THE MOTIONS OF THE ATMOSPHERE AND ESPECIALLY ITS WAVES.*

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The inadmissibility of those views according to which the motions of the atmosphere consist in the development of independent cyclones and anti-cyclones is, of late years, more and more plainly recognized. This conclusion has been arrived at, not so much through a severe criticism of the fundamental basis upon which these erroneous views had been established, as by the power of the facts that resisted introduction into this artificial system.

It is now nearly ten years since the theory of cyclones began to totter, and especially under the influence of a memoir published by Hann.† Following soon upon this the idea was developed by von Helmholtz ‡ (*Berlin Sitzungsb.*, 1888), "that in the atmosphere, by continuously acting forces, the formation of surfaces of discontinuity is possible, and that the anticyclonic motion of the lower strata and the extensive, gradually increasing cyclone of the upper strata that are to be expected around the north and south poles, resolve themselves into a great number of *irregularly* progressing cyclones and anticyclones with an excess of the former." Von Helmholtz further says§ (*Berlin Sitzungsb.*, 1889): "if as in the case of the earth, the lower stratum is the denser, then it can be shown that the disturbances must at first proceed as do the waves of water raised by the wind." In this way the origin of cyclones and anticyclones is traced back to the general atmospheric circulation which is itself dependent on the difference in temperature between the equator and the poles; but the cyclones and anticyclones when once formed can still be considered as independent phenomena complete within themselves.

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* Translated from *Verhandlungen der Gesellschaft Deutscher Naturforscher und Aerzte*, pp. 42–50 and 323–324, by Professor CLEVELAND ABBE.

† Comp. von Bezold, *Berliner Sitzungsberichte*, math.-nat. Kl., 1890, p. 831.

‡ Translated at pages 92, 93 of the *Mechanics of the Earth's Atmosphere.*

§ Translated at page 94 of the *Mechanics of the Earth's Atmosphere.*

The proof of this proposition was obtained by von Helmholtz by the application of the so-called theorem of the conservation of areas to a specific *portion* of the atmosphere, *i. e.*, a ring of air. But the application of this theorem in this manner is incorrect; it can only be applied to a free system of bodies (See *Met. Zeit.*, 1894, p. 114). Von Helmholtz applies this theorem to a ring of air cut out of the atmosphere; now such a ring of air does not form a free system but is subject to the conditions of its connection with the rest of the atmosphere, and the theorem of the conservation of angular momentum, or of areas, is not applicable to it. The application of this theorem to a ring-shaped portion of the atmosphere is only allowable in case the conditions within the atmosphere are of such a nature that they correspond entirely to the conditions at the boundary surface of a fluid and any other body. The conditions for the boundary surface of a fluid are: equal pressure on both sides of the boundary surface and equal values for the normal component of the velocities in the two masses that come in contact at the surface. (See Kirchhoff, *Mechanik*, 2d ed. 1877, p. 165; 3d ed. 1883, p. 164). These conditions are not fulfilled in the earth's atmosphere, which is in motion and not in static equilibrium. Therefore, that theory of atmospheric motions that depends upon their analysis into cyclones and anticyclones is deceptive and not so well established as it was believed to have been by the investigations of von Helmholtz.

Now, in order to find the point at which the prevailing views as to the motions of the atmosphere diverge from the reality it will be necessary to follow up a train of thought based upon indubitable well-established mechanical theorems. In doing this we will, as usual, begin with the consideration of an atmosphere that, without friction, surrounds the ideal figure of the earth, *i. e.*, a homogeneous ellipsoid of rotation, and whose particles are attracted by the mass of this ellipsoid, according to the general law of attraction of masses. It will be further assumed that, for a certain initial temperature, uniform throughout all its strata, the atmosphere has arrived at a uniform rate of revolution about the axis of rotation of the ellipsoid and with a velocity such that the ellipsoid represents the figure of equilibrium of an incompressible fluid.

