r along the curve, must be identically the same for congruent curves. For the exceptional case of constant values of r in the plane we come upon the "minimal lines" $x \pm iy = \text{const.}$

Here it is found that the lines of the two families separately are congruent. In space the minimal curves defined by $\sqrt{dx^2 + dy^2 + dz^2} = 0$ play a similar rôle.

It will be noted that here as in many important applications those invariants known as differential invariants occupy the most prominent position. Under the theory of this class of invariants belongs the entire theory of curvature both of curves and surfaces. Another chapter of the book, although offering no new results, shows also how Cayley's theory of invariants of binary forms can also be brought under this head, being the invariant theory of a linear homogeneous group, and closes by indicating a more general form of this same class of invariants for any group whatever.

We will not take up farther the applications given, but the reader will find extremely interesting and suggestive also the final chapter dealing with the integration theory of Riccati's equation, of systems of linear homogeneous differential equations, and finally that very large class of equations, differential equations with fundamental solutions.

J. M. BROOKS.

PRINCETON, *June* 8, 1895.

JORDAN'S COURS D'ANALYSE.

Cours d'Analyse de l'École Polytechnique, par M. C. JORDAN. Deuxième Édition, entièrement refondue. Tome deuxième, Calcul Intégral. pp. 627 + xviii.

IN the new edition of M. Jordan's Cours d'Analyse the second volume, like the first, differs greatly in form from that of the original edition. So much is this the case that the present work may be treated as a substantially new contribution to mathematical literature. The character of many of the alterations of previous proofs was foreshadowed by those made in the revised first volume: they are in the direction of increased carefulness and precision of statement. Especially is this the case in the chapter on Definite Integrals. Adopting the definition of $\int_a^b f(x)dx$ as a limit of sums, the author proceeds to wide generalizations of this definition. The first extension covers the case where $f(x)$ is integrable throughout the field ab except in the neighborhoods of certain points $c_1, c_2, \ldots, c_n$ which are finite in number, and this leads on naturally to the case where the points c are infinite in number. In this latter case it is proved that the points c must of necessity form an *ensemble parfait.* The division of the totality of convergent series into the two classes, viz., absolutely convergent and semi-convergent, finds its analogue in the classification of the definite integrals $\int_a^b f(x)dx$ (supposed determinate) as *absolutely convergent* when $\int_a^b |f|\, dx$ is finite, and *semi-convergent* in all other cases.

A valuable addition to the theory of definite integrals consists in the assignment of sufficient conditions for the change in order of the two integrations in $\int_b^B dy \int_a^A f dx$, where a, A, b, B are supposed finite. It is shown that

$$\int_b^B dy \int_a^A f dx \quad \text{and} \quad \int_a^A dx \int_b^B f dy$$

have finite, determinate, and equal values when the following three conditions are satisfied:

I. The points c of the field for which the function f ceases to be continuous are all situated on a limited number of continuous arcs $P_1Q_1, \ldots, P_nQ_n$ which are such that on each arc x on the one hand and y on the other increase or decrease constantly;

II. The integral $\int_x^{x+\lambda} f dx$, taken along a segment of a parallel to the axis of x which is situated arbitrarily in the field and whose length λ does not exceed a fixed number l, has always a determinate value whose absolute value is constantly less than ϵ_l, where ϵ_l is a quantity which depends solely on l and tends to zero with it;

III. The analogous integral $\int_y^{y+\lambda} f dy$, taken along a parallel to the axis of y, has likewise a determinate value whose absolute value is constantly less than ϵ_l', where ϵ_l' is a quantity precisely analogous to ϵ_l.

In § 70 the author proves that these three conditions can be replaced by conditions that are less restrictive, and in § 71 he takes up the case of an infinite field.

Twenty pages are allotted to the discussion of multiple integrals; this resumption of the work of the first volume is rendered necessary by the adoption of the generalized definition for definite integrals. For instance, the question arises whether the fundamental properties of definite integrals by excess or defect still subsist for the generalized integrals; this question is answered in the affirmative.

In § 95 the author reproduces Hurwitz's elegant proof of the theorem that e cannot be a root of an algebraic equation; we regret that he has omitted the proof of the corresponding theorem for π. The rest of chapter II does not differ greatly from the corresponding portions of the first edition.

In the first section of chapter III we have a discussion of differentiation under the sign of integration, in which use is made of the three conditions given above. This is followed by an account of various methods for evaluating special definite integrals. The second section is concerned with Eulerian integrals and the third with the Potential. Here some pages are assigned to Harmonic Functions, and Green's theorem is proved, as also some of the many associated theorems; mention is made of Dirichlet's problem and of the work of Schwarz, Neumann, and Poincaré, but limits of space prevent M. Jordan from undertaking any general account of their researches. The nature of the solution is, however, indicated in the special case of the sphere.

Chapter IV treats of the integrals of Fourier, trigonometric series, and the functions of Laplace.

The subject of complex integrals is considered in chapter V, the ordinary theorems in Cauchy's theory are explained fully and clearly, and several pages at the end of the chapter are devoted to Weierstrass's theorem on the construction of an integral function whose zeros are situated at assigned isolated points $a_0, a_1, a_2, \ldots, a_n, \ldots,$ and to Mittag-Leffler's celebrated theorem.

