where λ, μ, ν, ρ are tangential co-ordinates. The ratios of the differences of $\alpha, \beta, \gamma, \delta$ give the two degrees of freedom which the curve still possesses. Now by eliminating t we may write down

$$\left.\begin{aligned}(\beta-\gamma)\mu\nu+(\gamma-\alpha)\nu\lambda+(\alpha-\beta)\lambda\mu=0,\\(\beta-\delta)\mu\rho+(\delta-\alpha)\rho\lambda+(\alpha-\beta)\lambda\mu=0,\\(\gamma-\delta)\nu\rho+(\delta-\beta)\rho\mu+(\beta-\gamma)\mu\nu=0,\end{aligned}\right\}\qquad(10)$$

which are three tangential quadrics touching the osculating planes of the curve and not connected by a linear relation. Substituting, then, $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \frac{d}{du}$ for λ, μ, ν, ρ, respectively, in these three expressions and operating on the quadric given above, we get

$$\begin{aligned}(\beta-\gamma)f+(\gamma-\alpha)g+(\alpha-\beta)h=0,\\(\beta-\delta)m+(\delta-\alpha)l+(\alpha-\beta)h=0,\\(\gamma-\delta)n+(\delta-\beta)m+(\beta-\gamma)f=0,\end{aligned}$$

whence

$$\begin{aligned}(h-l)(g-f)(f-m)+(h-g)(m-h)(n-f)\\+(h-f)(h-l)(n-f)=0,\qquad(11)\end{aligned}$$

which is consequently the invariant relation connecting the cubic surface and the quadric when they are capable of being written in the forms (5). Thus it appears that a cubic surface and a quadric cannot, in general, be reduced to the forms (5), and that when they are reducible to these forms, the reduction can take place in a singly infinite number of ways, all the planes x_1, x_2, etc., involved being osculating planes of a given twisted cubic.

HAYWARD'S VECTOR ALGEBRA.

The Algebra of Coplanar Vectors and Trigonometry. By R. BALDWIN HAYWARD. Macmillan & Co., 1892. 8vo, pp. xxix + 343.

IT is a curious fact that while the English are the one nation which in elementary geometry clings to Euclid, the prototype of mathematical rigor, not only is most recent English mathematical work, however excellent in many respects, decidedly lacking in rigor of form, but many English mathematical writers of the present day show an entire lack of critical sense which if shown in elementary geometry would discredit a schoolboy.

The book we now have under consideration is an attempt to build up the theory of complex quantities and of the simpler functions of a complex variable in a systematic manner, presupposing only a moderate acquaintance with arithmetic and the elementary algebra of real numbers. In a work of this sort it is surely not pedantic to expect a certain amount of logical rigor. This need not perhaps be of the highest sort in which, as in the geometrical system of Euclid or the arithmetical system of Weierstrass and others, every proposition is an immediate deduction from one or more fundamental statements (definitions, axioms, etc.). Each proposition should, however, be a logical deduction from *something* which if not one of the formally stated axioms or definitions, must appear self-evident upon careful consideration.* Nothing of this sort is to be found in this book. Take, for instance, the matter of the "permanence of equivalent forms." This principle is stated, using the words of Peacock, as follows (page 4): "All results of algebra which are general in form, though they have been established as true only for restricted meanings of the symbols, must also be true when the symbols are general in value as well as in form." The language here used is so vague that no definite meaning can be attached to it. Moreover, when this principle is applied to the subject of infinite series it is modified by the requirement that the series in question should converge, but not otherwise. We may mention, for example, the treatment of the binomial theorem for negative or fractional exponents. The author never doubts that whenever we get a convergent series by applying the formula obtained for positive integral exponents this series will converge to the desired value. It will be seen that Mr. Hayward has either never heard of the classic researches of Abel (*Crelle*, vol. I., 1826), or regards them as utterly unnecessary. This one example may suffice. The language is constantly open to misinterpretation, and often, as has just been pointed out, the trouble is not merely in the form of expression.

It will be seen from what has already been said that the book is not one which can safely be placed in the hands of a beginner. To the more mature mathematical reader, however, who is able to sift the grain from the chaff it presents much which is of interest.

---

* By this is meant that we may in ordinary mathematical work make use of propositions which we cannot prove, but which our geometrical intuition tells us must be true; as, for instance, the proposition that a continuous curve which crosses a horizontal line twice must have a horizontal tangent at some intermediate point. Even such a slight departure as this from true mathematical rigor may, however, in some extreme cases lead us into error.

There are two ways in which the idea of number can be developed. The first, and from the scientific point of view undoubtedly the best, is the formal method of Weierstrass, Cantor, and others, which consists in defining numbers as symbols to which no concrete meaning is at first attached, but which can be combined with one another according to certain arbitrarily assigned laws. After the arithmetic of such numbers has been developed, it may be applied to various concrete (for instance, geometrical) questions.

