while the symbols S_n are determined by the harmonic elements of the initial distribution of velocity and condensation.*

9. *Free vibrations between two concentric spherical surfaces.* Since the radial velocity at the surface $r = r_1$ is zero, then

$$F_m(kr_1)\Big[S_n \sin kat + S'_n \cos kat\Big] + F_{-m}(kr_1)\Big[S''_n \sin kat + S'''_n \cos kat\Big] = 0;$$

and there is a similar equation involving r_2.

These must be satisfied for all values of θ, φ, t,

$$\therefore S_n F_m(kr_1) = -S''_n F_{-m}(kr_1);\; S'_n F_m(kr_1) = -S'''_n F_{-m}(kr_1),$$

with two similar equations in r_2,

$$\therefore \frac{S''_n}{S_n} = \frac{S'''_n}{S'_n} = -\frac{F_{-m}(kr_1)}{F_m(kr_1)} = -\frac{F_{-m}(kr_2)}{F_m(kr_2)} = \rho, \text{ say.} \quad (17)$$

The possible values of k, and of the wave length $2\pi/k$, are to be found from the third of these equalities ;† and then S''_n, S'''_n are known multiples of S_n, S'_n. Thus (3) takes the form

$$r^{1/2}\psi_n = [J_m(kr) + \rho J_{-m}(kr)]\;(S_n \sin kat + S'_n \cos kat), \quad (18)$$

an equation which, extended to the whole of space, gives a series of nodal spherical surfaces, of which $r = r_1$, and $r = r_2$ are a pair. At such surfaces the superposed divergent and convergent waves interfere.

ADDITIONAL NOTE ON DIVERGENT SERIES.

BY PROFESSOR A. S. CHESSIN.

In a previous note (pp. 72–75) it has been shown that every divergent series oscillating between finite limits can by a proper arrangement of its terms be made convergent. We will now extend those results to the case when one or both limits between which the series oscillates are infinite. To this end it suffices to consider, together with *regular* se-

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* The work is exemplified for the case $n = 1$, *Theory of Sound*, pp. 236, 237.

† *Annals of Mathematics*, vol. 9, No. 1, pp. 29, 30.

quences of numbers, such sequences as tend to infinity *in a determinate way.* The number N_i then may be $+\infty$ or $-\infty$ and the infinite series

$$a_{1,i} + a_{2,i} + \cdots + a_{n,i} + \cdots$$

of Theorem I. (p. 73) then tends to infinity *in a determinate way.* This series is *unconditionally divergent.* A similar change in the considerations of Theorem II. (p. 74) shows that the series

$$(u_1 + u_2 + \cdots + u_{\mu_{1,i}}) + (u_{\mu_{1,i+1}} + \cdots + u_{\mu_{k,i}}) + \cdots$$

tends to infinity in a determinate way, but it must be remembered that here each expression in parenthesis is to be considered as a *single* term, and that this determinateness would be lost at once if the brackets were dropped.

Example: $1-2+3-4+5-\cdots$

$$g=2; \quad N_1 = +\infty = 1+(-2+3)+(-4+5)+\cdots$$

$$N_2 = -\infty = (1-2)+(3-4)+\cdots$$

We will say that an infinite series tends to infinity only if it does so in a determinate way. It was also in this sense that the expression was used in Theorem III. (p. 75). We may now add

Theorem VI. Every oscillating series can by a proper arrangement of its terms be made convergent or tending to infinity.

To conclude, a remark must be made with regard to the different rearrangements of the terms of an infinite series. On p. 75 it was mentioned that by a proper arrangement, both commutative and associative, of the terms of a conditionally divergent series, this series can be made to converge to any arbitrarily assigned number. It is important to take into consideration whether the associative change precedes the commutative or whether the inverse takes place. Of course the remark just mentioned applies only to series such that $\lim_{m=\infty} (u_m) = 0$, if u_m be the general term of the series, this condition being implicitly assumed in the extension of Riemann's proposition to conditionally divergent series.

Consider now a divergent series as given in Theorem II. (p. 74) and let again

$$N_i = (u_1 + \cdots + u_{\mu_{1,i}}) + (u_{\mu_{1,i+1}} + \cdots + u_{\mu_{k,i}}) + \cdots$$

Here each expression in parenthesis is to be regarded as a *single* term, and the series may be either absolutely convergent or only semi-convergent. It seems, therefore, that we were not justified in saying (Theorem V.) that every conditionally divergent series can by a proper arrangement of its terms be made *semi-convergent;* that we should have used the word *convergent* instead. But the character of semi-convergence appears at once if we reverse the order of our changes in the original series, *i. e.*, if we first introduce a commutative and then proper associative changes. An oscillating series having necessarily an infinite number of positive and negative terms, the numbers N_i of Theorem II. will in general change their values if the order of terms in the given oscillating series be changed. Indeed, provided $\lim_{m=\infty} (u_m)=0$, it is possible by a proper commutative arrangement to make not only the N_i but also their number g quite arbitrary. This is why we have used the word semi-convergent, and not convergent, in Theorem V.

As an example, let us take again the series (2) of the first note :

$$\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{2}{3} - \tfrac{3}{4} + \tfrac{3}{4} - \cdots$$

which may be written as follows:

$$\sum_{1}^{\infty}{}_{n}\left\{\frac{2n-1}{2n}-\frac{2n}{2n+1}+\frac{2n}{2n+1}-\frac{2n+1}{2n+2}\right\}$$

If we now introduce the following communicative change

$$\sum_{1}^{\infty}{}_{n}\left\{\frac{2n-1}{2n}-\frac{2n}{2n+1}-\frac{2n+1}{2n+2}+\frac{2n}{2n+1}\right\}$$

we obtain $g = 3$ and

$$N_1 = \tfrac{1}{2} = \tfrac{1}{2} - \sum_{1}^{\infty}{}_{n}\left\{\frac{2n}{2n+1}+\frac{2n+1}{2n+2}-\frac{2n}{2n+1}-\frac{2n+1}{2n+2}\right\}$$

$$N_2 = -\tfrac{1}{2} = \sum_{1}^{\infty}{}_{n}\left\{\frac{2n-1}{2n}-\frac{2n}{2n+1}-\frac{2n+1}{2n+2}+\frac{2n}{2n+1}\right\}$$

$$N_3 = -\tfrac{3}{2} = \tfrac{1}{2}-\tfrac{2}{3}-\tfrac{3}{4}+\sum_{1}^{\infty}{}_{n}\left\{\frac{2n}{2n+1}+\frac{2n+1}{2n+2}-\frac{2n+2}{2n+3}-\frac{2n+3}{2n+4}\right\}$$

Thus the particular change here introduced has increased the number g by unity. It is easy to apply a number of other modifications by following the same process.