branches as its essential part? Botanists tell us that the question is badly framed, and that the life of the organism depends on the mutual action of its different parts.

DARBOUX'S MEMOIR ON CYCLIQUES.

Sur une Classe Remarquable de Courbes et de Surfaces Algébriques et sur la théorie des Imaginaires, par GASTON DARBOUX, Doyen de la Faculté des Sciences de Paris. Second tirage. Paris, A. HERMANN, 1896.

IT is a pity that M. Darboux did not make some additions to this work since its first publication in 1873. He is so full of ideas in a number of mathematical directions that it is a cause for regret that he has not devoted some more time to a subject which offered him once such a fruitful field for original investigation. However, those who know the work will be glad to make acquaintance with it again, and others who are tired of conic sections and quadrics may be gratified by finding in it a somewhat similar but still novel field of investigation.

Darboux calls the curves and surfaces which he treats of cycliques and cyclides, respectively. The latter word has been generally adopted in English as a name for the surfaces, but the former has been replaced by bicircular quartics and sphero-quartics, which two designations conveniently distinguish between the plane and spherical curves. The name cyclide had already been used for a particular form of the surface by Dupin, and soon came to be adopted generally in its present sense, while the different names of the curves arose from their being studied independently in Great Britain and France. At one time, from about 1865 until early in the seventies, these curves and surfaces were studied enthusiastically in France by Darboux, Laguerre, de la Gournerie, Moutard, Mannheim and others, while they received attention in England at the hands of Crofton and Clifford, and in Ireland secured very full and adequate treatment from Casey. All these mathematicians worked at the same time and nearly entirely independently of each other. Thus Darboux's book contains much that has been more fully worked out elsewhere, and, besides, of course, it has no references to many striking results that have been arrived at in this branch of geometry, as, for instance, Casey's ingenious method of rectification of the bicircular quartics and sphero-quartics. But these wants are made up for by Darboux's methods of investigation and by his curious paths of inquiry, from which new views and sidelights on the subject are continually obtained. It is this, as well as the incidental solutions of many problems not directly connected with the subject, that gives a charm to the work, and only those who have a preference for formal and logically ordered treatises will be displeased with its discursive character.

M. Darboux begins by considering the transformation of inversion, and in addition to the ordinary properties of this process shows that foci and focal curves, when inverted, possess the same relation to the inverted curve. Then he shows that every *cyclique* is transformed into a similar curve by inversion. Thus for an origin outside of its plane, the bicircular quartic, or plane quartic with the two circular points at infinity as nodes, is transformed into a sphero-quartic, or curve of intersection of a sphere and a quadric. In this way he is enabled to select the most suitable curves for his first investigation. Choosing the sphero-quartic he makes use of the known result that four cones of the second order can be described passing through the curve. Since the tangent planes of the cones intersect the sphere in circles having double contact with the curve, he thus deduces the four-fold generation of the sphero-quartic as the envelope of small circles orthogonal to a given circle J and having their centres on a sphero-conic F. Then by geometrical considerations the four circles such as J are shown to be mutually orthogonal and the four sphero-conics such as H to be confocal, and further the focal curves are shown to lie on four mutually orthogonal spheres having their centres at the vertices of the cones. In this way M. Darboux establishes these and many other properties of the curves by means of previously known results. M. Darboux obtains a number of properties of the bicircular quartics, sphero-quartics and allied curves of higher orders by the use of imaginaries. In particular we may mention the differential equation of the system of confocal bicircular quartics, viz:

$$\frac{du}{\sqrt{f(u)}} \pm \frac{dv}{\sqrt{f_1(v)}.} = 0.$$

where

$$f(u) = A(u-a_1)(u-a_2)\,(u-a_3)(u-a_4)$$
$$f_1(v) = B(v-b_1)\,(v-b_2)\,(v-b_3)\,(v-b_4),$$
$$u = x+iy, \quad v = x-iy,$$

and the anharmonic ratios of

$$f(u)=0, \quad f_1(v)=0$$

are equal.

