places. Table VI gives, chiefly to twelve significant figures, the values of $I_n(x)$ for integral values of $n$ from 0 to 11 and for values of $x$ at intervals of 0.2 from 0 to 6. This collection of tables will be highly appreciated by all who have to use Bessel's functions in numerical work. A table similar to table III but giving the roots of the equation $J_0(x)=0$ and the corresponding values of $J_1(x)$ would be a welcome addition in a future edition. Such a table would be useful in the numerous problems involving the development of unity in the interval from zero to $a$ in a series of the form $\sum_{s=0}^{s=\infty} A_s J_0(\lambda_s r)$ where $\lambda_s$ is the $s^{\text{th}}$ root of the equation $J_0(\lambda a)=0$.

A short bibliography which though confessedly incomplete will be found useful, and a drawing of the curves $y=J_0(x)$ and $y=J_1(x)$ close the volume.

Serious misprints seem to be rare. On page 14, however, there is one which deserves mention as it occurs in an important formula. The last term in formula (31) reads $-\sum_1^n \frac{1}{2s}$. The minus sign should be changed to plus.

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## MODERN METHODS OF ANALYTICAL GEOMETRY.

*An Introductory Account of Certain Modern Ideas and Methods in Plane Analytical Geometry.* By CHARLOTTE ANGAS SCOTT, D. Sc., Girton College, Cambridge; Professor of Mathematics in Bryn Mawr College, Pennsylvania. XII+288 pp., octavo. London and New York, Macmillan and Co. $2.50.

A minor excellence of this book, for which many readers will feel truly grateful, is the fact that it is written in the English of English speaking and writing people. Private abbreviations, cabalistic marks necessitating constant reference to an elusive "list of signs," Teutonisms, and Greek logomachy in the way of "tetrastigms," etc., are agreeably absent. The parvenu "join" is flattered with recognition, but this term is now in such general use that to protest further against it will be of little avail. It is in a measure a consolation that no one is as yet permitted to "enthuse" over this acquisition to the language. "Joining line" is a letter and a syllable shorter than "intersection," the correlative term. If we say "join" shall we not say "meet" or "cross"? "Double point" also might better be written in full. But these are trivial affairs.

The book modestly designated by the author as an "introduction," and certainly well adapted to the needs of the learner, is in fact a compact scholarly work on the more accessible principles and methods of modern analytical geometry. It exhibits to a marked degree that genial breadth of treatment and conciseness which are associated only with mature scholarship and extensive and accurate information. Everything fits naturally in place, nothing is cramped or forced out of its natural relations. In the limited space of less than three hundred pages the great ideas of geometry, including the doctrine of coördinates, projection and dualism, correspondence, and the absolute, are clearly and skillfully developed, and illustrated by application to an ample list of well-chosen examples, a sufficient number of the latter being reserved as an exercise for the reader. We notice with pleasure that the book is entirely free from the criticism, so often and so properly urged against many English mathematical works, that their authors exhibit no adequate conception of the distinction between a general principle and a particular example. Miss Scott is not only keenly alive to this distinction; she is also perfectly aware of the importance of choosing methods adapted to the subject, which is here primarily geometry, not algebra.

Historical and other references are only rarely given, on the assigned ground of the introductory character of the book. Some years ago a prominent French mathematician offered a different excuse for the same omission, viz., that he could not by giving references enhance the fame of the illustrious geometers who had preceded him. We apprehend that neither plea will be good in court. The beginner as well as the scholar should have every facility for acquainting himself with the history and classical works of his subject. And, as we have pointed out, in the present case "introductory" is not by any means synonymous with "elementary."

The author's mode of presentation, usually good, becomes occasionally very felicitous. At the beginning the justification of the line as an element, co-important with the point, of the plane is well stated. The infinite and imaginary elements—line at infinity, circular points, isotropic lines, etc., are skillfully managed. The chapter on the Absolute is an admirably lucid and elementary exposition. The author aims throughout at generality and elasticity of treatment. Her success in this respect is very pronounced, and is in fact one of the most acceptable features of the book.

The book covers of course to a large extent the same field as the treatises of Salmon to which the author very appreciatively refers, and to which the reader is directed for the study of many aspects of the subject. No mention is however made of the German editions of Salmon's works, which are in some cases greatly superior to the originals. There is ample room for new and modern books in various departments of geometry, and we will trust that Miss Scott's very successful present venture into this field will not be her last.

