ON A REMARKABLE COVARIANT OF A SYSTEM OF QUANTICS.

BY PROFESSOR H. B. NEWSON.

I wish to call attention in this paper to a remarkable covariant of a system of n quantics in n homogeneous variables. This covariant although not entirely new has not received the attention which it deserves. While the exposition here given is limited to the case of three quantics in three homogeneous variables, it will be seen that the methods are applicable to a system of n quantics in n variables. This covariant is here approached from the geometric point of view and the language used is that of the theory of higher plane curves. I shall attempt only to sketch the work in outline and shall omit all details as unsuitable to the place and object of this paper.

The covariant in question bears much the same relation to the Jacobian of the system of quantics as the Steinerian of a single quantic bears to the Hessian. In order therefore to get the most advantageous starting point for the discussion, we shall begin by restating two fundamental properties of the Jacobian.

Let there be given three curves, U, V and W, whose degrees are respectively m, n, and p. The Jacobian of these three curves, whose equation is obtained by equating to zero the functional determinant of U, V and W, has two well known polar properties as follows:

(1). *The Jacobian of U, V and W is the locus of all points common to the first polars of a point (x', y', z') with respect to U, V and W.*

(2). *The Jacobian of U, V and W is also the locus of the point (x', y', z'), whose polar lines with respect to U, V and W, meet in a point.*

The equation of the Jacobian is obtained in two different ways, each of which leads to one of the properties in question. The first polars of (x', y', z') with respect to U, V and W, are given by the following equations, using Salmon's notation (see Higher Plane Curves, Art. 61).

$$(1) \qquad \begin{aligned} x'U_1+y'U_2+z'U_3&=0,\\ x'V_1+y'V_2+z'V_3&=0,\\ x'W_1+y'W_2+z'W_3&=0. \end{aligned}$$

When these three equations are simultaneous their resultant vanishes, and we have the equation of the Jacobian and the proof of property (1).

The polar lines of (x', y', z') with respect to U, V and W are given by the equations

$$(2) \qquad \begin{aligned} xU_1' + yU_2' + zU_3' &= 0, \\ xV_1' + yV_2' + zY_3' &= 0, \\ xW_1' + yW_2' + zW_3' &= 0. \end{aligned}$$

When these three lines meet in a point the resultant of their equations vanishes, and we have again the equation of the Jacobian and the proof of the second property.

The Jacobian is a curve of order $(m+n+p-3)$.

But instead of confining our attention to the Jacobian we might also consider the locus of the point (x', y', z') whose first polars meet in a point. The equation of this locus is obtained by eliminating x, y and z from equations (1). It is easy to see that we obtain the same equation by eliminating x', y' and z' from equations (2), except that x, y, z are interchanged with x', y' and z'.

I propose to name this locus provisionally the Cremonian of U, V and W; I say provisionally because I do not know by whom the curve was first studied. It is specifically mentioned in Cremona's *Introduzione ad una teoria geometrica delle curve piane*, Art. 98. See also Hagen's *Synopsis der Hoeheren Mathematik*, vol. 2, page 201. In Higher Plane Curves, Art. 401, Salmon mentions this locus which is here called the Cremonian, but says: "We shall confine ourselves to the consideration of the case when the three curves are first polars of a given curve, in which case the Jacobian in the Hessian of that curve, and the other locus now mentioned is its Steinerian." It seems to me that the general case is too important to be dismissed in this way; in fact more important than the special case which is frequently mentioned.

The Cremonian has two important geometric properties corresponding to the two methods by which its equation may be obtained.

(1). *The Cremonian of U, V and W is the locus of the point (x', y', z') whose first polars with respect to U, V, and W have a common point; the locus of these common points is of course the Jacobian.*

(2). *The Cremonian of U, V, and W is also the locus of (x, y, z) the point of intersection of the polar lines of (x', y', z'), with respect to U, V and W; i. e., it is the locus of the point of intersection of the polar lines of the points on the Jacobian.*

If we select some point as C on the Cremonian, its first polars intersect in a point J on the Jacobian; the polar lines of the point J meet in a point on the Cremonian. It is not difficult to see that this last point must be the point C. Thus there are two relations connecting the points C and J, which may well be called corresponding points on the two curves. All the properties of the Jacobian and the Cremonian so far mentioned may be summed up in the following comprehensive theorem :

If the first polars of a point C with respect to three curves U, V and W meet in a point J; then the polar lines of the point J with respect to the same three curves meet in the point C; and vice versa. The locus of all points C is the Cremonian of the three curves, and the locus of all points J is the Jacobian of the three curves.

When we consider the process of eliminating x, y and z from the three equations (1), it is easy to see that the Cremonian is of order $(mn+mp+np-2m-2n-2p+3)$. And since the two curves have a one to one correspondence, we infer that the Cremonian and the Jacobian have the same deficiency.

The lines joining corresponding points on the Jacobian and Cremonian, envelope a curve for which I have no better name than the Hyper-Cayleyan; because it is generated in a manner similar to the method of generating the Cayleyan. It is evident that the Hyper-Cayleyan has a one to one correspondence both with the Cremonian and the Jacobian, and has therefore the same deficiency. The class of this Hyper-Cayleyan is, we know, equal to the sum of the orders of the Jacobian and the Cremonian, *i. e.*, it is given by the number $(mn+mp+np-m-n-p)$.

It is not a difficult task to prepare a table of the Plueckerian characteristics of the three curves, the Jacobian, the Cremonian and the Hyper-Cayleyan, similar to that for the Hessian, Steinerian and the Cayleyan given in Clebsch-Lindemann's *Vorlesungen ueber Geometrie*, vol. I., page 368. This table is also given in Hagen's *Synopsis der Hoeheren Mathematik*, vol. II., page 203.

In the most general case we suppose that the three curves, U, V, W, have no double points, are of different degrees, m, n, p, and have no points or points common to all three. A study of the special cases brings to light a large number of interesting theories. Thus we may consider the cases where one or more of the curves have double points, cusps or multiple points: also the cases where two and then three of the curves have points in common, which may be ordinary points on the curve or multiple points of various orders.

When U, V, W are all of the same degree, we know that the Jacobian of the three curves is the locus of the double points on all the curves of the system $U+kV+lW=0$. When U, V, W are all of the same degree and at the same time are first polars of three points with respect to a curve C of degree $(m+1)$ or n; then the Jacobian of U, V, W becomes the Hessian of C, the Cremonian of U, V, W becomes the Steinerian of C, and the Hyper-Cayleyan of U, V, W becomes the Cayleyan of C.

It is not difficult to extend the conception of the Cremonian here developed to spaces of higher dimensions. In three dimensions we must have given four surfaces, U, V, W, T; the Jacobian and Cremonian are then defined and their equations obtained in a manner analogous to that employed in two dimensions. The lines joining the corresponding points on the Jacobian and Cremonian surfaces form a system of ∞^2 lines in space. I have not examined the properties of this system of lines.

In a paper published in the *Kansas University Quarterly* vol. 3., No. 2, entitled *On the Hessian, Jacobian, Steinerian, etc., in Geometry of One Dimension*, the writer has developed some of the properties of the Cremonian of two binary quantics. In that paper the name Cremonian was not used; but the covariant in question was designated by the symbol $M(VW)$.

So far as my knowledge goes there has been no systematic and detailed study of the properties of the covariants, which are here called the Cremonian and Hyper-Cayleyan. It seems to me that they are worthy of serious investigation, and it is to be hoped that some one will undertake the task.

KANSAS STATE UNIVERSITY,
February 17, 1896.