BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY

THE SECOND SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Second Summer Meeting of the American Mathematical Society was held in the High School building at Springfield, Mass., on August 27 and 28. The society enjoyed the co-operation of the American Association for the Advancement of Science, the regular sessions of which began on August 29, and the members of the society were entitled to all the privileges extended to the members of the latter organization. Among those present were: Prof. A. L. Baker, Dr. J. H. Boyd, Prof. C. H. Chandler, Prof. L. L. Conant, Prof. C. L. Doolittle, Prof. W. P. Durfee, Prof. E. Frisby, Dr. G. D. Gable, Dr. J. W. L. Glaisher, Prof. A. Hall, Jr., Prof. Ellen Hayes, Dr. G. W. Hill, Prof. W. Woolsey Johnson, Mr. P. A. Lambert, Prof. J. McMahon, Prof. M. Merriman, Prof. F. Morley, Prof. H. B. Newson, Prof. W. F. Osgood, Prof. M. I. Pupin, Mr. R. A. Roberts, Mr. C. H. Rockwell, Prof. J. B. Shaw, Prof. W. E. Story, Prof. H. Taber, Prof. J. M. Van Vleck, Prof. E. B. Van Vleck, Prof. C. A. Waldo, Prof. H. S. White, Prof. J. M. Willard, Prof. C. B. Williams, Prof. F. S. Woods, and Prof. R. S. Woodward.

The President, Dr. G. W. Hill, occupied the chair. Two sessions were held each day, meeting respectively at 10 A.M. and 2.30 P.M. On the recommendation of the Council, Dr. Thomas Jefferson J. See, University of Chicago, Chicago, Ill., was elected to membership. Three nominations for membership were received.

The following papers were read:

1. *The periodic solution as a first approximation in the lunar theory.* Dr. G. W. HILL.
2. *The linear vector operator of quaternions.* Professor J. B. SHAW.
3. *A new application of quaternions to geometry.* Professor J. B. SHAW.

4. *On a generalization of Weierstrass's equation with three terms.* Professor F. Morley.

5. *Formulas for the sides of rational plane triangles.* Dr. Artemas Martin.

6. *Partial linear transformations of ternary quantics and their concomitants.* Professor J. McMahon.

7. *An introduction to the integrell calculus.* Professor W. H. Echols.

8. *On the expansion of a uniform function of a real variable without use of derivatives.* Professor W. H. Echols.

9. *On continuous functions without differential coefficients.* Mr. P. A. Lambert.

10. *Concerning Jordan's linear substitution groups.* Professor E. H. Moore.

11. *Algebraic symbols and* $\sqrt{-1}$. Professor A. L. Baker.

12. *An application of the method of conformal representation to the study of related differential equations.* Professor E. B. Van Vleck.

13. *On the differential equations of certain systems of conics.* Mr. R. A. Roberts.

14. *On bilinear forms.* Professor H. Taber.

15. *Elementary proof of the quaternion associative principle.* Professor A. S. Hathaway.

16. *Asymptotic lines on a circular ring.* Professor H. Maschke.

Dr. Hill's paper, since its presentation, has been published in the *Astronomical Journal,* No. 353, September 7, 1895, pp. 137–143. Its object is to obtain the values of the coefficients of the periodic inequalities having the multiples of the mean angular distance of the Moon from the Sun as arguments, when the inclination of the lunar orbit and the two eccentricities are neglected. It is very desirable to have these coefficients with a high degree of accuracy in order to effect their useful employment in the further determination of the motion of the perigee and node, and in fact of all the other coefficients of the periodic inequalities. This work has been done previously by the author in the *American Journal of Mathematics,* vol. 1, but in neglecting the lunar mass and the solar parallax. Mr. Ernest W. Brown has in the same journal supplemented these researches, but still leaving out of consideration the mass of the Moon.

Professor Shaw's first paper is a development of the linear vector operator of quaternions in the form of a scalar part and two vector parts, also as a tensor and a versor, and finally as a sum of nine operators. The forms are similar to those obtained in the articles of Professor Taber (*Amer. Jour. Math.,* vols. 12, 13), but are here developed entirely from quaternion expressions and not from matrices. It was stated that this paper would be offered to the *American Journal of Mathematics* for publication.

Professor Shaw's second paper applies the quaternion calculus to homogeneous geometry of two dimensions. In the expression $\zeta = xi + yj + zk$, x, y, z are proportional to the areas PBC, PCA, PAB respectively where the point P is referred to the fundamental triangle ABC. By this convention propositions of projective geometry are easily proved, and especially such propositions of modern geometry as those of the Lemoine-Brocard type. The author expected to contribute this paper to the *Annals of Mathematics*.