The sixth chapter, on elliptic functions, is of great importance. Recognizing the indisputable superiority of Weierstrass's methods over those of Jacobi and Abel, M. Jordan has recast his account of the theory of elliptic functions, and has taken for guides Schwarz and Halphen. In its new form this chapter covers over two hundred and thirty pages, and forms a very complete introduction to the subject of elliptic functions. It is impossible to give an adequate account of the subject-matter of this chapter within the limits of a review article; we will content ourselves with calling attention to a few special points. After some preliminary theorems on linear substitutions, the author defines *the principal periods* and proves several theorems on general elliptic functions. He then passes on to the Weierstrassian functions $\wp u, \zeta u, \sigma u,$ and develops their principal properties. The zeros e_1, e_2, e_3 of the cubic expression $4z^3 - g_2 z - g_3$ are numbered according to the order in which they are met when the perimeter of the triangle of zeros is described in the direct sense, and three periods

$$2\omega_1 = \int_{e_2}^{e_3} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}, \quad \text{etc.},$$

are selected, where the paths of integration are straight lines from lower to upper limits. It is then proved that the periods $2\omega_1$, $2\omega_3$, $2\omega_2$ form a principal triangle, and that the ratios ω_2/ω_1, ω_3/ω_2, ω_1/ω_3 have their imaginary parts positive. Attention is paid to the special cases

(1) J real and greater than 1, where $J = \dfrac{g_2^3}{g_2^3 - 27g_3^2}$;
(2) J real and less than 1;
(3) $J = 1$;
(4) $J = 0$.

The significance of the first case consists in the fact that e_1, e_2, e_3 are then situated on a straight line which passes through the origin.

After discussing the functions $\sigma_1 u$, $\sigma_2 u$, $\sigma_3 u$, M. Jordan connects these functions and the parent function σu with the four single theta-functions. As the nomenclature for the theta-functions is far from fixed, he considers that it is not too late in the day to introduce a new notation. According to a very common form of notation, θ_1 is the odd function and θ, θ_2, θ_3 the even functions; but now that Weierstrass's functions have passed into common use, this notation has become objectionable, for it makes σu correspond not to θu but to $\theta_1 u$. The suggested change of notation leads to a parallelism of notation between the theta-functions and the sigma-functions.

In § 392 a practical method is given for effecting the numerical calculations relative to the elliptic functions when these are supposed to be defined by their periods or by their invariants. Within the limits of section VI a luminous and sufficient account is given of periodic functions of the second and third kinds; in section VII there is a full treatment of differentiation with respect to parameters; and in the latter portion of the chapter a considerable number of pages are devoted to the subjects of division and transformation.

The final chapter treats of Abelian integrals; it is a rapid sketch of the theory of these integrals, based principally on Riemann's classical memoir, on the Traités d'Analyse of Picard and Laurent, and on Forsyth's Theory of Functions. By suitable transformations the basis-equation $f(x, y) = 0$ is reduced to a form which yields simple branch-points that are situated in the finite part, of the plane and then a canonical surface is constructed. It is proved that the Abelian integrals have periods that are associated with the $2p$ cross-cuts that are needed to reduce the surface to simple connection; some pages are assigned to Abel's Theorem and to the theorem of Riemann-Roch, and the final section treats of the problem of inversion and of the properties of the special theta-functions that are needed for the purposes of this inversion.

It will be seen from what we have said that this second volume contains a great wealth of material, and that much that has been previously dark is cleared up by M. Jordan's new researches. It may safely be affirmed that no students of the *methods* of the Differential and Integral Calculus can afford to neglect the Cours d'Analyse in its new form. From beginning to end the reader feels that he is being guided by a master-hand.

J. HARKNESS.

ON A THEOREM CONCERNING p-ROWED CHARACTERISTICS WITH DENOMINATOR 2.

BY PROFESSOR E. HASTINGS MOORE.

MR. PRYM'S book, *Untersuchungen über die Riemann'sche Thetaformel und die Riemann'sche Charakteristikentheorie* (Leipzig, 1882), has as a brief third part, *Beweis einiger Charakteristikensätze.* I recall the terms and theorems in question:

A p-rowed characteristic * is a complex

$$\begin{bmatrix} \epsilon_1 \, \epsilon_2 \ldots \epsilon_p \\ \epsilon_1' \epsilon_2' \ldots \epsilon_p' \end{bmatrix}$$

whose $2p$ elements $\epsilon_1 \ldots \epsilon_p'$ are integers taken modulo 2. The notation $[\epsilon]$ is used.

A characteristic is even or odd according as

$$\sum_{\nu=1}^{\nu=p} \epsilon_\nu \epsilon_\nu' = 0 \text{ or } 1. \quad (\text{mod. } 2.)$$

(Theorem I.) There are in all 2^{2p} p-rowed characteristics, of which $g_p = 2^{p-1}(2^p + 1)$ are even and $u_p = 2^{p-1}(2^p - 1)$ are odd.

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* The elements of the complex are the numerators of fractions having the common *denominator* 2 which enter in the definition of the theta function of p variables.