Let this initial equilibrium of the atmosphere be now disturbed by introducing a different distribution of temperature, viz., one depending only on altitude above the ideal surface of the earth and on latitude, but uniform along any circle of latitude and every circle concentric therewith.

The *first* question now is whether a new state of equilibrium can develop, perhaps by means of a new rate of rotation that shall be different in the various strata of air vertically above each zone of latitude but be uniform in each circle concentric with and lying in the same plane as the circle of latitude.

The conditions for the equilibrium of a rotating fluid are:

$$\mu(R+r\,\omega^2)=\frac{\partial p}{\partial r}, \qquad \mu Z=\frac{\partial p}{\partial z},$$

where the axis of z is the axis of rotation; μ is the density of the element of mass; r is the distance of the element of mass from the axis of z; ω is the angular velocity; R and Z are the components of the forces parallel to r and z; p is the elastic pressure. Let V be the potential of the attraction of the mass of the terrestrial ellipsoid; T the absolute temperature of the element of fluid mass. Let the characteristic equation of elasticity for gases be:

$$(1) \qquad \frac{p}{T}=A\mu\ ,$$

The equations of equilibrium can now be transformed into:

$$\frac{1}{T}\left(\frac{\partial V}{\partial r}+r\,\omega^2\right)=\frac{A}{p}\,\frac{\partial p}{\partial r}, \qquad \frac{1}{T}\,\frac{\partial V}{\partial z}=\frac{A}{p}\,\frac{\partial p}{\partial z}\,.$$

By integrating the second equation there results:

$$(2) \qquad A\log p=\int\frac{1}{T}\cdot\frac{\partial V}{\partial z}\,dz+F(r).$$

By differentiating this with respect to r, substituting the value of $\frac{A}{p}\cdot\frac{\partial p}{\partial r}$ thus obtained in the first equation and executing a partial integration we obtain:

$$(3) \qquad r\omega^2=T\left\{\int\left(\frac{\partial V}{\partial z}\cdot\frac{\partial\frac{1}{T}}{\partial r}-\frac{\partial V}{\partial r}\cdot\frac{\partial\frac{1}{T}}{\partial z}\right)dz+F'(r)\right\}.$$

$F(r)$ and $F'(r)$ are determined by the surface conditions, in the present case by the circumstances existing in the highest layer of the atmosphere in which a uniform temperature and a velocity of rotation equal to that of the earth will be assumed. Thus, p, μ and ω are determined by the preceding equations. Under the fundamental assumption that no other external forces than the attraction of the mass of the ellipsoidal earth are acting on the atmosphere, it follows that the changes in the distribution of temperature that produce changes in the motions of the atmosphere can produce no changes in the total moment (M) with reference to the axis of z. Therefore, a state of equilibrium can be brought about in the above-mentioned manner only for such distributions of temperature in the atmosphere as, by means of the above equations, give values for μ and ω that shall satisfy the equation

$$(4) \qquad \int \mu .\, r^2 .\, \omega .\, d\text{v} = M,$$

the integration extending over the whole volume of the atmosphere. Therefore, a condition of equilibrium can only exist when the temperature T satisfies the equations (1), (2), (3) and (4). Such a condition of equilibrium cannot occur with every distribution of temperature, but, in general, there must exist movements in the direction from the equator to the pole, and inversely, besides a different rate of rotation for the different strata of air.

The *second* question now becomes this: Can a steady condition of motion occur in the atmosphere with an unvarying distribution of temperature? That is to say, under the assumed conditions, can the movement at each individual point in the atmosphere remain always the same?

Under the original assumption that the initial angular velocity is the same throughout the whole atmosphere, and that the temperature changes only with the distance from the pole, and with the altitude above the ideal surface of the earth, it must happen that whenever any steady condition does occur, the conditions must be the same at every point of any circle described about the axis of rotation. Therefore, throughout such a circle there can be no change of pressure with reference to either time or space [*i. e.* neither chronological nor geographical].

