THE THREE GREAT PROBLEMS OF ANTIQUITY, CONSIDERED IN THE LIGHT OF MODERN MATHEMATICAL RESEARCH.

Vorträge über ausgewählte Fragen der Elementargeometrie. F. KLEIN. Ausgearbeitet von F. TÄGERT. Leipzig, Teubner, 1895. pp. 66.

AMONG the minor mathematical works published during the past year, one of the most interesting is Klein's Festschrift for the third meeting of the association for the advancement of mathematical and scientific teaching in the Gymnasia. The author has himself explained, in his paper *Ueber Arithmetisirung der Mathematik*, that his main object in writing this pamphlet was to emphasize the necessity for strict logical developments as a corrective to the tendency to rely too exclusively on intuitive proofs. Intended primarily for teachers of the more elementary mathematics, this pamphlet is confined to the three great problems of antiquity, as they appear in the light of modern research—or rather to the mathematical investigations for which these problems have furnished the text. These problems, (1) the duplication of the cube, (2) the trisection of an angle, (3) the quadrature of the circle, presented themselves at a very early stage in the development of mathematics, and they naturally present themselves correspondingly early in the mathematical development of the individual.

The Greek mathematicians, striving to solve these problems—problems for each of which the solution was impossible, within the domain of the geometry of the straight line and circle—were led to investigate and discover nearly all that lay within their originally unconsciously imposed boundaries. "Let it be granted, that a straight line may be drawn from any one point to any other point," "let it be granted, that a circle may be described from any centre, at any distance from that centre," these indicate the limitations of elementary geometry as understood by the Greeks, limitations recognized and formulated in the brightest period of Greek geometry by Euclid. It has been left to later times to recognize that other not less important limitations are implied in the definitions and axioms.

Still striving after the solution of these problems, Archytas, Eudoxus and their successors were led to enlarge their boundaries, admitting other methods, in the first instance the construction of conics, by which they solved the problems that they classed as "solid," and then the construction of higher curves, "linear loci," for use in treating those questions that were neither plane nor solid.

In this pamphlet the essential nature of the problems is sharply defined; the domain of Greek geometry is investigated, as to its contents and its boundaries; the correspondence with different parts of algebra is traced, and it is shown, in part all too briefly, how the successively enlarged domain of algebra corresponds with the different parts of geometry, the admissible algebraic operations being represented by the postulated geometrical constructions. Rational algebraic operations are represented by linear constructions; operations depending on the extraction of the square root can all be performed by means of straight lines and circles, in fact, by means of straight lines and a single circle, a possibility shown by Poncelet. These operations can also be performed by means of circles only, for Mascheroni wrote a book on geometry in which the only admissible construction is that of a circle, with any centre, passing through any assigned point. Klein refers to Hutt for an account of this book; a paper by Cayley in the *Messenger of Mathematics*, vol. 14, p. 179, may be more available for some readers. This paper gives sufficient account of the work to enable one to see the course of the proofs; the constructions depend chiefly on the conception of symmetry, points on a line being determined by the intersections of equal circles whose centres are symmetrically placed with respect to the line.

The pamphlet is divided into two parts, the first of which deals with algebraic numbers, the second with transcendental numbers. In the first part, Ch. I. explains the nature of a system of conjugate irrational quantities, and proves that there exists only one irreducible equation satisfied by one, and consequently by all, of the quantities of any such system. The chapter contains also a discussion of the nature of the algebraic equations that can be solved by the extraction of square roots; this discussion may be described as elementary, inasmuch as all the steps are minutely explained, but it gives the rigorous proof of the important theorem that an irreducible equation cannot be solved by the extraction of square roots, and hence cannot be constructed by means of straight lines and circles, unless its degree be a power of 2. In order to solve a cubic equation the domain of algebra must be enlarged by adjoining the operation of taking the cube root; hence problems whose algebraic expression leads to an equation of the third degree cannot be solved by elementary geometry. In Ch. II. it is shown that the first two of the three problems under discussion are of this type; they are therefore outside the domain of elementary geometry.