Professor Morley's paper is published in the present number of the BULLETIN. It contains a simple generalization of the formula

$$\begin{aligned} &\sigma(u + u_1)\ \sigma(u - u_1)\ \sigma(u_2 + u_3)\ \sigma(u_2 - u_3) \\ +\ &\sigma(u + u_2)\ \sigma(u - u_2)\ \sigma(u_3 + u_1)\ \sigma(u_3 - u_1) \\ +\ &\sigma(u + u_3)\ \sigma(u - u_3)\ \sigma(u_1 + u_2)\ \sigma(u_1 - u_2) = 0. \end{aligned}$$

In Dr. Martin's paper a large number of formulæ are deduced for calculating rational numbers which represent the sides of triangles having a rational area. Among them are, for the case of a right-angled triangle,

$$x = 2pq, \qquad y = p^2 - q^2, \qquad z = p^2 + q^2,$$

and, for the case of an oblique triangle,

$$\begin{aligned} x &= (p^2 + q^2)(r^2 - s^2), \\ y &= 2rs(p^2 + q^2) \pm 2rs^2(p^2 - q^2), \\ z &= (p^2 + q^2)(r^2 + s^2) \pm 2rs(p^2 - q^2), \end{aligned}$$

in which x, y, z denote the sides, and p, q, r, s any entire numbers whatever. The paper, which contains many numerical applications, will be printed in the *Mathematical Magazine*. In the absence of Dr. Martin his paper was read by the Secretary.

Semi-invariants of a ternary quantic satisfy some, but not all, of the six differential equations which characterize invariants. Professor McMahon's paper shows how to distinguish between those semi-invariants which are the sources of covariants and those which are the sources of semi-covariants, and gives a simple method of deriving from the latter all the coefficients of the semi-covariant, or of a semi-contravariant if desired. A systematic geometrical interpretation of these three kinds of semi-concomitants is presented and appears to be particularly useful in Cartesian co-ordinates. This paper is intended for the *Annals of Mathematics*.

In the absence of Professor Echols his two papers were presented by the Secretary. The first one was the second section of an essay "On the differell and integrell calculus," the preceding section having been read at the May meeting of the Society. Professor Echols approaches the infinitesimal calculus from the calculus of finite differences. He considers the latter, however, in a greatly generalized form. A differell is defined to be the limit of a ratio whose terms are the nth differences of the function and the independent variable. An integrell is a differell of negative index. The applications presented consisted chiefly in the expansion of functions in terms of differells and integrells. In the second paper some of the more novel results were translated into the language of ordinary calculus.

Mr. Lambert attempted to show that functions of which Weierstrass's derivativeless function is the type may be so considered geometrically that their curves will have determinate tangents. In order to secure agreement of the analytic result with the geometric he found it necessary to replace Weierstrass's supposition in regard to h

$$\tfrac{1}{2}\frac{\pi}{a^n} < h < \tfrac{3}{2}\frac{\pi}{a^n},$$

by the limitation

$$0 < h < \epsilon\frac{\pi}{a^{2n}},$$

n increasing indefinitely, and ϵ being any finite quantity.

In Professor Moore's paper a tactical configuration is established which exactly defines Jordan's linear substitution group when taken fractionally. This configuration is self-reciprocal. The properties of this configuration and of other allied configurations similarly related to certain important subgroups of the main group are developed. Professor Moore's paper will appear in the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY. It was presented at the meeting by the Secretary.

Professor Baker's paper contained a detailed discussion of the character of the operations of algebra and the theory of imaginary quantities. It will be published in the *American Journal of Mathematics.*

In Professor Van Vleck's paper an aggregate of regular linear differential equations of the second order is taken, each of which has a polynomial solution. The requirement is made that these equations shall have a common group arising from their four common branch points, or in Riemann's phraseology, that they shall be related. This requirement necessitates the introduction of one accessory branch point into each equation. These accessory points do not, however, give rise to any substitutions of the group. The paper outlines a method by which the position of the accessory points may be investigated, as well as the distribution of the roots of the various polynomials between the four (real) branch points and in the imaginary domain. One result is the determination of the distribution of the roots of all polynomials satisfying differential equations of the second order with exactly four branch points which with their exponent differences are given.

Mr. Roberts' paper gives the differential equations of certain systems of conics in a plane, viz., conics having double contact with two fixed conics, etc., and in space, of conics touching six fixed planes, conics having double contact with three quadrics inscribed in the same developable, circles having double contact with two confocal quadrics, etc. These results are principally deduced by means of elliptic integrals and the first-class of hyper-elliptic integrals, and from them flow certain theorems concerning doubly infinite porisms of curvilinear polygons. The paper appears in the present number of the BULLETIN.

