes more than once in $a\,b$ it vanishes at least three times, and, therefore, if R_n' is positive it has at least one minimum, if negative, one maximum. In either case R_n' and R_n''' have the same sign while $R_n'' = 0$, and this is seen to be impossible when we consider the following relation obtained by differentiating the differential equation for R_n:

$$\xi R_n'''(\xi) + (n+2) R_n''(\xi) + R_n'(\xi) = 0.$$

We thus get the theorem:

Between two successive positive roots of $J_n(x)$ *lies one and only one root of* $J_{n+2}(x)$.

This root is greater than the root of $J_{n+1}(x)$ *which lies in the interval in question.*

It is clear that the method here used may be applied to a large class of similar cases.* For example the following theorem may be deduced directly from Bessel's equation:

Between two successive roots of $J_n(x)$, *between which* $x > |n|$, *lies one and only one root of* $J_n'(x)$ *and one and only one root of* $J_n''(x)$. *The root of* $J_n''(x)$ *is greater than the root of* $J_n'(x)$.

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* The principal theorem used applies even to the case of certain partial differential equations. It may be stated as follows (cf. for $n=1$ Sturm p. 160).

In the partial differential equation:

$$\sum_{i=1}^{i=n}\left({}_i \frac{\partial^2 u}{\partial x_i^2} + b_i \frac{\partial u}{\partial x_i}\right) + cu = 0.$$

a_i, b_i, c *are real functions of the real variables* $x_1, \ldots x_n$, *and throughout a certain region,* $a_1, \ldots a_n$ *are positive. If throughout this region* c *is positive* u *cannot have a positive minimum or a negative maximum; if* c *is negative* u *cannot have a positive maximum or a negative minimum.*

For an example of the use of this theorem when $n=2$ cf. Picard Traité d' Analyse, Vol. ii., p. 34.

We will now proceed to a comparison of the method just explained with the one used by Van Vleck. It seems to me that the former is of a distinctly more elementary character, inasmuch as it involves the sort of discussion which we must always go through with when we wish to get an idea of the nature of a real function of a real variable, viz., the location of maxima and minima, points of inflection, etc. Van Vleck's method, on the other hand, is not only extremely elegant but is in touch with the important idea of *related* * differential equations so that it can be applied at once to hypergeometric functions and in fact to the solutions of any regular linear differential equation of the second order and possibly, therefore, ultimately to irregular equations of the second order in general, since these may be regarded as the limiting forms of regular equations. The method contained in the foregoing pages, while applying as readily to irregular as to regular equations, depends essentially on the properties of derivatives which are merely special cases of related functions (having in general accessory points †), and appears, therefore, at first sight to have a more limited range of application than Van Vleck's. The two methods appear, however, on closer examination, to be coextensive, at least in their application to Bessel's functions and hypergeometric functions. Two very simple examples may suffice here.

1. The theorem above proved that between two successive positive roots of $J_n(x)$ lies just one root of $J_{n+2}(x)$ follows at once by Van Vleck's method from the relation:

$$J_{n+2} J_{-n} - J_{-(n+2)} J_n = \frac{4(n+1)\sin(n+1)\pi}{\pi x^2},$$

or when n is an integer:

$$J_{n+2} Y_n - Y_{n+2} J_n = \frac{2(n+1)}{x^2}.$$

2. If we remember that

$$\frac{d}{dx} F(\alpha, \beta, \gamma, x) = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1, x)$$

and that $F(\alpha, \beta, \gamma, x)$ satisfies the differential equation:

* In my lectures I have been in the habit of translating the German term *verwandt* by *kindred* rather than *related* as this last term is constantly used in an entirely untechnical sense to denote any kind of connection or similarity in form.

† Thus $J_n'(x)$ has accessory points at $x = \pm n$.

$$\frac{d^2y}{dx^2}+\frac{\gamma-(\alpha+\beta+1)x}{x(1-x)}\frac{dy}{dx}-\frac{\alpha\beta}{x(1-x)}y=0,$$

we obtain by the method above explained the theorem:

Between two successive roots of $F(\alpha, \beta, \gamma, x)$ which lie in the interval from 0 to 1 lies one and only one root of $F(\alpha+1, \beta+1, \gamma+1, x)$; and, provided that the point $x=\gamma/(\alpha+\beta+1)$ does not lie between the roots of $F(\alpha, \beta, \gamma, x)$ in question, one and only one root of $F(\alpha+2, \beta+2, \gamma+2, x)$.

The second part of this theorem might easily be obtained by Van Vleck's method, while the first part is merely a special case of his theorem concerning contiguous hypergeometric functions. The general theorem can easily be obtained in the same way.

It is clear that what is essential to the application of our method is that the two related functions should be so prepared by multiplication by suitable factors and, if necessary, by a change of the independent variable that one is a derivative of the other.