Such a distribution of pressure, however, cannot coexist with motions having a meridional component. This can be proved as follows:

Assuming that the temperature in the atmosphere depends only on the latitude and altitude, it has been shown that, in the steady condition, the motion and pressure must be the same for equal latitudes and altitudes. If, therefore, meridional components of motion exist, the individual particles of air must move on closed surfaces of revolution about the earth's axis. Such a surface will contain, at any moment, all those particles that at any previous time crossed over any particular parallel circle of this surface; and the meridional component of motion must have the same direction along any meridional section of the surface. The space enclosed by one of these surfaces may enclose a part of the axis of rotation or not. The *first* case cannot occur, because on account of the general agreement in direction of the meridional component of motion, there would arise a steadily increasing accumulation of atmosphere at one pole of the surface of revolution, while from the opposite pole the adjacent air would steadily depart without any corresponding replacement by other air.

In the *second* case when the motion of the air occurs on a closed surface that does not intersect the earth's axis, we may refer back to the equation of continuity for fluid motion. This equation, referred to rectangular coördinates and for the case of steady motion, reads

$$\frac{\partial \mu \frac{dx}{dt}}{\partial x}+\frac{\partial \mu \frac{dy}{dt}}{\partial y}+\frac{\partial \mu \frac{dz}{dt}}{\partial z}=0.$$

Transform this equation to another system of coördinates r, φ, z, connected with x, y, z by the equations

$$x=r\cos\varphi, \quad y=r\sin\varphi, \quad z=z\,;$$

take the origin of coördinates at the centre of the earth, the axis of z as the earth's axis of rotation and recall that under the conditions here assumed, the derivatives with respect to φ are equal to zero, and we obtain:

$$\frac{\partial \mu \frac{dr}{dt}}{\partial r}+\frac{\partial \mu \frac{dz}{dt}}{\partial z}+\frac{\mu}{r}\cdot\frac{dr}{dt}=0\,;$$

whence it follows that

$$\mu r\frac{dr}{dt}=-\frac{\partial F(r,z)}{\partial z}\,, \qquad \mu r\frac{dz}{dt}=+\frac{\partial F(r,z)}{\partial r}$$

and

$$\frac{\partial F(r,z)}{\partial z}\cdot\frac{dz}{dt}+\frac{\partial F(r,z)}{\partial r}\,\frac{dr}{dt}=0.$$

But a function $F(r,z)$ varying continuously in such a way that the equation $F(r,z)=C$ shall represent closed surfaces lying only within or without a semi-ellipsoidal shell, does not exist. Therefore in no case, when the distribution of temperature causes meridional components of motion can a condition of steady motion and stationary pressure exist within the atmosphere.

Excepting the memoirs of von Helmholtz, already referred to, the assumption of a stationary condition has hitherto been adopted in all investigations into the motions of the atmosphere, *e. g.*, Ferrel, "Meteorological Researches," Part I., 1877, and Oberbeck (*Berlin. Sitzungb.* 1888).* Certain errors in the two memoirs by myself (*Met. Zeits.*, 1893, pp. 1, 131) also result from this erroneous assumption. Such an assumption has certainly led to impossible conclusions, but the fundamental nature of the error has, so far as I know, not been recognized hitherto.

Understanding, therefore, that in an atmosphere under these ideal assumptions, a steady condition cannot occur, it follows that the movements of the real atmosphere under conditions that imply continuous values and such as depend only on the latitude and the altitude above the earth's surface, must consist of oscillations and *regularly* progressing waves. These waves have no relation to the billows considered by von Helmholtz, for the existence of these latter depends on the formation of surfaces of discontinuity whereas the present discussion is based wholly upon continuous conditions of motion and pressure.

The waves originating in the difference of temperature between the equator and the poles may be classified into two groups. The first group includes waves that advance in the direction of the meridian, and that may partly resolve into standing oscillations with nodes at the poles; the second group consists of waves that advance in the direction of the circles of latitude. The atmospheric pressures and motions resulting from the combination of these two groups of intersecting waves, give rise to the phenomena hitherto called cyclonic and anticyclonic.