This equation gives the complete theory of the arrangement of the 16 foci of the curve, for by the definition of a focus these points are found from $du=0$, $dv=0$. Thus we get $f(u)=0$, $f_1(v)=0$, which determine 16 points; and these from the equality of the anharmonic ratios must lie by fours on four circles.

By the use of imaginaries M. Darboux also obtains properties of curves comprised in the equation

$$R_1 R_2 R_3 \cdots = k\, r_1 r_2 r_3 \cdots ,$$

where $R_1, R_2, \cdots, r_1, r_2 \cdots$, are the distances from fixed points, and he shows that such curves can be written in this form in an infinite number of ways. Similar results are obtained for the sphere, and theorems concerning closed polygons formed by the imaginary generators of the sphere are proved. He thus gives a new demonstration of Poncelet's theorems for polygons of an even number of sides and arrives at an important theorem concerning polyhedra formed by generators of a quadric, viz: that if they form a polygon having its vertices on the curve of intersection with another quadric, then there exist an infinite number of polygons formed in the same way. M. Darboux observes that this result was given by Moutard in the *Nouvelles Annales de Mathématiques* in 1864. I think, however, that Steiner and Hart were acquainted with it previously. In 1846 Steiner gave in Crelle's Journal a theorem concerning polygons inscribed in a binodal quartic. This is to the effect that, if a certain condition is satisfied, an infinite number of polygons can be described, such that one set of alternate sides all pass through one node, while the other set of alternate sides pass through the other node. Now let the nodes be the circular points, then if we invert from a point outside the plane, the lines through the nodes become the imaginary generators of a sphere, and the curve becomes a sphero-quartic. Thus we see the identity of Steiner's theorem with that just mentioned. I may notice that Professor A. R. Forsyth, who has treated this problem from an analytical point of view, makes no mention of those who had gone before him. (*Proceedings of the London Mathematical Society*, vol. 14, p. 35.)

M. Darboux further notices that the same property exists when we substitute for the circumscribing quadric quadrics inscribed in the developable formed by the inscribed quadric and one of the circumscribing quadrics, or which is the same thing, if we suppose the vertices to lie on confocal quadrics; and he observes that this proposition seems new. Hart, however, was previously aware of the fact that there could be an infinite number of geodesic polygons circumscribed about a line of curvature of a quadric so that these vertices lie on other lines of curvature. Now, if we suppose the quadric passing through the inscribed line of curvature to coincide with the given one, the geodesics become simply the generators. Thus Darboux's result is comprised in Hart's theorem. (*Cambridge and Dublin Mathematical Journal*, vol. 4, p. 192.)

Proceeding to consider the surface M. Darboux makes use of analysis. With the help of the inscribed quadrics, viz.: quadrics such as V when the equation of the surface takes the form $S^2 = V$, where S is a sphere, he deduces the generation of the cyclide as the envelope of a sphere orthogonal to a given sphere J and having its centre on a quadric F. Further he shows that this generation is fivefold, the five spheres such as J being mutually orthogonal, and the five quadrics such as F being confocal, while the five sphero-quartics FJ are the focal curves. This mode of treatment, it may be observed, has been adopted by Dr. Salmon in his geometry of three dimensions. The remarkable form assumed by the equation of confocal cyclides in terms of the five spheres is worth noticing. If X, Y, Z, U, V represent the squares of the tangents drawn from a point to the spheres divided by the radius of each sphere, then $X^2 + Y^2 + Z^2 + U^2 + V^2 = 0$ identically; and the equations

$$\frac{X^2}{\lambda - a_1} + \frac{Y^2}{\lambda - a_2} + \frac{Z^2}{\lambda - a_3} + \frac{U^2}{\lambda - a_4} + \frac{V^2}{\lambda - a_5} = 0$$

