tribute to the development of science, the really new impulses can be traced back to but a small number of eminent men. But the work of these men is by no means confined to the short span of their life; their influence continues to grow in proportion as their ideas become better understood in the course of time. This is certainly the case with Riemann. For this reason you must consider my remarks not as the description of a past epoch, whose memory we cherish with a feeling of veneration, but as the picture of live issues which are still at work in the mathematics of our time.

THE MULTIPLICATION OF SEMI-CONVERGENT SERIES.

BY PROFESSOR FLORIAN CAJORI.

IN *Math. Annalen*, vol. 21, pp. 327-378, A. Pringsheim developed sufficient conditions for the convergence of the product of two semi-convergent series, formed by Cauchy's multiplication rule, when one of the series becomes absolutely convergent, if its terms are associated into groups with a finite number of terms in each group. The necessary and sufficient conditions for convergence were obtained by A. Voss (*Math. Annalen*, vol. 24, pp. 42–47) in case that there are two terms in each group, and by the writer (*Am. Jour. Math.*, vol. 15, pp. 339–343) in case that there are p terms in each group, p being some finite integer. In this paper it is proposed to deduce the necessary and sufficient conditions in the more general case when the number of terms in the various groups is not necessarily the same.

Let $U_n = \sum\limits_0^n a_n$ and $V_n = \sum\limits_0^n b_n$ be two semi-convergent series, and let the first become absolutely convergent when its terms are associated into groups with some finite number of terms in each group. Let r_n represent the number of terms in the $(n+1)$th group, and let g_n represent the $(n+1)$th group embracing r_n terms. Let, moreover, a_{R_n} represent the first term in the group g_n, where $R_0 = 0$ and $R_n = r_0 + r_1 + r_2 + \ldots + r_{n-1}$, then

$$g_n = (a_{R_n} + a_{R_n+1} + a_{R_n+2} + \ldots + a_{R_n+r_n-1}) \quad \text{and} \quad U_n = \sum_0^n g_n.$$

Since, by a theorem of Mertens, the product of an absolutely convergent series and a semi-convergent series, formed by Cauchy's multiplication rule, is absolutely convergent, it follows that

$$UV = \sum_0^\infty (g_n b_0 + g_{n-1} b_1 + \ldots + g_0 b_n).$$

If the product $\sum_0^n (a_n b_0 + a_{n-1} b_1 + \ldots + a_0 b_n)$ of $\sum_0^n a_n$ and $\sum_0^n b_n$ is convergent, then, by a theorem of Abel, it converges to UV. Hence it follows that the necessary and sufficient condition for the convergence of this product is that

$$\sum_0^\infty (g_n b_0 + g_{n-1} b_1 + \ldots + g_0 b_n) = \sum_0^\infty (a_n b_0 + a_{n-1} b_1 + \ldots + a_0 b_n). \quad \text{(I)}$$

Taking $n = R_m + t$, where m is some integer less than n and where t may have any value $0, 1, 2, \ldots, (r_m - 1)$, we get