Pure geometry travels flying light, without baggage, and lives on the country. The analytical arm of the service carries a heavy, but nowadays perfectly mobile, siege train with many supply depots. And first of all the accoutrements in the way of coördinates must be well appointed. A coördinate is any quantity that helps to determine an element, and an element is any geometric form. The simplest elements of plane geometry are the point and the line. On each of these a geometry can be constructed; and these two geometries are identical in substance and differ only in guise. How many coördinates a given element may have is a matter of taste. Cartesian geometry, content with bare necessaries, considers only the point as element and assigns to this two coördinates. With three coördinates for the point, (or line) however, the algebraic evolutions become much more symmetric, inasmuch as all equations are then homogeneous. At the the same time, and this is the real advantage, the whole coördinate system becomes projective, and accompanies the corps of figures to which it is attached in their passage from one plane to another. The three coördinates of point or line are referred to the sides and vertices of a triangle of reference, most conveniently the same triangle in both cases. In the Cartesian geometry, as viewed in this aspect, the coördinate axes are two sides of this triangle of reference, while the third side is the conventional "line at infinity." The latter is a line, geometrically because its projection is a line, and analytically because the equations of all lines infinitely removed tend to the same form. Since it is true that every quadrangle can be projected into every other, it might seem desirable to employ four coördinates instead of three. In fact, however, one vertex of the quadrangle is already in demand as an adjustment—unit point, *vide infra*—for the coördinate system attached to the other three.

If a plane figure is projected into a new plane, the triangle of reference of the original plane becomes a triangle which may be used as a triangle of reference in the new plane. This being the case the equations of the two figures are identical. If, however, it is desired to change the triangle of reference in a plane, this is effected by an integral homogeneous linear transformation of coördinates with non-vanishing discriminant. The equations of transformation admit of a second interpretation, viz.: the triangle of reference may be regarded as fixed while the plane undergoes distortion. In the latter case any figure is distorted into a projective figure. The discussion of the relation between projection and linear transformation is elaborated in a special chapter.

Since two coördinates are sufficient to determine a point or line, the three coördinates actually employed in each case are connected by an identity. The latter is linear for point coördinates, but, curiously enough, for line coördinates it is quadratic. In this violation of the principle of dualism we have an epitome of the one-sided character of our system of metric, leading to no little inconvenience until it finally comes to be recognized as a one-sided limiting form of a more general conception in which the principle of dualism retains its validity in all cases, and at the same time all properties are projective. Properties are then still divided into two classes according as they belong to a figure in itself or express its relations to the absolute conic. Until this step is taken, however, it follows from the identities above mentioned that we have only one singular (infinitely distant) line, but two infinitely distant points. Closely connected with this is the fact that there is in our ordinary geometry no natural unit of length, while there is a natural unit angle, viz., the complete circumference.

A coördinate, as an algebraic quantity, may, and not only may but inevitably will, assume imaginary as well as real values, and its imaginary values are infinitely more numerous than its real ones. The common conception of geometry, therefore, requires to be idealized by the introduction of imaginary *geometric* elements. Writers on analytical geometry do not ordinarily concern themselves with the development of this idea and the author follows the usual practice. Any extensive treatment of the imaginary may well be left to special treatises, but some of the more elementary notions might properly have been mentioned here. For example, what is called the equation of a plane curve really represents a surface of which the curve is only a section; two such surfaces intersect in a finite number of distinct points; isolated points of a curve are happily explained in this connection. The nature of the continuity of the surface is readily ascertained, and the theories of deficiency and of birational transformation are intimately related to this subject. But this is already going rather far,—beyond the limit that the author has set for the work.

