grals have periods that are associated with the $2p$ cross-cuts that are needed to reduce the surface to simple connection; some pages are assigned to Abel's Theorem and to the theorem of Riemann-Roch, and the final section treats of the problem of inversion and of the properties of the special theta-functions that are needed for the purposes of this inversion.

It will be seen from what we have said that this second volume contains a great wealth of material, and that much that has been previously dark is cleared up by M. Jordan's new researches. It may safely be affirmed that no students of the *methods* of the Differential and Integral Calculus can afford to neglect the Cours d'Analyse in its new form. From beginning to end the reader feels that he is being guided by a master-hand.

J. HARKNESS.

ON A THEOREM CONCERNING p-ROWED CHARACTERISTICS WITH DENOMINATOR 2.

BY PROFESSOR E. HASTINGS MOORE.

MR. PRYM'S book, *Untersuchungen über die Riemann'sche Thetaformel und die Riemann'sche Charakteristikentheorie* (Leipzig, 1882), has as a brief third part, *Beweis einiger Charakteristikensätze.* I recall the terms and theorems in question:

A p-rowed characteristic * is a complex

$$\begin{bmatrix} \epsilon_1 \; \epsilon_2 \ldots \epsilon_p \\ \epsilon_1' \epsilon_2' \ldots \epsilon_p' \end{bmatrix}$$

whose $2p$ elements $\epsilon_1 \ldots \epsilon_p'$ are integers taken modulo 2. The notation $[\epsilon]$ is used.

A characteristic is even or odd according as

$$\sum_{\nu=1}^{\nu=p} \epsilon_\nu \epsilon_\nu' = 0 \text{ or } 1. \quad (\text{mod. } 2.)$$

(Theorem I.) There are in all 2^{2p} p-rowed characteristics, of which $g_p = 2^{p-1}(2^p + 1)$ are even and $u_p = 2^{p-1}(2^p - 1)$ are odd.

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* The elements of the complex are the numerators of fractions having the common *denominator* 2 which enter in the definition of the theta function of p variables.

The sum (= difference) of two characteristics is a third characteristic whose elements are respectively the sum (= difference) (reduced modulo 2) of the corresponding elements of the two characteristics. Hence arises a theory of the additive composition and decomposition of characteristics. The results of this theory are invariant for any permutation of rows either with or without interchange of lines made simultaneously and similarly in all the characteristics involved.

(Theorem II.) Every p-rowed characteristic, except $[0] = \begin{bmatrix} 00\ldots0 \\ 00\ldots0 \end{bmatrix}$, decomposes in 2^{2p-1} ways into two (distinct) characteristics. $[0]$ decomposes in 2^{2p} ways into two (equal) characteristics.

(Theorem III.) Every p-rowed characteristic, except $[0]$, decomposes in $x_p = g_{p-1} = 2^{p-2}(2^{p-1}+1)$ ways into two even characteristics, in $y_p = u_{p-1} = 2^{p-2}(2^{p-1}-1)$ ways into two odd characteristics, and in $z_p = g_{p-1} + u_{p-1} = 2^{2p-2}$ ways into an even and an odd characteristic.

Mr. Prym proves Theorem III by the use of three equations:

$$(1) \quad g_p = 2x_p + z_p, \qquad (2) \quad u_p = 2y_p + z_p,$$

$$(3) \quad 0 = x_p + y_p - z_p;$$

of these one sees (1) and (2) at once, while (3) is obtained easily, but less intuitively.

Now it is possible to replace the equation (3) by a direct determination

$$(3') \qquad y_p = u_{p-1};$$

that is, marking $(p-1)$-rowed characteristics by a superscript *,

$[\epsilon]$ *being any p-rowed characteristic, except* $[0]$, *there are exactly as many decompositions of* $[\epsilon]$ *into two odd characteristics* $[\zeta]$, $[\eta]$,

$$(4) \qquad [\epsilon] = [\zeta] + [\eta],$$

as there are odd $(p-1)$*-rowed characteristics* $[\zeta^*]$.

Whenever a certain number occurs in the enumeration of two distinct sets of objects, it is often possible and, when possible, always desirable † to effect a direct connection between the two sets of objects. I establish such a connection in the

† For example: KRONECKER: Ueber bilineare Formen mit vier Variabeln (p. 4), Abhandlungen der Akademie zu Berlin, 1883.

case ($y_p = u_{p-1}$) in hand between the, say, totality of *pairs* of odd characteristics $[\zeta]$, $[\eta]$ for a given $[\epsilon]$ different from $[0]$ and the totality of odd characteristics $[\zeta^*]$.