The second method is to define numbers as symbols standing for certain concrete conceptions; for instance, the ordinary real numbers would be best defined as lengths measured upon a straight line. The obvious advantage of this method is the concreteness which it gives to the whole subject; an advantage so great that this appears to be the only practicable way of treating irrational numbers until the student has attained a very great degree of maturity. The plan which Mr. Hayward follows is the application of this method to the treatment of the imaginaries of ordinary algebra. The first subject taken up is therefore the geometrical one of *vectors,* the case in which the vectors lie in a single plane being considered almost exclusively. After devoting one chapter to the addition and subtraction of vectors, the multiplication of coplanar vectors is next taken up, the definitions here laid down being of course not those of quaternions. A particular vector is chosen as unit, its length being the unit of length and its direction being that from which the inclination of other vectors is measured. The product of two vectors is then defined as that vector which can be obtained from the multiplicand by the same amount of turning and stretching as is necessary to pass from the unit vector to the multiplier. The commutative, associative, and distributive laws are established, and the allied questions of division and raising to real powers are discussed. It is then shown—and here the question of complex numbers first comes in—that if we define the complex number $a + b\sqrt{-1}$ to be the vector whose components in the direction of the unit vector and in the direction obtained by turning this through the positive angle of 90° are a and b respectively, the complex quantities thus defined will obey all the ordinary laws of algebra. It will be seen that for Mr. Hayward complex numbers *are* vectors, whereas it seems much better to say merely that they *may be represented by* vectors. This may seem a small matter, but we are inclined to think that it is worth while to bring it clearly before the student at this point. *

* This is admirably done by De Morgan in his "Trigonometry and Double Algebra," 1849, a book which, as Mr. Hayward states in his preface, has served him in many respects as a model.

In chapter III comes the introduction of the trigonometric functions of a real angle, which are discussed graphically with some fulness. Many of the properties of these functions (addition theorems, etc.) are deduced in the following chapter directly from De Moivre's theorem. This method of obtaining these formulæ is not only extremely elegant, but has the advantage of avoiding, without any effort on the part of the reader, the necessity of considering separately the various cases which arise according as the angles lie in one or another quadrant.

"Vector Indices and Logarithms" is the title of chapter V, one of the least satisfactory chapters of the book. For instance, on page 108 the value of a certain constant (η) is determined on the assumption that it exists—an assumption which is nowhere justified. Later in the chapter we are pleasantly surprised by finding a rigorous, though perhaps not the best, treatment of the exponential limit.

Next follows a chapter on the hyperbolic functions, which is marred, as is in fact the whole book, by the use of new terms in place of universally accepted ones. The author "hopes to carry mathematicians with him in giving the name of *excircle* to the *rectangular hyperbola*." The hyperbolic functions are of course spoken of as excircular functions, though the ordinary notation for them is not changed. Other innovations in terminology are *project* and *traject* for the real and imaginary parts of a complex number respectively; and *logometer* (introduced by De Morgan) for the general logarithm.

A readable chapter on the roots of unity made clear by a constant reference to figures closes what may be regarded as the first part of the book.

This is followed by four chapters on infinite series and products. The first of these begins with an instructive and somewhat detailed graphical discussion of the geometric series. Then comes a rather brief treatment of the convergency of series of complex terms in general, including a treatment of the question of infinitely slow convergency. It goes without saying that we welcome the recent appearance of this subject in any form in English books. It is, however, an extremely difficult subject for the student to grasp, and should be amply illustrated by the graphical method for series with real terms before those with complex terms are taken up. Moreover, it is a subject in which the statements must be made with the greatest care if they are to convey the correct idea. Would it not have been better if Mr. Hayward had passed beyond the standpoint of Stokes and Seidel (now nearly fifty years old) and treated the subject, as it is now treated on the Continent, from the point of view of uniform convergency, which is both more general and less likely to suggest false ideas?

In the following chapters—Expansions and Summations; Series of Products (i.e., infinite products); Series of Partial Fractions—many well-known matters are taken up and are often illustrated in an interesting graphical manner. As in the case of the exponential limit so again on page 279 we are surprised at finding a sound treatment of the development of *sin u* in an infinite product. Our surprise is, however, lessened when we note from the remark near the bottom of page 278 that Mr. Hayward does not appreciate the proof which he gives, that the product converges to the *right* value, but thinks it would be sufficient merely to prove that it converges.

The last chapter, on Rational and Integral Functions, is a slight introduction (geometrical, of course, as we believe such an introduction should be) to what is ordinarily spoken of as the theory of functions. A well-known but incomplete proof that every equation has a root will be found here which deserves special mention as it has crept into even so good a book as Harkness and Morley's Theory of Functions. Then follows Cauchy's theorem for determining the number of roots within a given contour; the proof given being, as Mr. Hayward says, simpler than that usually given. It must, however, be said that the possibility of factoring a rational integral function into linear factors has here been otherwise proved, while it is ordinarily desired to prove this fact as a corollary to Cauchy's theorem. Finally, the question of conformal representation is taken up, especially for cubic functions with real coefficients; but only the images of the lines parallel to the axes of reals and of imaginaries in the plane of the independent variable are considered. Among other things an interesting general theorem, due, we believe, to F. Lucas, concerning the asymptotes of these curves is given.

As has already been sufficiently emphasized, Mr. Hayward does not treat his subject with that degree of mathematical rigor which it seems to us to demand. On the other hand, the constant use of geometrical illustration gives the book a freshness and interest which many a mathematical writer might well envy.

MAXIME BÔCHER.

HARVARD UNIVERSITY.