If the heights of the waves that advance in the direction of the circles of latitude vary with the latitude, then this would cause so-called cyclonic and anticyclonic phenomena in the atmosphere, even though no waves moving along the meridians were formed. In any case, however, the preceding considerations lead to the following conclusion: A large part of the atmospheric phenomena called cyclones and anticyclones, having a progressive motion along the earth's sur-

* Translated in the "Mechanics of the Atmosphere," pp. 176, 178.

face, belongs to the general atmospheric circulation that develops in consequence of the differences of temperature between the equator and the poles; that is to say such phenomena would be found in the terrestrial atmosphere even if there were no differences of temperature along the individual latitude circles and no friction or resistances of any kind. These phenomena are therefore not to be considered as local disturbances of an equilibrium of the atmosphere produced by the temporary distribution of temperature between the equator and the poles, but present themselves as the regular, normal and mechanically necessary phenomena of an atmosphere for which, in general, no such state of equilibrium can exist except in very definite and restricted cases.

In several important memoirs (*Vienna Sitzungsb., math. nat. Klasse*, vol. 99, Abt. IIa, p. 204 ; Vol. 101, Abt. 2a, p. 597; Vol. 102, Abt. 2a, p. 11 and p. 1369.) M. Margules has treated of atmospheric waves, by submitting to analytical investigation the *possible motions* of the air in a rotating spheroidal shell under certain assumptions made in order to simplify the calculations. But the results of Margules's investigations ought not to be applied without further consideration to an atmosphere of great height. According to Margules in the case of an advancing wave in a frictionless medium the strongest wind should occur simultaneously with the extremes of pressure.

Under the assumption, as before, of no friction and a distribution of temperature dependent only upon latitude and altitude there will be formed, besides the waves, a stationary zonal distribution of pressure corresponding to atmospheric currents parallel to the circles of latitude. This zone partially balances the disturbance of equilibrium due to a part of the temperature distribution, and must respond to the conditions under which equilibrium can occur in the atmosphere. If the meridional section shows that there are maxima and minima in this distribution of pressure, as appears to be actually the case, then these, in connection with the waves running along the zone of latitude, also lead to the formation of anticyclonic and cyclonic phenomena.

There have been many attempts to explain or compute this stationary distribution of pressure, as it is presented to us, mixed up with another pressure distribution due to thermal anomalies, in the mean values of atmospheric pressure, and that, too, in the belief that in these mean pressures, we have the simple result of the difference of temperature between the equator and the poles. The attempts to deduce this evident zonal distribution of pressure, even in its general features, from general dynamic considerations must even *a priori* be considered as failures. It is as impossible for general considerations to lead to this result as for them to determine the form of an uniformly rotating fluid, whose particles are subject only to the general attraction of masses, as that of an ellipsoid.

The mathematical figure of the rotating fluid and of the surfaces of equal pressure in our atmosphere (therefore also the location and value of the zonal maxima and minima of pressure on the earth's surface, where it intersects the surfaces of equal pressure), can be found if not by observation, then only by expressing numerically the relations to each other of the quantities that come into consideration.

Such numerical computations of zonal distributions of pressure have been executed especially by Ferrel and Oberbeck. Both these authors, as already remarked, assume a stationary condition (as to temperature, motion and pressure). This would, perhaps, not prevent their computation of the stationary zonal part of the distribution of pressure from being, at least, approximately satisfactory. But Ferrel applies the theorem of the conservation of areas to an individual particle of air, which is not allowable; Oberbeck assumes the surface of the earth to be spherical, and neglects the differences of density in the atmosphere—which are precisely those that bring about the motions of the air.* Hence the problem to determine numerically the stationary zonal distribution of pressure is not yet resolved.