represent confocal cyclides. The remarkable fact that spheres such as $l_i X + m_i Y + n_i Z + p_i U + q_i V = 0$, cut orthogonally if $\Sigma l_1 l_2 = 0$, shows at once that the three confocal cyclides drawn through a point cut orthogonally. There is in these equations an analogy to linear expressions passing through a point and cones, if we may call them so, in four dimensions. In fact just as bicircular quartics are the inverses of sphero-conics, so cyclides may be regarded as the inverses of the sphero-quadrics of four dimensions. This system of coördinates enables us to express the coördinates of a point on a cyclide in terms of two parameters. For solving for X in terms of the parameters of three surfaces of the system we get

$$X^2=\frac{(a_1-\lambda_1)\,(a-\lambda_2)\,(a_1-\lambda_3)}{f'(a_1)},$$

where

$$f(\lambda)=(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)(\lambda-a_4)(\lambda-a_5),$$

and similar values for Y, Z, U, V. Hence if we suppose λ_3 to be a constant for a given surface we obtain, from

$$X=\frac{1}{r}(x^2+y^2+z^2-2\,\alpha\,x+2\,\beta\,y-2\,\gamma^2+\delta)$$

and similar equations, x, y, z expressed linearly in terms of the five radicals $\sqrt{\{(a_i-\lambda_1)\,(a_i-\lambda_2)\}}$. Thus by the theory of hyperelliptic integrals if we write

$$\frac{d\lambda_1}{\sqrt{f(\lambda_1)}}+\frac{d\lambda_2}{\sqrt{f(\lambda_2)}}=d\,u_1,\ \frac{\lambda_1\,d\lambda_1}{\sqrt{f(\lambda_1)}}+\frac{\lambda_2\,d\lambda_2}{\sqrt{f(\lambda_2)}}=d\,u_2,$$

we have the coördinates expressed linearly in terms of the inverse hyperelliptic functions of u, v, called $al(u_1, u_2)_1$, $al(u_1, u_2)_2$, etc., by Weierstrass, just as the coördinates of a point on a quadriquadric curve can be expressed linearly in terms of sn$u$, cn$u$, dn$u$.

In the notes and additions, which comprise a considerable portion of the work, M. Darboux gives a number of miscellaneous theorems concerning curves and surfaces. In particular he studies the system of coördinates obtained by taking as variables the parameters of the tangents which can be drawn from a point to a given conic. Thus if the conic is $\beta^2-4\,\alpha\,\gamma=0$, a tangent may be written

$$\alpha\,\rho^2+\beta\,\rho+\gamma=0,$$

so that if ρ_1, ρ_2 are the parameters of the tangents drawn from α, β, γ, we have

$$\alpha=-\frac{\beta}{\rho_1+\rho_2}=\frac{\gamma}{\rho_1\rho_2}\,.$$

By means of these coördinates he obtains several interesting results regarding polygons. For instance, if a curve of the n^{th} degree pass through the intersection of the n tangents, $A_1, A_2 \cdots A_n$ with the n tangents $B_1, B_2, \cdots B_n$, its equation is

$$A_1\,A_2 \cdots A_n=k\,B_1\,B_2 \cdots B_n.$$

Since

$$A_i=\alpha a_i^2+\beta a_i+\gamma=\alpha\,(a_i-\rho_1)(a_i-\rho_2) \text{ and } B_i=\alpha\,(b_i-\rho_1)(b_i-\rho_2),$$

this equation takes the form

$$\frac{\varphi\,(\rho_1)\,\varphi\,(\rho_2)}{\psi\,(\rho_1)\,\psi\,(\rho_2)}=k.$$

But this form can be assumed in a singly infinite number of other ways, as we see by writing it

$$\frac{\{\psi(\rho_1)+\lambda\varphi(\rho_1)\}}{\{k\lambda\psi(\rho_1)+\varphi(\rho_1)\}}\frac{\{\psi(\rho_2)+\lambda\psi(\rho_2)\}}{\{k\lambda\psi(\rho_2)+\varphi(\rho_2)\}}=\frac{1}{k}$$