$$\begin{aligned}
&\sum_0^n (g_n b_0 + g_{n-1} b_1 + \ldots + g_0 b_n) - \sum_0^n (a_n b_0 + a_{n-1} b_1 + \ldots + a_0 b_n) = \\
&\qquad b_0 \{ a_{n+1} + a_{n+2} + \ldots + a_{R_{n+1}-1} \} \\
&\qquad + b_1 \{ a_n + a_{n+1} + \ldots + a_{R_n - 1} \} \\
&\qquad + b_2 \{ a_{n-1} + a_n + \ldots + a_{R_{n-1}-1} \} + \ldots \\
&\qquad + b_n \{ a_1 + a_2 + \ldots + a_{R_1 - 1} \} = \\
&b_0 \{ a_{R_m+t+1} + a_{R_m+t+2} + \ldots + a_{R_{m+1}-1} \} + b_0 \{ g_{m+1} + g_{m+2} + \ldots + g_n \} \\
&+ b_1 \{ a_{R_m+t} + a_{R_m+t+1} + \ldots + a_{R_{m+1}-1} \} \\
&\qquad + b_1 \{ g_{m+1} + g_{m+2} + \ldots + g_{n-1} \} + \ldots \\
&+ b_t \{ a_{R_m+1} + a_{R_m+2} + \ldots + a_{R_{m+1}-1} \} + b_t \{ g_{m+1} + g_{m+2} + \ldots + g_{n-t} \} \\
&\qquad + b_{t+1} \{ g_m + g_{m+1} + \ldots + g_{n-t-1} \} \\
&+ b_{t+2} \{ a_{R_m - 1} \} + b_{t+2} \{ g_m + g_{m+1} + \ldots + g_{n-t-2} \} + \ldots \\
&+ b_{r_{m-1}+t} \{ a_{R_{m-1}+1} + a_{R_{m-1}+2} + \ldots a_{R_m - 1} \} \\
&\qquad + b_{r_{m-1}+t} \{ g_m + g_{m+1} + \ldots + g_{n-r_{m-1}-t} \} \\
&\qquad + b_{r_{m-1}+t+1} \{ g_{m-1} + g_m + \ldots g_{n-r_{m-1}-t-1} \} + \ldots \\
&+ b_{R_m - R_s + t + 2} \{ a_{R_s - 1} \} + b_{R_m - R_s + t + 2} \{ g_s + g_{s+1} + \ldots + g_{R_s - 2} \} + \ldots \\
&+ b_{R_m - R_{s-1} + t} \{ a_{R_{s-1}+1} + a_{R_{s-1}+2} + \ldots + a_{R_s - 1} \} \\
&\qquad + b_{R_m - R_{s-1} + t} \{ g_s + g_{s+1} + \ldots + g_{R_s - 1} \} \\
&\qquad + b_{R_m - R_{s-1} + t + 1} \{ g_{s-1} + g_s + \ldots + g_{R_{s-1} - 1} \} + \ldots \\
&+ b_{R_m + t - 1} \{ a_2 + a_3 + \ldots + a_{r_0 - 1} \} + b_{R_m + t - 1} \{ g_1 \} \\
&+ b_{R_m + t} \{ a_1 + a_2 + \ldots + a_{r_0 - 1} \}. \qquad \text{(II)}
\end{aligned}$$

We proceed to show that all the terms on the right, involving g's, approach the limit zero as n increases indefinitely. Notice that the line in which any group g_y occurs for the first time is the one involving $b_{R_m-R_y+t+1}$, provided that we agree to take b_0 whenever $R_m - R_y + t + 1$ gives a negative number. Observe, moreover, that the line in which g_y occurs for the last time is the line involving b_{n-y}, and that g_y occurs once in all the intervening lines. This enables us to express all the terms in (II) which contain g's, as follows (letting $m = 2s$ or $2s + 1$):

$$\begin{aligned} &g_1\{b_{R_m-R_1+t+1}+b_{R_m-R_1+t+2}+\ldots+b_{n-1}\} \\ &+ g_2\{b_{R_m-R_2+t+1}+b_{R_m-R_2+t+2}+\ldots+b_{n-2}\}+\ldots \\ &+g_s\{b_{R_m-R_s+t+1}+b_{R_m-R_s+t+2}+\ldots+b_{n-s}\} \\ &+ g_{s+1}\{b_{R_m-R_{s+1}+t+1}+b_{R_m-R_{s+1}+t+2}+\ldots+b_{n-s-1}\}+\ldots \\ &+g_n\{b_0\} \equiv E. \end{aligned}$$

Since $\sum\limits_0^n g_n$ is absolutely convergent and $\sum\limits_0^n b_n$ is convergent, we can choose a positive finite quantity β and an infinitesimal ϵ_s, approaching the limit zero as s increases indefinitely, so that

$$|E| < \epsilon_s\{|g_1|+|g_2|+\ldots+|g_s|\} + \beta\{|g_{s+1}|+|g_{s+2}|+\ldots+|g_n|\}.$$