Since in Ch. I. the possibility of solving an apparently insoluble equation, if the degree be a power of 2, has presented itself, in Ch. III. this question is considered more carefully. It is not possible (*i. e.*, within the assigned limits of construction) to trisect *any* angle, but we can trisect a complete angle, we can divide the circumference of a circle into three equal parts. Can we divide the circumference into *any* number of equal parts? This inquiry opens out an extensive field of modern research, and may well serve to introduce the would-be mathematician to some comprehension of what modern mathematics means. The division into 3, 4 or 5 equal parts, with the bisection of each of these parts any number of times, gives the extent of the Greek achievement in this line, and was supposed to give the limit of all possible achievement. The list was however extended by Gauss, who showed that the division can be performed for any prime number of the form $2^{2^\mu}+1$, and for any number whose prime factors other than 2 are of this form, and non-repeated. Moreover, these numbers give all for which the division is possible.

In the first place, assuming for a moment that the prime must be of the form 2^h+1, it is shown that a necessary but not sufficient condition that this represent a prime is $h=$ a power of 2, 2^μ. It is remarked that the values 0, 1, 2, 3, 4, for μ give prime numbers 3, 5, 17, 257, 65537; that the values 5, 6, 7 for μ do not give prime numbers; and that the investigation has not been carried any further. At this point Klein suggests that possibly 4 is the highest value for μ that gives a prime. In the *Récréations mathématiques* of E. Lucas, however, the remark is made (vol. II., p. 235) that Eisenstein enunciated the theorem, "The number of primes of the form $2^{2^\mu}+1$ is infinite," giving no proof, though Lucas suggests that possibly he had one. The theorem may of course be purely conjectural.

It is now to be shown that the equation $\dfrac{z^p-1}{z-1}=0$,

i. e. $$z^{p-1}+z^{p-2}+\cdots+z+1=0,$$

which with the ordinary use of the complex variable is the equation for p-section of a circle of unit radius, is irreducible. This, by the Gauss lemma, is made to depend on the proof that the expression cannot be resolved into factors with integral coefficients. Here Klein follows Eisenstein's method, which, while perfectly simple, is not satisfactory in point of form, for the proof is made to depend on the somewhat irrelevant transformation $z = x + 1$, by which it is shown that the contrary supposition is untenable. When it has been shown that the equation is irreducible the rather lengthy argument of Ch. I. shows that the degree of the equation, *i. e.*, $p-1$, must be a power of 2.

As regards the actual theorem, a different proof is to be found in the December number of the BULLETIN, in the paper by Dr. James Pierpont. This paper deals with the complete theorem under consideration, giving an admirably simple proof, which sets the essential part of the theorem in evidence. It is to be shown that the equation

$$F(x) = x^{(p-1)p^{\alpha-1}} + \cdots + 1 = 0,$$

where m, the number of parts into which the circle is to be divided, $= p^{\alpha}$, cannot be rationally resolved; this is accomplished by following a method of Kronecker, preferable to that of Eisenstein, and presenting no greater difficulty; it is then shown that considering irrational factors of $F(x)$, the irrationalities depending only on the extraction of a square root, the only possible way in which $F(x)$ can split up the first time is into two conjugate factors of equal degree. Applying the theorem to each of these, and then to their factors, and so on, it is seen that the degree of F must be a power of 2.

The theorem having been established, the solution of the cyclotomic equation follows. It is possible to divide the roots into two periods whose sum and product are integral numbers; hence the two periods are the roots of a quadratic equation with integral coefficients. Taking one of the roots of this quadratic (that is, one of the two periods first formed), this can again be divided into two periods, given as the roots of a quadratic whose coefficients are rational in the roots of the first quadratic. In this way we obtain the solution of the cyclotomic equation by the solution of a chain of quadratics, the coefficients in any one of which depend on the roots of the preceding one. The process is fully explained in Ch. IV., being applied to the equation

$$z^{16} + z^{15} + \cdots + z + 1 = 0,$$

from the solution of which there is obtained a linear construction for a regular polygon of 17 sides in a given circle.* The equations are so carefully worked out and so minutely explained that the chapter will throw much light on the difficulties that some readers will find in the first few pages of the paper. But seeing how much importance attaches to the fact that the equation is irreducible until the domain of rationality is extended, we could wish that Klein had thought it worth while to give the successive factors;

