"cosine circle" of a triangle (p. 316), that is, he noticed as a special case of one of his theorems that there is only one point p (the point now called the symmedian) through which if lines be drawn so that the intercepts made on them by the pairs of sides of a triangle are bisected at p, the ends of the intercepts lie on a circle, whose centre is of course p.

Finally we have the handling of two integrals, the second of which

$$\int \frac{\Sigma \pm (c_1 x_2 dx_3)}{[\frac{1}{2} c_1 g'(x_1) + \frac{1}{2} c_2 g'(x_2) + \frac{1}{2} c_3 g'(x_3)] \sqrt{f(x, x)}}$$

(where $f(x, x)$ and $g(x, x)$ are ternary quadratic forms, x_1, x_2, x_3, are the coordinates of a point on $g(x, x) = 0$, c_1, c_2, c_3, the coordinates of a fixed point on the same conic), is transformed into

$$-\int \frac{d\lambda}{\sqrt{g(\lambda)}}$$

where $g(\lambda)$ is the cubic which determines the line-pairs of the pencil defined by f and g. This covers, for example, the problem of transforming the elliptic integral $\int (xdx)/\sqrt{a_x^4}$ to Weierstrass's normal form.

FRANK MORLEY.

ON DIVERGENT SERIES.

BY PROFESSOR A. S. CHESSIN.

THAT every semi-convergent series can by a proper arrangement of its terms be made divergent is a well-known fact. It will be shown in this note that, conversely, every divergent series which does not tend towards infinity (series oscillating between finite limits) can by a proper arrangement of its terms be made convergent.

Only series with real terms will be considered since the investigation of series with complex terms, at least with regard to the substance of this note, can be reduced to that of series with only real terms.

THEOREM I. — *An infinity of numbers being given within a limited interval, we know that there will be at least one infinite accumulation of numbers of the given totality within the given interval. Let, in general, $N_1, N_2, \cdots, N_g$ be the numbers about which these infinite accumulations take place. It is always possible to form g distinct convergent series having for their respective sums the numbers $N_1, N_2, N_3, \cdots, N_g$.*

In fact, in the neighborhood of a number N_i there is an infinity of numbers of the given totality either on both sides of N_i or only on one side of it. In the last case, if $\alpha_{\kappa, i}$ be one of these numbers, we can always find another one among them $\alpha_{\kappa+1, i}$ such that $\alpha_{\kappa+1, i} > \alpha_{\kappa, i}$ or $\alpha_{\kappa+1, i} < \alpha_{\kappa, i}$ according as these numbers are on one side of N_i or on the other. We thus obtain a *regular* sequence of increasing or decreasing numbers such that $\lim (\alpha_i) = N_i$. In the other case we can always find two numbers $\alpha_{1, i}$ and $\beta_{1, i}$ among the given ones in the neighborhood of the number N_i such that there will be an infinity of numbers of the given totality within the interval $(\alpha_{1, i}, \beta_{1, i})$. In this interval we can again find two numbers $\alpha_{2, i}, \beta_{2, i}$ such that $\alpha_{2, i} > \alpha_{1, i}$ and $\beta_2 < \beta_{1, i}$, and that the interval $(\alpha_{2, i}, \beta_{2, i})$ shall contain an infinity of numbers of the given totality. Continuing this process, we arrive at two *regular* sequences of numbers

$$\alpha_{1 i}, \alpha_{2, i}, \alpha_{3, i}, \cdots$$
$$\beta_{1 i}, \beta_{2, i}, \beta_{3, i}, \cdots$$

the first having only increasing, the second only decreasing terms. Both define the same number N_i,

i.e. $$\lim (\alpha_i) = \lim (\beta_i) = N_i.$$

Let us now put

$$\alpha_{1, i} = a_{1, i}$$
$$\alpha_{2, i} - \alpha_{1, i} = a_{2, i}$$
$$\alpha_{3, i} - \alpha_{2, i} = a_{3, i}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

then $$\alpha_{n, i} = a_{1, i} + a_{2, i} + \cdots + a_{n, i}$$
and therefore
$$N_i = a_{1, i} + a_{2, i} + \cdots + a_{n, i} + \cdots$$

for all the values $1, 2, 3, \cdots, g$ of i; q. e. d.

Suppose now that we have a divergent series which does not tend towards infinity

$$u_1 + u_2 + \cdots + u_n + \cdots,$$

and let us form the sequence of numbers

$$\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots$$

where $\gamma_m = u_1 + u_2 + \cdots + u_m$. This sequence contains an infinity of numbers within a limited interval. Let then

$$N_1, N_2, \cdots, N_g$$

be the numbers about which infinite accumulations of the numbers (γ) take place. According to Theorem I. we can form g convergent series having for their respective sums the numbers $N_1, N_2, \cdots, N_g$.

