the minute finish and careful proportion of parts that we feel justified in expecting from him. And yet revision and consolidation might have seriously interfered with the graphic simplicity of these chapters, and left them less adapted to their special purpose. Any English-speaking association with aims similar to those of the German association for which the pamphlet was prepared would do a service in publishing a thoroughly good translation of this inspiring work and circulating it as widely as possible.

CHARLOTTE ANGAS SCOTT.

BRYN MAWR COLLEGE,
January 15, 1896.

PAINLEVÉ'S LECTURES ON DYNAMICS.

I. *Leçons sur l'intégration des equations différentielles de la Mécanique et Applications.* Par P. PAINLEVÉ. pp. 291; 4to (lithographed).

II. *Leçons sur le Frottement.* Par P. PAINLEVÉ. pp. VIII.+ 111; 4to (lithographed). Paris, A. Hermann, 1895.

The publication of Mr. Painlevé's lectures on the integration of the differential equations of dynamics will be welcomed by everyone interested in the progress of theoretical mechanics. It is a long time since Jacobi's *Vorlesungen über Dynamik* appeared, and a strong need was felt of a systematic work which would contain the more recent researches in this important branch of mathematical science. Mr. Painlevé has admirably supplied this need. The lectures are not intended for beginners in theoretical mechanics, but rather as a supplementary course for those already familiar with its elements.

The first three lectures contain the fundamental definitions and principles of dynamics, the propositions relating to the first integrals of the motion of rigid systems and the theory of the motion of a solid body, the whole followed by a number of examples. Despite the brevity of this exposition it is clear and instructive, owing to several interesting remarks.

The fourth lecture deals with the general equations of the motion of systems. Lagrange's equations with the multipliers λ (also called Lagrange's equations of the first form) are derived from the consideration of the virtual work, which leads the author to a classification of systems into those with friction and without. In the next lecture this distinction is further discussed and a proof is given of Gauss's theorem that the constraint of a system is minimum when it is one without friction. Then the motion of systems with friction is studied and applied to the motion of a point on a surface. However the general theory of such systems constitutes the subject of a more detailed study in another of Mr. Painlevé's courses to which we will return later.

In the next two lectures, after establishing d'Alembert's equation and the principle of virtual velocities, the author passes to Lagrange's equations (of the second form), concluding with an enumeration of the principal systems without friction. All following lectures are restricted to such systems only.

Lectures VII.–IX. deal with the applications of Lagrange's equations, the cases being discussed separately when the equations of constraint do or do not contain the time explicitly. The author further proves Liouville's theorem relating to a large class of systems the motion of which can be found by quadratures. Several applications of this theorem are given. The classic problem of the motion of a point attracted by two fixed points and a number of other well chosen examples are treated in the course of these three lectures, the solutions being given with great simplicity and elegance.

Bour's differential equations of relative motion in the second Lagrangian form with examples taken from Gilbert's interesting memoir, on the applications of Lagrange's equations to the problem of relative motion, are the subject of the tenth lecture.

The next lecture begins with Lejeune-Dirichlet's theorem about the stability of the equilibrium of a system after which the author applies Lagrange's equations to the study of small oscillations of systems.

The remaining six lectures (XII.–XVII.) deal with the integration of the differential equations of dynamics principally after the works of Hamilton and Jacobi. The subjects of lectures XII. and XIII. are the canonic equations and Jacobi's theory of the last multiplier. Bour's canonic equations for relative motion are indicated but not discussed. The theory of the last multiplier is first applied to the motion of rigid systems in general. The application of the same theory to the canonic equations constitutes the subject of the fourteenth lecture, which is illustrated by a number of examples.

The next lecture contains Hamilton-Jacobi's method of integrating the differential equations of dynamics and the several simplifications which may occur in practice are discussed. A new demonstration is given of Liouville's theorem together with a generalization of the same due to P. Stäckel.

The sixteenth lecture presents the greatest novelty and interest. It deals with the more modern researches, the impulse to which was given first by Bertrand's principle of similitude in mechanics and quite recently by the remarkable paper of Stäckel (Crelle 1891) on the analytic equivalence of dynamic problems. The reader will find in this lecture a good deal of Mr. Painlevé's own researches chiefly contained in his important memoir on the transformation of the equations of Dynamics (J. de Liouville 1894). In the beginning of this lecture the author proves that the paths of a system depend in general on $2k-1$ arbitrary constants, k being the degree of freedom of the system, and he gives the condition necessary and sufficient in order that this number reduce to $2k-2$. The case is considered next, when these paths coincide with the geodesics. Then, after a discussion of the general case when the acting forces are independent of the velocities, the author passes to a detailed study of the real paths of a system; the principle of least action and Hamilton's principle are touched upon in the course of this lecture.