$$z^{16}+z^{15}+\cdots+1=\Big(z^8-\eta_0 z^7+(2-\eta_0)z^6+(3-\eta_0)z^5+(1-2\eta_0)z^4+(3-\eta_0)z^3+(2-\eta_0)z^2-\eta_0 z+1\Big)\Big(z^8-\eta_1 z^7 \text{ etc.},\cdots\Big),$$

$$z^8-\eta_0 z^7+\cdots+1=\Big(z^4-\eta_0' z^3+\frac{\eta_0'\eta_0-\eta_1'+1}{2}z^2-\eta_0' z+1\Big)\Big(z^4-\eta_1' z^3+\frac{\eta_1'\eta_0-\eta_0'+1}{2}z^2-\eta_1' z+1\Big),$$

$$z^4-\eta_0' z^3+\frac{\eta_0'\eta_0-\eta_1'+1}{2}z^2-\eta_0' z+1=\Big(z^2-\eta_0'' z+1\Big)\Big(z^2-\eta_1'' z+1\Big);$$

this would have thrown some additional light on the significance of the early part of the paper.

Chapter V. indicates briefly the points of interest in connection with the construction of higher equations. The cubic and biquadratic are solved by means of conics, and thus conics are needed for the solution of problems whose algebraic expression leads to such equations. An example that naturally presents itself is the construction of the regular heptagon by means of a parabola and a rectangular hyperbola; and it is at once perceived that if conics are admitted, many more regular polygons can be described. Apparently Descartes knew that the regular heptagon and nonagon can be described by conics, for when discussing constructions by curves of higher order, he says that these can be applied to construct regular polygons of 11 or 13 sides. Now the regular polygons with 3, 4, 5, 6, 8, 10, 12

* In this connection it is perhaps worth while drawing attention to the very simple construction for the regular 17-ic given by Mr. H. W. Richmond in the *Quarterly Journal*, vol. 26, p. 206, 1893. The construction is simple in practice; theoretically it is not as simple as the one here given, for the equations are combined in such a way as to require circles in their construction. It might probably with slight modifications give a Mascheroni construction, which Klein remarks (p. 27) has not yet been considered for this case.

sides were described by the Greeks, and supposing Descartes to be aware that for 7 and 9 sides only conics are needed, his idea was apparently that the first regular polygons for whose construction higher curves are required are those of 11 and 13 sides. (The 13-ic can however be described by conics.) The omission of any explicit mention of the 7-ic and 9-ic proves nothing in the case of Descartes, for at the point where he might naturally have given it he had already devoted several pages to the construction of cubic equations, and was probably, as he himself says in another place with less reason, "bored already with writing so much about it." Such omissions may surely be forgiven for the sake of his concluding remark: "Et j'espère que nos neveux me sauront gré, non seulement des choses que j'ai ici expliquées, mais aussi de celles que j'ai omises volontairement, afin de leur laisser le plaisir de les inventer."

The first part of the paper has made clear the meaning of algebraic numbers, irrationalities successively introduced in the solution of algebraic equations. For the discussion of the third great problem, the quadrature of the circle, it is necessary to show that algebraic numbers form only a part, in point of fact an insignificant part, of the numbers that claim consideration. The chapter devoted to this cannot fail to interest many hitherto unfamiliar with Cantor's work. It is first to be shown that there is a (1, 1) correspondence between positive integers and all real algebraic numbers, that is to say, that the real algebraic numbers can be counted. To prove this it is shown, by a particular grouping of the irreducible equations whose roots give the algebraic numbers, that the numbers can be arranged in a singly infinite series of groups, those contained in any one group being finite in number; and that thus all the real algebraic numbers, positive and negative, can be arranged in an absolutely determinate order, which is not the order of magnitude; and so can be spoken of as 1st, 2nd, 3rd, etc., that is, put in (1, 1) correspondence with the positive integers.

