KINETIC STABILITY OF CENTRAL ORBITS.*

BY PROFESSOR W. WOOLSEY JOHNSON.

1. IN the chapter on Central Orbits of Tait and Steel's Dynamics of a Particle, Fourth Edition, p. 125, occurs an investigation of the apsidal angle of a nearly circular orbit. When the attraction varies inversely as the nth power of the distance, the expression found becomes imaginary when n exceeds 3, and the remark is made that "the investigation furnishes a simple example of the determination of the conditions of *Kinetic Stability*, which we cannot discuss in this elementary treatise." It may not be without interest to show that an investigation of a no less elementary character will furnish a satisfactory discussion of this interesting subject, so far as it relates to central forces.

2. The usual polar equation of the central orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}, \tag{1}$$

in which P is the attraction acting on a unit of mass.

The first integral of this equation found in the usual way is

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2}{h^2}\int\frac{Pdu}{u^2}.$$

Since $\int\frac{Pdu}{u^2} = -\int Pdr = -V + C$, V being the potential function, this equation may be written

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2}{h^2}(C - V) - u^2 = \psi(u), \tag{2}$$

the function $\psi(u)$ depending not only on the given law of force, but also upon the values given to the two constants h and C.

3. In discussing the function $\psi(u)$ we have only to consider positive values of u, (the reciprocal of r the distance from the centre of force,) and it is evident from equation (2) that the values of u for every point of an actual orbit must be such as to make $\psi(u)$ positive, except the maxima and minima values

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which make $\psi(u) = 0$. Thus, if $\psi(u)$ is negative for all positive values of u, there is no orbit having the given values of C and h; and if $\psi(u)$ is always positive there is no maximum or minimum distance, but an orbit extending to the centre of force in one direction and to infinity in the other.

In the general case, putting $\psi(u) = 0$, we have the apsidal values of u. Let the positive roots of this equation be u_1, u_2, etc., which at first suppose to be all distinct. If we draw circles about the centre of force with the reciprocals of these quantities as radii, the plane will be separated into spaces in which $\psi(u)$ is alternately positive and negative (for $\psi(u)$ cannot become infinite for a finite value of u). In each of the spaces where $\psi(u)$ is positive an orbit exists with the given values of C and h, having its apsides upon the circle or circles which bound the space.

4. Now suppose the values of C and h to be so related that $\psi(u) = 0$ has a pair of equal roots, say u_0. As they approach the special values which fulfil the condition, $\psi(u) = 0$ will have two roots near to u_0. If in the annular space between the corresponding circles $\psi(u)$ is positive, there exists an orbit which, as the annular space contracts, approaches the circular form, and finally becomes a circular orbit described like any other orbit with kinetic stability. The differential equation is satisfied by the constant value $u = u_0$, and for all neighboring values of u $\psi(u)$ is negative.

If, on the other hand, $\psi(u)$ is negative in the narrow annular space, there are no orbits approximating the circular form. $u = u_0$ still satisfies the differential equation, but $\psi(u)$ is positive for neighboring values of u on either side, and when $u = u_0$ the body may be regarded as being at an apse of either of the orbits which exist in the two spaces in which $\psi(u)$ is positive, which are now brought into juxtaposition. The circular orbit is only a singular solution, and not a case of the general integral of the differential equation, and is described with *Kinetic Instability*.

5. To fix the ideas as to the mode in which the condition of equal roots is approached, as well as to obtain convenient expressions for the condition and for the criterion of stability, let us assume u_0 as a possible value of u, say that of a point of projection, and then express C and h in terms of two other parameters. For this purpose take v_0, the initial velocity, and γ, the angle between the direction of the initial radius vector r_0 and that of projection.