Hence if a curve pass through the n^2 points where n tangents to a conic meet other n tangents, it must pass through an infinity of such systems of n^2 points. Further, if a curve of the n^{th} degree pass through all the points of intersection of $n+1$ tangents to the conic, it may be written

$$\frac{a_1}{A_1}+\frac{a_2}{A_2}+\cdots\frac{a_{n+1}}{A_{n+1}}=0$$

which becomes

$$\Sigma\frac{a_i}{(a_i-\rho_1)\ (a_i-\rho_2)}=0$$

or, if we multiply by $\rho_1-\rho_2$

$$\Sigma\frac{a_i}{a_i-\rho_1}=\Sigma\frac{a_i}{a_i-\rho_2}$$

that is

$$\frac{f(\rho_1)}{\varphi(\rho_1)}=\frac{f(\rho_2)}{\varphi(\rho_2)}$$

which may also be written

$$\frac{f(\rho_1)}{\varphi(\rho_1)+kf(\rho_1)}=\frac{f(\rho_2)}{\varphi(\rho_2)+kf(\rho_2)},$$

which is of the same form as the preceding with indeterminate k. Hence the curve will pass through all the vertices of an infinite number of polygons of $n+1$ sides circumscribed to the conic.

In this way M. Darboux proves the theorem originally obtained by Lüroth (*Math. Annalen* vol. 1.) viz: that if a quartic curve pass through all the points of intersection of five lines it cannot be the general curve of that degree, but must satisfy an invariant relation. M. Darboux might have noticed also that a general quartic curve cannot pass through the 16 points where four tangents of a conic meet four other tangents.* The invariant condition satisfied in this case is a problem suggested here for solution. Again by the use of these coördinates M. Darboux gives a very simple proof of Poncelet's theorem concerning polygons.

* It may be observed that this result can also be put in the form : the twelve vertices and eight points of contact of two quadrilaterals circumscribed about a conic lie on a quartic curve which is not a general one.

It may be noticed that W. K. Clifford arrived at these results in precisely the same way (*Proceedings of the London Mathematical Society*, vol. 7, p. 29), and when he found that he had been anticipated by Darboux, expressed his opinion that this, viz: the work of Darboux that we are now considering, is a book which it is almost inexcusable in a geometer not to have read, marked, learned and inwardly digested.

There are many other interesting investigations in the book, especially a study at considerable length of the intersection of a sphere with a cyclide; an extension also of Ivory's theorem to confocal cyclides is worthy of notice. If A, B, C are three points on the cyclide and A', B', C' are three corresponding points on a confocal cyclide, M. Darboux shows that the relation

$$AB'.BC'.CB'=BA'.CB'.BC'$$

connects the distances between the points.

R. A. Roberts.

BESSEL FUNCTIONS.

A Treatise on Bessel Functions and their Applications to Physics. By Andrew Gray and G. B. Mathews. Macmillan & Co. 1895. 8vo, x+292 pp.

The transcendental functions to which Bessel's name has been attached are not only of the highest importance in mathematical physics, second perhaps only to the trigonometric and exponential functions, they are also of great interest to the student of pure mathematics both from the formal side and from the point of view of the theory of functions. There has, however, up to this time been no connected treatment of these functions in the English language, with the exception of the utterly inadequate treatment contained in the last sixty-five pages of Todhunter's book, *The Functions of Laplace, Lamé and Bessel*, published twenty years ago. The German monographs by C. Neumann and Lommel make no attempt to cover more than small portions of the subject, and the same is true to an even greater extent of the sections devoted to Bessel's functions in Heine's *Kugelfunctionen*, Basset's *Hydrodynamics*, Rayleigh's *Sound* and elsewhere. Messrs. Gray and Mathews have therefore filled a real gap in mathematical literature.

The authors make it clear in their preface that their own