The great unifying and fertilizing principle of geometry is the doctrine of projection, or more generally of correspondence. The great advances which analytic geometry has made during the present century are due entirely to the adoption of projective ideas and methods. With this has gone a recasting of the entire notation of the science into projective form by the aid of anharmonic ratio. And thus finally all essential distinction between synthetic and analytic geometry has long since disappeared. Still our modern works on analytic geometry are very distinctively analytic, and writers on pure geometry, almost without exception, avoid carefully any explicit mention of coördinates. But it is an easy prediction that this fictitious boundary line will not long be observed. The author of the *Introduction* gives liberal recognition to the methods of pure geometry; but we could wish that she had gone still further in this direction. For example an alternative, purely geometrical, proof of the characteristic property of the complete quadrangle and quadrilateral (p. 43) might well have been given. We miss, too, the many easy geometric deductions from the notion of projective ranges and pencils, including ranges on a conic and tangents of a conic, especially in the latter case the geometric proof of Pascal's and Brianchon's theorems, which in point of conciseness and exact adaptation to the subject are simply perfection. One matter in particular is so important and so pertinent to *analytic* geometry, that we cannot but regard its omission as unfortunate. It is well known that the anharmonic ratio is the simplest *invariant*. It is also true that the anharmonic ratio is a *universal coördinate*, all other coördinates being merely disguised forms of anharmonic ratios. In fact, the anharmonic ratio is referred to three elements at which it takes the values $\infty$, 1, 0, respectively, and if any coördinate $x$ of an element $P$ takes these three values at the elements, $A$, $B$, $C$, then $x = (ABCP)$. For example, the Cartesian coördinate $x$ of a point on a line is an anharmonic ratio referred to the point at infinity $A$, the origin $C$ and the unit point $B$, $CB$

being the arbitrary unit of length. Again the ratios of the trimetric coördinates of a point, $P$, referred to a triangle, $ABC$ are

$$\frac{x_1}{x_2}=(C-ADBP), \frac{x_2}{x_3}=(A-BDCP), \frac{x_3}{x_1}=(B-CDAP),$$

when $D$ is the unit point, *i. e.*, the point for which $x_1 : x_2 : x_3 = 1$. On this principle depends the possibility of making analytic geometry thoroughly projective, the coördinates themselves admitting of projective definition.

Geometric properties are primarily classified as projective and non-projective. Beside these the terms "descriptive" and "metrical" are in common use and are given prominence by the author. According to Cayley (Salmon's Higher Plane Curves, Chapter I) properties are descriptive if they have no reference to the line at infinity or the circular points; otherwise they are metrical. In this use "descriptive" is identical with "projective" and "metrical" with "non-projective." The author's definition of "descriptive" is the analytical equivalent of Cayley's, but this definition is not strictly adhered to inasmuch as properties expressed in terms of anharmonic ratios are afterward described as metrical (p. 147). It seems to us that the term "descriptive" which is commonly employed in conjunction with "projective" and non-projective, as a species of *tertium quid*, might well be reserved to designate the technical art to which it was originally applied. "Metrical" has an intrinsic value, and in the sense of "involving measurement" could hardly be dispensed with. That its use is however liable to be attended with some ambiguity is evidenced by the statement (p. 147). "Theorems stated with reference to cross-ratio are metric theorems in projective form," which to our mind rather gains in force if the words "metric" and "projective" are interchanged.

In pure geometry projection is a matter complete in itself, but in analytic geometry it has its counterpart in the theory of linear transformation. The latter subject may be fairly said to lie at the very center of modern mathematics. From the formal standpoint projective geometry is merely an interpretation in space of linear transformation, an interpretation so interesting and valuable that when, with an increased number of variables in the equations, ordinary space no longer suffices for the purpose, no hesitation is felt in positing a hyper-space with any number of dimensions. In fact the study of geometry has, for many, its chief value in the fact that it furnishes an abundance of simple and current illustrations of principles, whose abstract foundation belongs to other more difficult mathematical sciences. For example, as the author very appropriately points out, all the collineations of a given space form a *group*. This group has a variety of *subgroups*, and it has also its peculiar *invariants* and *covariants*.

The chapter on Projection and Linear Transformation is followed by one on Correspondence, in which the general notion of correspondence is explained and applied to the general (1, 1) quadric correspondence, to reciprocation, and briefly to the birational transformation of a curve into itself. Following this is the chapter on the Absolute, and finally a very short but well-written introduction to Invariants and Covariants. Descriptive and metrical properties, and the theory of anharmonic ratio and involutions are treated at length, but compactly in the earlier chapters. There are also separate chapters on Dualism and on Unicursal Curves.

The book will be found to gain continually in interest on repeated reading, and this is due especially to its eminently suggestive character. The author succeeds admirably in conveying to the reader, if he be fit, an accurate knowledge and command of general principles. We know of no introductory work which is better adapted in this particular for the use of those who desire not merely to learn but also to master geometry.

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COLUMBIA UNIVERSITY,
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