The wave motion, existing even in an ideal atmosphere, will be composed of groups of waves of different lengths and altitudes, as well as of different periods. In reference to the waves moving along the latitude circles, it is easy to see that the length of these must be a simple fraction of the whole circumference of the circle. In this case the pressure and motion correspond to the sum of a stationary zonal distribution, and of that due to the individual waves in their temporary positions. The progressive movement of the individual maxima and minima is therefore dependent upon the progress of the individual waves; the variations in the

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*Oberbeck neglects the changes of temperature that depend upon the time in order to confine himself to steady motions and stationary conditions; he also considers the atmosphere as an incompressible liquid, thereby neglecting vertical differences of density due to pressure, but he does not neglect the horizontal differences of density due to temperature. C. A.

depths of the minima and heights of the maxima, as well as the distribution of pressure in their neighborhood, depend on the differences in phase that the waves have with respect to each other at the places where the minima or maxima occur. Therefore, for example, the minimum deepens when the deepest parts of the different waves approach each other. It is easy to see that the isobars that surround the minima and maxima do not in general have a circular form. This form can only occur when the differences of pressure in the direction from west to east are the same as those that simultaneously occur in the direction from north to south, a case that can only seldom occur, as is confirmed by experience, and which, in consequence of the further progress of the waves, is soon converted into other relations. Hence it is seen that when circular cyclonic and anticyclonic phenomena are made the foundation of our views as to atmospheric motions and other forms of isobars are considered as departures from these normal conditions—this is not accordant with the actual process of nature.

In the atmosphere surrounding the earth, on account of the irregular distribution of water and land, the zonal distribution of atmospheric temperature hitherto considered does not exist, and, consequently, the actual distribution of the pressure and the movement of the air will differ from that corresponding to such a zonal distribution of temperature. It will, however, be proper to analyze the relations actually existing into two components, one corresponding to the mean temporary zonal distribution of temperature and the other depending on any departure from this mean temperature over a greater or smaller region.

The phenomena resulting from this latter departure, even when they extend over large regions, still have a local character and are dependent on the nature of the respective terrestrial surfaces beneath them, and will, therefore, not take part in those motions that are caused by the average zonal temperature conditions but will be added to or superposed upon them. The analysis of the phenomena into these two subdivisions appears so much the more justifiable since the local temperature irregularities diminish with altitude, and the zonal temperature distribution becomes by so much the more important. Under circumstances that are easy to perceive, this summation or superposition of the progressive waves due to the general circulation, and of the pressure and motion due to local irregularities of temperature, can also produce progressive cyclonic and anticyclonic phenomena.

It must not be forgotten that the movements depending on the local irregularities of temperature cannot remain perfectly stationary but must also contain undulatory phenomena, for the pressure and motion cannot in general assume such a status that the motion of the particles of air follows the intersections of the surfaces of equal partial pressure with the level surfaces due to the force of terrestrial gravitation.

But this latter constitutes the general condition by reason of which the motion of the air in the rotating atmosphere may remain the same, that is to say steady, in any finite region and during any finite time. Now this condition is, in general, not fulfilled in the atmosphere (neither by a purely zonal distribution of temperature, nor by the partial motions that result from local departures of temperature from the zonal averages), and, hence winds that are steady as to direction and force cannot persist during any considerable interval of time, but atmospheric billows or even waves of small length must be formed. The preceding deductions lead us to the variations of pressure and the gusts within storms that have lately been demonstrated by observations, even for great altitudes, especially by Vallot on Mont Blanc and by Pernter on Sonnblick.

These latter and the atmospheric waves that become visible in the cloud formations known as "mackerel" sky, are probably the shortest waves that we have occasion to observe. The atmospheric waves of next greater extent are those that play an important role in the formation of thunderstorms, and a remark by Erk (*Met. Zeit.* 1894, p. 271) in reference to the thunderstorm of June 7, 1894, offers an excellent illustration of these. Of about the same order of dimensions are the tornadic storms whose great intensity depends upon the conjunction of several wave troughs. From the point of view here presented, we can also understand how it happens that such phenomena disappear and again reappear at other points in the prolongation of the direction of their earlier paths. This phenomenon can be regarded as a consequence of the interference of systems of waves progressing with different velocities.*

We must, therefore, think of the atmosphere as filled with numerous systems of waves of extremely different sizes and especially, first, the regular waves that correspond to a certain average zonal distribution of temperature and, second, the irregular waves that result from the local

*I must consider this explanation as wholly inapplicable to the tornadoes of the United States. C. A.

deviations from this distribution of temperature and which extend upward to only a slight altitude above the earth's surface.