Observe that C denotes the total energy of a unit mass in the orbit and that h is the double area described by the radius vector in a unit of time. Hence

$$C = V_0 + \tfrac{1}{2}v_0^2, \qquad h = r_0 v_0 \sin\gamma, \tag{3}$$

and $\psi(u)$ becomes

$$\psi(u) = \frac{2u_0^2}{v_0^2 \sin^2 \gamma}(V_0 - V) + \frac{u_0^2}{\sin^2 \gamma} - u^2,$$

which makes $\psi(u_0)$ positive. Now supposing C, and therefore v_0, to be fixed, h is restricted to be not greater than $r_0 v_0$, which is its maximum value, corresponding to $\gamma = 90°$ and making u_0 an apsidal value. The value of $\psi(u)$ when the point of projection is an apse is therefore

$$\psi(u) = \frac{2u_0^2}{v_0^2}(V_0 - V) + u_0^2 - u^2, \tag{4}$$

which makes $\psi(u_0) = 0$.

6. If now C (and v_0) has such a value that when $\gamma = 90°$ we reach the case of equal roots, we must have $\psi'(u_0) = 0$. Differentiating equation (4),

$$\psi'(u) = \frac{2u_0^2}{v_0^2}\left(-\frac{dV}{du}\right) - 2u;$$

or, since $-\dfrac{dV}{du} = \dfrac{P}{u^2}$,

$$\psi'(u) = \frac{2u_0^2 P}{v_0^2 u^2} - 2u, \tag{5}$$

and

$$\tfrac{1}{2}\psi'(u_0) = \frac{P_0}{v_0^2} - u_0.$$

For equal roots we must therefore have (as well as $\gamma = 90°$,)

$$v_0^2 = P_0 r_0; \tag{6}$$

in other words, the centrifugal force must equal the attraction at the initial point.

7. Passing now to the criterion of stability, observe that when the circular orbit is stable $\psi(u_0)$, of which the value is zero, is a maximum value of $\psi(u)$, and this requires $\psi''(u_0)$ to be negative.

Differentiating (5) and putting $P_0 r_0$ for v_0^2 by (6), we find

$$\psi''(u) = \frac{2u_0^3}{P_0}\left[\frac{1}{u^2}\frac{dP}{du} - \frac{2P}{u^3}\right] - 2,$$

whence

$$\tfrac{1}{2}\psi''(u_0) = \frac{u_0}{P_0}\frac{dP}{du}\bigg]_0 - 3.$$

It follows that, while with any law of central attraction a circular orbit is possible with any radius r_0, it will be $\left\{\begin{matrix}\text{stable}\\ \text{unstable}\end{matrix}\right\}$ according as $\left.\frac{u_0}{P_0}\frac{dP}{du}\right]_0$ is $\left\{\begin{matrix}\text{less}\\ \text{greater}\end{matrix}\right\}$ than 3.*

8. The case in which $P = \mu u^3$ is peculiar, since the criterion is then identically equal 3. The special case occurs when $C = 0$, the orbit being an equiangular spiral unless $h^2 = \mu$, which makes $\gamma = 90°$, when it becomes a circle, and the circle must be regarded as described with kinetic instability.

LAGRANGE'S PLACE IN THE THEORY OF SUBSTITUTIONS. †

BY DR. JAMES PIERPONT.

IN the present brief note I cannot vindicate Lagrange's right to the title of creator of the theory of substitutions; but I hope, by presenting a few examples of his methods, to show the importance of considering him from this point of view. Lagrange was led to the study of this theory by his attempts to solve equations of degree higher than the fourth. Speaking of the inherent difficulties which this thorny subject offered to the investigator, he remarks: ‡

"The theory of equations is of all parts of analysis the one, we would think, which ought to have acquired the greatest degree of perfection, by reason both of its importance and of the rapidity of the progress that its first inventors made; but although the mathematicians of later days have not ceased to apply themselves, there remains much in order that their efforts may meet with the success that one could desire. In regard to the resolution of literal equations one has hardly advanced further than one was in Cardan's time, who was the first to publish the resolution of equations of the third and fourth degree. The first successes of the Italian analysts in this branch seem to have marked the limit of possible discoveries: at least it is certain that all attempts that have been made up to the present to push back the limits of this branch of algebra have hardly served for other purposes than

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* An equivalent criterion is otherwise derived in Thomson and Tait's Natural Philosophy, § 350.

† Read before the Yale Mathematical Club.

‡ Lagrange: Nouveaux Mémoires, Acad. Sciences Berlin, years 1770–71. Also, Œuvres, vol. III, pp. 205–421, Réflexions sur la résolution algébrique des équations.