The last lecture relates to conservative systems. After establishing Poisson's theorem, the author returns once more to Hamilton-Jacobi's canonic equations and gives a concise but clear exposition of Jacobi and Mayer's methods for the integration of partial differential equations of the first order, concluding with the application to Jacobi's equation of Legendre's transformation, which is a particular case of Jacobi and Lie's transformations of contact.

It is to be regretted that the author did not add a chapter on the theory of perturbations which is so closely connected with the subject of his lectures. However, if these lithographed pages in which the exposition was necessarily limited are only the precursor of a more complete printed course, we may hope that the theory of perturbations will still find its place among Mr. Painleve's interesting lectures.

We now pass to the second of Mr. Painlevé's courses. The *lectures on friction* are a valuable contribution to rational mechanics. The author proposes to solve the following problem: *to develop for systems with friction a general theory similar to the one existing for systems without friction*, as far as the empiric laws of friction allow such a general treatment. The lectures before us deal first with discontinuous systems, and then with such continuous systems which can be considered as formed of infinitesimal elements that remain unchanged throughout the motion,* the other systems (to which belong extensible strings and membranes, compressible fluids and so on) being reserved by the author for another course to appear later. It must be remarked moreover that all cases where the conditions of constraint to which the systems are subject are over-abundant were systematically avoided by the author, since the motion of such systems in general depends on their interior constitution and belongs therefore to the theory of elasticity.

After recalling his general definition of systems with friction given in the fourth lecture on the integration of the differential equation of dynamics, derived from the consideration of the virtual work, the author examines first two special cases, namely, the motion of a point on a curve and on a surface, then passing to the general form of the laws of friction. We arrive here at a most interesting part of this course, both for its novelty and importance. The notion of partial friction is introduced and the author gives a detailed study of the laws of friction which result from the combination of the laws of partial friction (*combinaison des liaisons*). An important result is obtained when the conditions of constraint can be grouped into two parts: one, G_1, with friction and the other G_2, without. For, in such cases it is advantageous not to introduce the reactions due to the group G_1; to this purpose it suffices to form Lagrange's differential equations with regard to the parameters which would define the position of the system if only the conditions of constraint G_2 were taken into consideration.

After discussing the question of the compatibility of the conditions of constraint in general, the author classifies these conditions for continuous systems into three groups. To the first belong the conditions which are determined by a single relation between the parameters defining the position of the system. The author distinguishes nine types in this class. To the second group, consisting of five types, belong the conditions which are determined by two rela-

* To this class belong inextensible strings and membranes, incompressible fluids and so on. The position of such systems cannot be determined by means of a finite number of parameters. But, although Mr. Painlevé considers in his lectures only systems the position of which can be determined by means of a finite number of parameters, the general propositions developed by him can be immediately extended to the class here mentioned.

tions between the parameters defining the system. Finally, the conditions determined by three relations form the third group and are all conditions of constraint without friction.

This general theory is followed by numerous applications worked out completely, and with the elegance and clearness which are characteristic of the two courses of lectures we have before us.

It is only fair in concluding this review to remark to Mr. Hermann's credit that the reading of these two volumes is not in the slightest degree trying to the eyes, which unfortunately could not be said with regard to, for instance, Mr. Klein's lithographed courses.

ALEXANDRE S. CHESSIN.

JOHNS HOPKINS UNIVERSITY,
January 24, 1896.

A GEOMETRIC PROOF OF A FUNDAMENTAL THEOREM CONCERNING UNICURSAL CURVES.

BY PROFESSOR W. F. OSGOOD.

1. IF $f(x,y)=0$ is the equation of an irreducible curve of deficiency 0, then, as is well known, the coördinates can be expressed as rational functions of a parameter λ:*

$$x=r_1(\lambda) \qquad y=r_2(\lambda)$$

where not only to a given value of λ corresponds one and only one point of the curve, but conversely to a given point (x,y) on the curve corresponds in general one and only one value of λ.† λ can be expressed as a rational function of x and y.

If a multiple-leaved Riemann surface spread out, say, over the x-plane be used to represent geometrically the above locus, $f(x,y)=0$, the deficiency of this surface will likewise be 0, and as is shown in the elements of Riemann's theory of functions,‡ there exist single-valued functions on such a surface having but a single pole, and that of the first order, and taking on every value once and only once on the surface. Call such a function λ. Being single-valued on the surface it will be a rational function of x and y: $\lambda=R(x,y)$,

*SALMON, *Higher Plane Curves*, p. 30; CLEBSCH-LINDEMANN, *Geometrie* vol. I., p. 883.

† CAYLEY has given to such curves the name *unicursal*.

‡ KLEIN, *Modulfunctionen*, vol. I., p. 493 et seq. PICARD, *Traité d' analyse*, vol. II., Ch. XVI.