As s increases indefinitely, the right member of this inequality approaches the limit zero. Hence E approaches zero, and the condition that (I) be satisfied, for $n = R_m + t$, is that the sum of the terms in (II) which do not involve g should approach the limit zero as n increases indefinitely. Neglecting, as we may, a finite number of terms, this condition can be expressed thus:

$$\lim_{m=\infty} \sum_{i=1}^{i=m} (b_{R_m-R_i+t+2}a_{R_i-1} + b_{R_m-R_i+t+3}\{a_{R_i-2} + a_{R_i-1}\} + \ldots + b_{R_m-R_{i-1}+t}\{a_{R_{i-1}+1} + \ldots + a_{R_i-1}\})=0. \quad \text{(III)}$$

In using this formula we must observe that, if a group contains only two terms, so that $R_i - R_{i-1} = r_{i-1} = 2$, then, for the particular value or values of i which give $r_{i-1} = 2$, the outermost parenthesis in (III) represents only one term, $b_{R_m-R_i+t+2}a_{R_i-1}$. If, for some particular value or values of i, $r_{i-1} = 1$, then, for those values of i, the parenthesis does not represent anything whatever. As t may have any value $0, 1, 2, \ldots, (r_m - 1)$, we see that $R_m + t$ represents any value of n, and (III) *embodies r_m equations which together constitute the necessary and sufficient conditions for the existence of* (I) *and, therefore, for the convergence of the product of the two given series $\sum\limits_0^n a_n$ and $\sum\limits_0^n b_n$.*

As the numerical value of r_m may vary for every new value of m, the number of conditions is indicated by the highest value taken by r_m, as m increases indefinitely.

Another set of necessary and sufficient conditions can be deduced from conditions (III), viz.:

Cauchy's multiplication rule is applicable to $\sum\limits_0^n a_n$ *and* $\sum\limits_0^n b_n$, *if* ONE *of the conditions* (III) *is satisfied and the nth term of the product-series always approaches zero.*

We first prove that, if the $(n+1)$th term in the product-series approaches the limit zero, as n increases indefinitely, and if the tth condition is satisfied, then the $(t+1)$th condition is satisfied. We have

$$\begin{aligned}\operatorname*{Lim}_{m=\infty}\Bigg[\sum_{i=1}^{i=m}\Big(&b_{R_m-R_i+t+1}a_{R_i-1}+b_{R_m-R_i+t+2}\{a_{R_i-2}+a_{R_i-1}\}+\dots\\ &+b_{R_m-R_{i-1}+t-1}\{a_{R_{i-1}+1}+\dots+a_{R_i-1}\}\Big)\\ &+\sum_{i=2}^{i=m}g_{i-1}b_{R_m-R_{i-1}+t}-\sum_{x=0}^{x=R_m+t}a_{R_m+t-x}b_x\Bigg]\\ =\operatorname*{Lim}_{m=\infty}\sum_{i=1}^{i=m}\Big(&b_{R_m-R_i+t+2}a_{R_i-1}+b_{R_m-R_i+t+3}\{a_{R_i-2}+a_{R_i-1}\}+\dots\\ &+b_{R_m-R_{i-1}+t}\{a_{R_{i-1}+1}+\dots+a_{R_i-1}\}\Big).\end{aligned}$$

Remembering that $\sum\limits_0^n g_n$ is absolutely convergent, it will be seen that the first member of the equation approaches the limit zero as m increases indefinitely; hence the second member approaches the limit zero. Consequently, if the tth condition is satisfied, then the $(t+1)$th is. If the $(t+1)$th is satisfied and the next following term in the product-series approaches the limit zero, then the $(t+2)$th is satisfied, and so on.

If all groups g_n contain the same number p of terms, then we may write $R_m = pm$ and $R_i = pi$. If we take $m - i = i'$ and substitute, we obtain the conditions (II) given in the *Am. Jour. of Math.*, vol. 15, p. 341.

By repeating the reasoning given on p. 342 of the article just referred to, we may derive from the necessary and sufficient conditions (III) of the present article the sufficient conditions developed by A. Pringsheim (*Math. Annalen*, vol. 21, p. 334), only now we are no longer restricted by the assumption that the number of terms shall be the same in all the groups.

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