The given $[\epsilon]$ is not $[0]$, and hence must in at least one row be different from $\frac{0}{0}$. We call one such row the last row, and have $\frac{\epsilon_p}{\epsilon_p'} = \frac{0}{1}, \frac{1}{0}$, or $\frac{1}{1}$, and need to consider indeed only $\frac{\epsilon_p}{\epsilon_p'} = \frac{0}{1}$ or $\frac{1}{1}$. We write

$$(5) \qquad [\epsilon] = [\epsilon^* \epsilon^{(p)}], \quad [\zeta] = [\zeta^* \zeta^{(p)}], \quad [\eta] = [\eta^* \eta^{(p)}],$$

thus separating † each p-rowed characteristic into a $(p-1)$-rowed and a one-rowed characteristic. From (4) we have

$$(6) \qquad [\epsilon^*] = [\zeta^*] + [\eta^*], \quad [\epsilon^{(p)}] = [\zeta^{(p)}] + [\eta^{(p)}].$$

$[\zeta]$ and $[\eta]$ are both odd; so $[\zeta^*]$, $[\zeta^{(p)}]$ are one odd, one even, and likewise $[\eta^*]$, $[\eta^{(p)}]$ are one odd, one even. The $2^2 = 4$ one-rowed characteristics are three even, $\frac{0}{0}, \frac{1}{0}, \frac{0}{1}$, and one odd, $\frac{1}{1}$. Since $[\epsilon^{(p)}]$ is not $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $[\zeta^{(p)}]$ and $[\eta^{(p)}]$ are one or both even, and hence $[\zeta^*]$ and $[\eta^*]$ one or both odd, say $[\zeta^*]$ surely odd.

Now we let $[\zeta^*]$ be *any* odd $(p-1)$-rowed characteristic and determine $[\eta^*]$ so that

$$[\epsilon^*] = [\zeta^*] + [\eta^*].$$

If $[\eta^*]$ is even, we are able to determine in exactly one way the $[\zeta^{(p)}]$, $[\eta^{(p)}]$ leading to a pair of odd characteristics $[\zeta]$, $[\eta]$ with sum $[\epsilon]$; this pair corresponds to the odd $[\zeta^*]$. If, however, $[\eta^*]$ is also odd, we are led to exactly two pairs $[\zeta]$, $[\eta]$; if $[\epsilon^*] = [0]$, then $[\zeta^*] = [\eta^*]$ and the two pairs are the same (with the unessential interchange of the $[\zeta]$, $[\eta]$), so that this one pair corresponds to the odd $[\zeta^*] = [\eta^*]$; on the other hand, if $[\epsilon^*] \neq [0]$, the $[\zeta^*]$, $[\eta^*]$ are distinct, and the two pairs are distinct, and these two pairs correspond to the two odd $[\zeta^*]$, $[\eta^*]$ (for, clearly, if we take the $[\eta^*]$ as $[\zeta^*]$, we reach the same two pairs as before). *This is the direct connection which was to be established.*

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† Mr. PRYM uses this separation to establish the recursion formulæ

$$g_p = 3g_{p-1} + u_{p-1},$$
$$u_p = 3u_{p-1} + g_{p-1},$$

which lead to Theorem II.

The statements made are easy consequences of the decompositions

$$[\epsilon] = [\zeta] + [\eta] \qquad [\epsilon] \neq [0]$$

for $p = 1$:

$$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix},$$

$$\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix},$$

that is, every one-rowed mark not [0] decomposes in one way into two even characteristics (in two ways, if order of components be material) and in one way into an even and an odd characteristic.

We may treat similarly the equations

$$(3'') \quad x_p = g_{p-1}, \qquad (3''') \quad z_p = g_{p-1} + u_{p-1},$$

and thus obtain the three determinations of Theorem III independently of one another and each in this highly satisfactory way.

THE UNIVERSITY OF CHICAGO, *June* 17, 1895.

NOTE ON THE TRANSITIVE SUBSTITUTION GROUPS OF DEGREE TWELVE.

BY G. A. MILLER, PH.D.

§ 1. *Primitive groups.*

IN the *Comptes Rendus*, vol. 75, page 1757, Camille Jordan states that there are *three* primitive groups of degree twelve, excluding the groups which contain the alternating group. In working over this region lately we found the following *four* multiply transitive primitive groups of this degree, excluding the two groups containing the alternating group. Since all such groups include transitive groups of degree eleven, we may prove that the following list is exhaustive by proving that no more than four such groups could be based upon these transitive groups. This may be done as follows:

No group can be based upon

$$(abcdefghijk)\text{cyc}^*,$$

* A list of the transitive groups of degree eleven is given by Professor COLE, *Quarterly Journal of Mathematics*, vol. 27, p. 49.