Very different from the two methods so far discussed and in some respects much more far-reaching is the method suggested by Mr. Porter. In order to simplify matters we will restrict ourselves in this article to positive values of n. Before beginning with a discussion of the method it will be well to note that the proof given at the beginning of this paper established the fact that, *k being any positive quantity, between two successive positive roots of $J_n(x)$ (n positive) cannot lie more than one root of $J_{n+k}(x)$.*

The theorem of Sturm quoted at the beginning of this paper was proved by Mr. Porter by means of the following two more fundamental theorems also due to Sturm:

I. *p and q being continuous real functions of the real variable x, between two successive roots of a solution of the differential equation:*

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0$$

will lie one and only one root of any linearly independent solution.

II. *If, φ_1 and φ_2 being continuous real functions of the real variable x, $\varphi_1<\varphi_2$, and if y_1 and y_2 both vanish at a point x_0 and satisfy respectively the equations:*

$$\frac{d^2y}{dx^2}=\varphi_1\cdot y, \tag{1}$$

$$\frac{d^2y}{dx^2}=\varphi_2\cdot y, \tag{2}$$

then, if y_2 has n roots to the right (left) of x_0, y_1, will also have at least n roots there and the k^{th} root of y_2 from x_0 will be greater (less) than the k^{th} root of y_1 from x_0.*

Mr. Porter's proof now is as follows:

Retaining the notation of the last theorem, let us suppose that between two successive roots of a solution y_1 of (1) lie more than one root of a solution $\bar{y}$ of (2). Let y_2 be a solution of (2) which vanishes at one of the two roots of y_1 in question. Then by Theorem I y_2 must vanish between any two successive roots of $\bar{y}$ and therefore at least once between the roots of y_1 in question; but this is impossible by Theorem II.

I have reproduced this proof not merely because I consider the method used instructive, but because it is capable of extension to cases which could not readily be attacked by the method given by Sturm.

Before going farther, I should like to present a proof of Theorem I just stated which I have been in the habit of giving in my lectures and to which, though it will be familiar to many, I am unable to give a reference†.

Let a and b be two successive roots of a solution y_1 of the differential equation. Let y_2 be any solution linearly independent of y_1 and consider the ratio y_1/y_2. If y_2 did not vanish between a and b this ratio would be continuous throughout the interval and, since it vanishes at the extremities, its derivative $(y_1' y_2 - y_1 y_2')/y_2^2$ would vanish in the interval. This is impossible as the vanishing of $y_1' y_2 - y_1 y_2'$ is obviously the condition that y_1 and y_2 should be linearly dependent. y_2 must then vanish at least once between two successive points where y_1 vanishes, and we see that it cannot vanish more than once, for if it did, by a similar proof y_1 would vanish between two successive points where y_2 vanishes and therefore a and b would not be *successive* roots of y_1.‡

* Sturm, p. 125.

† Two methods of proof are given by Sturm, of which the second, depending, as it does, on the formula $y_1 y_2' - y_2 y_1' = ce^{-\int p dx}$ has a certain analogy to Van Vleck's method above referred to.

‡ Various modifications of this proof can be given, one of which will be suggested by a method used by Van Vleck on p. 76 of the paper above quoted. If p and q are analytic functions y_1/y_2 is the "Schwarzian s-function" and the following form of the proof is instructive:

If x moves along its axis of reals s will do the same (p, q, y_1, y_2 being supposed real for the real values of x in question). It is a fundamental property of the s-function that the representation of the x-plane on the s-plane is conformal except at the singular points of the differential equation. As x moves always in one direction from one real root of y_1 to the

It is clear that the proof just given really establishes the following generalized form of the theorem:

If within the interval ab (excluding the ends) the coefficients of the differential equation:

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

are continuous, and if there exists a solution y_1 which does not vanish between a and b and such that its ratio to a linearly independent solution y_2 approaches zero as we approach each end of the interval, then y_2 vanishes once and only once between a and b.

In particular if p and q are analytic functions a and b might be regular points with real exponents, and y_1 a solution corresponding to the largest of these exponents at each point. Here either a or b, or both, may, of course, be not merely regular but non-singular points.

Although the theorem just stated is capable of important general application, it will be sufficient for our purposes to deduce from it the following theorem concerning Bessel's functions:

If n and k are positive and a is the smallest positive root of $J_n(x)$ then $J_{n+k}(x)$ cannot vanish in the interval from o to a.

For if $J_{n+k}(x)$ vanishes at the point b between o and a the solution of Bessel's equation with parameter $n + k$ which vanishes when $x = a$ would vanish again by the theorem just proved between o and b (say at c) and therefore by the theorem numbered II above, $J_n(x)$ would vanish between c and a, *i. e.*, a would not be the smallest positive root of $J_n(x)$. It will be noticed that we have merely repeated Mr. Porter's proof under slightly different circumstances.

Since, then, as n increases the smallest positive root $J_n(x)$ increases* and, since between two successive positive roots of $J_n(x)$, lies at most one root of $J_{n+k}(x)$ (k positive) it fol-

next without passing a singular point s must move always in the same direction (else the representation would not be corformal) from $s = 0$ back to $s = 0$ and must, therefore, pass through the point $s = \infty$ once and only once. This establishes the theorem. This proof is implicitly contained in Klein's paper: "Ueber die Nullstellen der hypergeometrishen Reihe." *Math. Ann.*, vol. 37.