The next step is to show how numbers can be constructed that shall not be contained in this orderly series. The number being required to lie within certain limits, so that there are given a certain number of decimal places, *e. g.*, 5, the digits in the following places have to be selected so that the number differs from all of the series. For a reason explained, the digit 9 is avoided. The 6th digit is chosen to be different from the 6th of the first algebraic number, and thus the number constructed will certainly be different from this;

the 7th digit is chosen to be different from the 7th of the second algebraic number, by which we ensure that the number written down is not the second, and so on. Hence we are assured of the existence of numbers that are not the roots of any algebraic equation; that is, the existence of transcendental numbers is proved, and it is shown how they can be written down. Moreover, since the choice of the digit to be written in any assigned place is restricted only by the exclusion of two digits, 9 and one other, we may choose any one of the 8 that are left, zero being admissible in the same way as any other. Hence between any two algebraic numbers there are 8^{∞} transcendental numbers, (not ∞^8 as stated in the pamphlet,) and real algebraic numbers form only a small part of all numbers.

The possibility of determining whether a given number is transcendental or not cannot be considered generally. A terminating or circulating decimal is obviously algebraic; a non-terminating and non-circulating decimal obviously cannot be given in full. Transcendental numbers can therefore only be given by some law of formation, as is seen in the case of e and π; but since algebraic numbers can also be given in this way, the fact that a number is so given proves nothing. The impossibility of giving any general criterion is brought home by the nature of the proofs for e and π. The connection of the two numbers, exhibited in the fundamental equation $e^{i\pi}+1=0$, gives the transcendency of π as a corollary from the theorem:—The quantity e is not a root of any equation in which both coefficients and exponents are algebraic numbers. To the consideration of e and π chapters III. and IV. are devoted; these give the most recent proofs, published as late as 1893.

Since the number π is transcendental it follows that no algebraic curve can be found in which the relation of derived straight lines is that of the circumference of a circle to the diameter. Hence the quadrature of the circle, the third great problem of antiquity, cannot be solved by means of algebraic curves.

Thus in these few pages some of the most striking results of modern mathematics are made accessible to many who would otherwise hardly have heard of them. But while reading this brilliant exposition it is difficult to avoid cherishing a lurking regret, which is possibly very ungracious, that Klein could not himself spare time to arrange his work for publication; for though we have here in full measure the incisive thought and cultured presentation which together make even strict logic seem intuitive, yet at times we miss the minute finish and careful proportion of parts that we feel justified in expecting from him. And yet revision and consolidation might have seriously interfered with the graphic simplicity of these chapters, and left them less adapted to their special purpose. Any English-speaking association with aims similar to those of the German association for which the pamphlet was prepared would do a service in publishing a thoroughly good translation of this inspiring work and circulating it as widely as possible.

CHARLOTTE ANGAS SCOTT.

BRYN MAWR COLLEGE,
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PAINLEVÉ'S LECTURES ON DYNAMICS.

I. *Leçons sur l'intégration des equations différentielles de la Mécanique et Applications.* Par P. PAINLEVÉ. pp. 291; 4to (lithographed).

II. *Leçons sur le Frottement.* Par P. PAINLEVÉ. pp. VIII.+ 111; 4to (lithographed). Paris, A. Hermann, 1895.

The publication of Mr. Painlevé's lectures on the integration of the differential equations of dynamics will be welcomed by everyone interested in the progress of theoretical mechanics. It is a long time since Jacobi's *Vorlesungen über Dynamik* appeared, and a strong need was felt of a systematic work which would contain the more recent researches in this important branch of mathematical science. Mr. Painlevé has admirably supplied this need. The lectures are not intended for beginners in theoretical mechanics, but rather as a supplementary course for those already familiar with its elements.

The first three lectures contain the fundamental definitions and principles of dynamics, the propositions relating to the first integrals of the motion of rigid systems and the theory of the motion of a solid body, the whole followed by a number of examples. Despite the brevity of this exposition it is clear and instructive, owing to several interesting remarks.

The fourth lecture deals with the general equations of the motion of systems. Lagrange's equations with the multipliers λ (also called Lagrange's equations of the first form) are derived from the consideration of the virtual work, which leads the author to a classification of sys-