THEOREM II.— *The g convergent series derived from the sequence* (γ) *can be obtained directly from the given divergent series merely by associating its terms in a proper way* (without deranging their places in the series).

In fact, we have seen in Theorem I. that we can pick out among the numbers (γ) in the neighborhood of the number N_i an infinity of numbers (γ) forming a regular sequence

$$\gamma_{\mu_{1,i}}, \gamma_{\mu_{2,i}}, \cdots, \gamma_{\mu_{n,i}}, \cdots,$$

and that $\lim (\gamma_{\mu_i}) = N_i$, for all the values $1, 2, \cdots, g$ of i.

In this sequence there is at least one index $\mu_{k,i} > \mu_{1,i}$; for otherwise the number of terms following the term $\gamma_{\mu_{1,i}}$ would be at the most equal to $\mu_{1,i}$. In like manner we prove that there must be at least one index $\mu_{e,i} > \mu_{k,i}$, and so on. We thus obtain the regular sequence

$$\gamma_{\mu_{1,i}}, \gamma_{\mu_{k,i}}, \gamma_{\mu_{e,i}}, \cdots$$

in which $\mu_{1,i} < \mu_{k,i} < \mu_{e,i} < \cdots$; and $\lim (\gamma_{\mu_i}) = N_i$.

But $\gamma_{\mu_{e,i}} = u_1 + u_2 + \cdots + u_{\mu_{e,i}}$.

If therefore we associate the terms of the given series in the following manner:

$$(u_1 + u_2 + \cdots + u_{\mu_{1,i}}) + (u_{\mu_{1,i}+1} + \cdots + u_{\mu_{k,i}}) + (u_{\mu_{k,i}+1} + \cdots + u_{\mu_{e,i}}) + \cdots,$$

this series will be convergent and have for its sum the number N_i; q. e. d.

Remark. — It is obvious that g is always greater than unity, the numbers N_i being all finite, otherwise the given series (u) would be convergent.

Examples:

(1) In the famous series

$$1 - 1 + 1 - 1 + 1 - \cdots$$

$$g = 2;\ N_1 = 0 = (1 - 1) + (1 - 1) + \cdots$$
$$N_2 = 1 = 1 + (-1 + 1) + (-1 + 1) + \cdots$$

(2) In the series

$$\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{2}{3} - \tfrac{3}{4} + \tfrac{3}{4} - \tfrac{4}{5} + \cdots$$

$$g = 2;\ N_1 = -\tfrac{1}{2} = (\tfrac{1}{2} - \tfrac{2}{3}) + (\tfrac{2}{3} - \tfrac{3}{4}) + (\tfrac{3}{4} - \tfrac{4}{5}) + \cdots$$
$$N_2 = \tfrac{1}{2} = \tfrac{1}{2} + (-\tfrac{2}{3} + \tfrac{2}{3}) + (-\tfrac{3}{4} + \tfrac{3}{4}) + \cdots$$

A divergent series which remains divergent whatever be the arrangement of its terms will be called *unconditionally divergent.*

THEOREM III. — *Every unconditionally divergent series tends towards infinity.*

In fact, in such a series $g = 1$, but this is impossible in a divergent series unless $N_1 = \pm \infty$.

It is easy to see that *conversely, every series which tends towards infinity is unconditionally divergent.*

A divergent series which becomes convergent after a proper arrangement of its terms will be called *conditionally divergent.*

THEOREM IV. — *Every divergent series which does not tend towards infinity is conditionally divergent, and conversely, a conditionally divergent series cannot tend towards infinity.*

THEOREM V. — *Every semi-convergent series can by a proper arrangement of its terms be made conditionally divergent, and conversely, every conditionally divergent series can by a proper arrangement of its terms be made semi-convergent.*

Remark. — Riemann has proved that by a proper arrangement (commutative, not associative) of the terms of a semi-convergent series this series can be made to converge to any arbitrarily assigned number. It follows from the above that *by a proper arrangement* (both commutative and associative) *of the terms of a conditionally divergent series this series can be made to converge to any arbitrarily assigned number.*

THE JOHNS HOPKINS UNIVERSITY,
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A SIMPLE PROOF OF A FUNDAMENTAL THEOREM OF SUBSTITUTION GROUPS, AND SEVERAL APPLICATIONS OF THE THEOREM.

BY DR. G. A. MILLER.

THEOREM. — *The average number of elements in all the substitutions of a group is $n - \alpha$, n being the degree of the group, and α the number of its transitive constituents.**

We shall first prove the theorem for $\alpha = 1$, *i.e.* for the transitive groups.

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* FROBENIUS, *Crelle*, vol. 101, p. 287.