* We have, strictly speaking, only proved that this root does not decrease as n increases. It is, however, easy to see that it cannot remain constant, for $J_n(x)$ is an analytic function of n and x and, therefore, if $J_n(x)=0$ x is an analytic function of n and if it were constant for any continuous set of values of n it would be constant for all values of n. This, however, is not the case, as the smallest root of $J_n(x)$ is evidently greater than n, since $J_n(x)$ cannot have a positive maximum or a negative minimum when $x < n$.

lows at once that *as n (which is supposed positive) increases, all the positive roots of* $J_n(x)$ *increase.**

From this it appears that, when k is sufficiently small, between two successive positive roots of $J_n(x)$ lies one and only one root of $J_{n+k}(x)$, and the question arises how large k can be allowed to become without this condition of affairs changing. It is clear that the theorem can cease to be true only when one of the roots of $J_{n+k}(x)$ passes a root of $J_n(x)$, and this can occur only when each of the subsequent roots of $J_{n+k}(x)$ has passed one of the roots of $J_n(x)$ as otherwise we should have two roots of $J_{n+k}(x)$ between two successive roots of $J_n(x)$. It is, therefore, the large roots of $J_{n+k}(x)$ which will first reach the next roots of $J_n(x)$ and from the fact that the large roots are given approximately by the formula:

$$\frac{\pi}{4}(2n - 1 + 4p),$$

where p takes on in succession all large integral values, it is clear that as long as $k < 2$ no root of $J_{n+k}(x)$ will have reached the next root of $J_n(x)$, while when $k > 2$ the large roots of $J_{n+k}(x)$ will have passed those of $J_n(x)$. What happens when $k = 2$ can be determined by the use of more exact asymptotic values for the large roots. We will not, however, stop to consider this point, as we have already obtained the result we need by another method (p. 207). We have thus obtained the theorem :

If n is positive and $o < k \leqq 2$ *between two successive positive roots of* $J_n(x)$ *lies one and only one root of* $J_{n+k}(x)$ *and vice versa.*

This theorem contains (when $k = 1$) the theorem established by Van Vleck as a special case. It brings out, however, clearly, at least when n is positive, (and it is herein that I see the chief importance of the method suggested by Mr. Porter) that *the theorem in question has no necessary connection whatever with the subject of related functions.*

The method just explained gives us, when carried a step farther, the following theorem:

If n is positive and $2p < k \leqq 2p + 2$, *where p is any positive integer, in each of the intervals bounded by successive positive roots of* $J_n(x)$ *lies one and only one root of* $J_{n+k}(x)$ *except in p of the intervals in which no root of* $J_{n+k}(x)$ *lies. Exceptions occur when and only when roots of* $J_n(x)$ *and* $J_{n+k}(x)$ *coincide, in which case one of the p intervals above mentioned is replaced by the two intervals which are separated by the common root in question.*

* This theorem may also be deduced by following more closely the methods of Sturm. Cf. the remark of Schläfli in the foot-note on p. 137 of vol. 10, of the *Math. Ann.*

The precise position of these p intervals can be determined when k is an integer either by Van Vleck's method or by the method explained at the beginning of this paper. If, for instance, $k = 3$ we may proceed as follows. We easily find that R_n satisfies the relation:

$$\xi^2 R_n''' + [\xi - (n + 1)(n + 2)] R_n' - (n + 2) R_n = 0.$$

At two successive points where $R_n = 0$ R_n''' will therefore have opposite signs unless between the points is question $\xi = (n + 1)(n + 2)$; and we have the theorem:

$J_{n+3}(x)$ vanishes once and only once between two successive positive roots of $J_n(x)$ except between the two roots which include between them the point $x = 2\sqrt{(n+1)(n+2)}$ in which interval $J_{n+3}(x)$ does not vanish at all.

Bessel's equation is clearly only a first example to which the methods of Sturm, which we have discussed, can be profitably applied. Further considerations of this sort, however, with reference especially to Bessel's functions with negative subscripts and to the theory of hypergeometric functions I will reserve for a future occasion. I shall be satisfied if the foregoing discussion helps to emphasize the importance of Sturm's paper.

HARVARD UNIVERSITY,
January 4, 1897.

ON THE TRANSITIVE SUBSTITUTION GROUPS WHOSE ORDERS ARE THE PRODUCTS OF THREE PRIME NUMBERS.

BY DR. G. A. MILLER.

[Read at the January meeting of the Society, 1897.]

ALL the regular groups of these orders have been determined by Cole and Glover and by Hölder.* It is the object of this paper to determine all the transitive groups that are simply isomorphic to these regular ones. As every substitution group of a given order is simply isomorphic to one and only one regular group, we shall thus find all the possible non-regular transitive groups whose orders are the products of any three prime numbers. At the same time we shall be

* A regular substitution group may be said to be determined by the simply isomorphic abstract or operation group and *vice versa*.