As another class of progressive changes of atmospheric pressure, there must be mentioned certain sudden changes of pressure whose initial development must be ascribed to the formation of discontinuities in the atmosphere. Although, as before stated, the formation of surfaces of discontinuity in the atmosphere, as imagined by von Helmholtz as an application to a ring of air of the theorem of conservation of areas, does not correspond to the principles of mechanics, still other possibilities can be imagined that would lead to such discontinuities, as, for example, the sudden precipitation of the aqueous vapor contained in any portion of the atmosphere. But such discontinuities will always extend over small regions only. The sudden change of pressure due to these can, indeed, continue and advance like a wave produced by concussion, even when the original cause of the development of the discontinuity itself has disappeared. The phenomenon described in the *Annalen d. Hydrographie*, 1889, p. 242, and which was traced from Sylt to Pola may belong to this class.

Since we, at present, do not possess a reliable mathematical or analytical presentation of the atmospheric pressures and motions, even under the simplest assumptions, therefore, the problem that first occurs is to analyze the atmospheric relations as derived from observations into those more regular features that belong to the general atmospheric circulation and those due to local departures and to investigate each of these by itself.

The more accurate knowledge of the regular phenomena of the general atmospheric circulation will reveal to us the connection of phenomena that are, geographically speaking, widely separated from each other, and will lead us to certain periodicities in the weather. It will thus be seen whether, besides the regular seasonal variations in the influence of the solar heat on the earth, there be still other causes lying outside of our planet that affect the terrestrial atmosphere.

Thus the question will then be answered, to what extent the solar spot period, by a change in radiant heat or in electrical phenomena, affects the processes in our atmosphere. For it may here be stated that within any rotating fluid envelope, whose parts are attracted, according to the general law of the attraction of masses, by a solid of revolution within the fluid, there cannot exist any stationary condition, or steady motion, when any influences, distributed zonally, prevent a rotation with uniform average, angular velocity of the whole envelope, or at least of those parts that are at equal distances from the axis of rotation. If, therefore, such influences exist on the sun, then wave formations must occur on it, and the sun spots, which have already been considered as vortex phenomena, as well as their variability, would thus find an explanation similar to that of the cyclones and anticyclones of the terrestrial atmosphere. It thus becomes quite possible that the regular phenomena of the general atmospheric circulation of the earth may, for purely mechanical reasons, have the same period as those on the sun. In this case there would not need to exist any direct connection between the sun spots and our weather, even though the same period should be established beyond doubt by observation. On the other hand, a further development of the theory may, perhaps, lead to conclusions as to certain mechanical relations, especially as to the velocity of rotation of the sun.

Even the discussion as to the influence of the moon on our weather, which is still by no means settled, will, in this way, be brought to an end, for it must then be possible to satisfactorily show whether the otherwise regular processes of atmospheric circulation are influenced by the changes in the moon's position, especially by its motion in declination.

It is to be hoped that the method here indicated may lead meteorology out of the region of vacillating ideas that now control it into a broader field, and place it among the exact sciences, where everything is reduced to numerical computation, and thus, to an important extent, further its application to daily practice.

DEUTSCHE SEEWARTE, HAMBURG.

SOME POINTS IN THE ELEMENTS OF THE THEORY OF FUNCTIONS.

BY PROFESSOR W. F. OSGOOD.

I. *A New Definition of an Analytic Function.* Cauchy defined $f(z)$ to be an analytic function of z when $f(z)$ is continuous and

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

converges toward one and the same limit when Δz con-