Professor Taber's paper contains a theorem concerning the transformation into itself by an orthogonal substitution of an alternate bilinear form with cogredient variables, and treats of the consequences of this theorem. Thus, denoting by G the group of linear substitutions which transforms into itself an alternate bilinear form with cogredient variables, the theorem in question is that every orthogonal substitution of the group G is the second power of a substitution of this group. From this theorem may be derived: (*a*) The conditions necessary and sufficient that a given substitution of the group G may be generated by the repetition of an infinitesimal substitution of the group. (*b*) A proof that every substitution of the group G can be obtained by the composition of three linear substitutions given by Cayley's rational representation for the substitutions of this group. (*c*) A representation of the coefficients of any orthogonal substitution of the group G as rational functions of the minimum number of parameters, provided the alternate bilinear form is orthogonal. Further it follows from the theorem that every linear substitution which transforms into itself a general bilinear form F is the second power of such a substitution, and can be generated by the repetition of an infinitesimal substitution which also transforms F into itself, provided, if F' denotes the form conjugate to F, that the forms $F + F'$ and $F - F'$ are both of non-evanescent determinant. The paper also contains a solution of the problem to determine the linear transformations between two alternate bilinear forms, depending upon an algebraic equation of degree equal to the number of variables of the forms. The paper will be contributed to the *Quarterly Journal of Pure and Applied Mathematics.*

In Professor Hathaway's paper the proof consists in identifying, by means of elementary geometry, the product of several versors with the composition of a set of rotations through angles double those of the corresponding versors. The obvious associative principle of the composition of the rotations proves the corresponding associative principle of multiplication of versors. This paper will appear in the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY. It was presented to the Society by Professor Shaw.

Professor Maschke's paper contains a very elegant application of elliptic functions to curves drawn on the surface of a circular ring. This paper appears in the present number of the BULLETIN. It was presented to the Society by Professor Morley.

At the afternoon session, August 28th, two topics were presented to the Society for general discussion:

(1) "A general subject catalogue or index of mathematical literature."

(2) "The mathematical curriculum of the college and scientific school."

The first discussion was opened by the Secretary, who gave a brief account of the "*Répertoire bibliographique des sciences mathématiques,*" in course of publication by the Mathematical Society of France. The discussion was continued by Professors Morley, Woodward, and McMahon. On motion by Professor McMahon, it was resolved that the Council be requested to consider the desirability of offering to the Mathematical Society of France the coöperation of this Society and of drawing up a plan for such coöperation.

The second discussion was opened by Professor Shaw, who presented a table of statistics showing the character of the mathematical instruction in 101 representative colleges and scientific schools. The discussion was continued by Professors White, Morley, Van Vleck, Doolittle, Chandler, Pupin, and Woodward. It seemed to be generally held that the work of the preparatory schools as a whole is not sufficiently thorough to serve as a satisfactory basis for collegiate courses, that a greater proportion of the students' time should be given to mathematical study, that greater stress should be laid on the fundamental subjects, that elementary portions of applied mathematics should be earlier introduced and more extensively taught, and that spherical trigonometry should be in great part, or altogether, dropped from the required curriculum.

At the close of the discussion the thanks of the Society were tendered to the Springfield Local Committee for the accommodations and hospitality which the Society had enjoyed, and the meeting was adjourned.

THOMAS S. FISKE.

COLUMBIA COLLEGE, *September* 10, 1895.

THE MEETING OF THE AMERICAN ASSOCIATION.

THE forty-fourth annual meeting of the American Association for the Advancement of Science was held at Springfield, Mass., August 29–September 4, President E. W. Morley, of Cleveland, Ohio, presiding. The attendance was not large, only 367 members being registered. This is 110 less than the number attending the Brooklyn meeting last year. There were 113 new members and 58 fellows elected this year.

Excursions were given to Forest Park, Springfield; to Amherst, Northampton, and Mount Holyoke College, and to Holyoke.

The officers of Section A were as follows: Edgar Frisby, Washington, D. C., Vice-President; Asaph Hall, Jr., Ann Arbor, Mich., Secretary; C. A. Waldo, Greencastle, Ind., Councilor; J. E. Kershner, Lancaster City, Pa.; C. A. Doolittle, Philadelphia, Pa.; L. L. Conant, Worcester, Mass.; C. H. Rockwell, Tarrytown, N. Y., together with the Vice-President and Secretary, sectional committee; J. R. Eastman, Washington, D. C.; S. C. Chandler, Cambridge, Mass.; J. M. Van Vleck, Middletown, Conn., with the Vice-President and Secretary, committee to nominate officers of the section. A. N. Skinner, D. C., Press Secretary.

The work of the Section was devoted almost wholly to astronomy, only two papers in pure mathematics being presented. The tendency of the Section to devote itself to the former class of work was commented on and deplored, and it was hoped that this tendency might be arrested before it had gone too far, and the balance between mathematical and astronomical work restored at the next meeting. This feeling manifested itself to a certain extent in the choice of officers, for the ensuing year, mentioned below.

The papers presented to Section A were as follows:

1. *Development of some useful quaternion expressions, with applications to geometry of three and four dimensions.* By JAMES BYRNIE SHAW, Jacksonville, Ill.

2. *The constant of aberration.* By C. L. DOOLITTLE, Philadelphia, Pa.

3. *On the constant of nutation.* By S. C. CHANDLER, Cambridge, Mass.