EDWARDS' DIFFERENTIAL CALCULUS.

An Elementary Treatise on the Differential Calculus, with applications and numerous examples. By JOSEPH EDWARDS, M.A., formerly Fellow of Sidney Sussex College, Cambridge. Second edition, revised and enlarged. London and New York, Macmillan & Co. 1892. 8vo, pp. xiii + 521.

WHEN a mathematical text book reaches a second edition, so much enlarged as this, we know at once that the book has been received with some favour, and we are prepared to find that it has many merits. We are at once struck by Mr. Edwards' lucid and incisive style; his expositions are singularly clear, his words well chosen, his sentences well balanced. In the text of the book we meet with various useful results, notably in the chapter on "some well known curves," and moreover the arrangement is such that these results are easy to find; and in addition to these, numbers of theorems are given among the examples, and, this being a feature for which we are specially grateful, in nearly every case the authority is cited. Recognizing these merits, however, we notice that the book has many defects, some proper to itself, some characteristic of its species; and just because it is so attractive in appearance, it seems worth while examining it in detail, and pointing out certain specially vicious features.

A book of this size may fairly be required to serve as a preparation for the function theory; at all events, the influence of recent Continental researches should be evident to the eyes of the discerning. Mr. Edwards' preface strengthens this reasonable expectation, for he promises us "as succinct an account as possible of the most important results and methods which are up to the present time known." But we soon find that the "important results and methods" are those of the Mathematical Tripos; and in our disappointment we utter a fervent wish that instead of the "large number of university and college examination papers, set in Oxford, Cambridge, London, and elsewhere," Mr. Edwards had consulted an equally large number of mathematical memoirs published, principally, elsewhere. The Mathematical Tripos for any given year is not intended for a *Jahrbuch* of the progress of mathematics during the past year; and as long as so many will insist on regarding it in that light, text books of this type will continue to be published.

Nothing in this book indicates that Mr. Edwards is familiar with such works as Stolz's *Allgemeine Arithmetik,* Dini's *Fondamenti per la teorica delle funzioni di variabili reali,* or Tannery's *Théorie des fonctions d'une variable.* In support of our contention we may instance the definitions of function, limit, continuity, etc. On page 2, Lejeune Dirichlet's definition of a function is adopted. According to this very general definition, there need be no analytical connection between y and x; for y is a function of x even when the values of y are arbitrarily assigned, as in a table. That Mr. Edwards does not adhere to this definition is evident from his tacit assumption that *every* function $\varphi(x)$ can be represented by a succession of continuous arcs of curves. Whatever definition is adopted for a continuous function y of x, it is evident that to small increments of x must correspond small increments of y; but Weierstrass has proved that there exist functions which have this property, but which have nowhere differential coefficients. The well known example of such a function is

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n x \pi),$$

where a is an odd integer, b a positive constant less than 1, and ab greater than $1 + 3\pi/2$. According to the accepted definition, this function of x is continuous; according to Mr. Edwards' definition, it is not continuous, inasmuch as it cannot be represented by a curve $y = f(x)$ with a tangent at every point.

We acknowledge that Mr. Edwards displays a considerable degree of consistency in his view of the meaning of a continuous function, but we insist that after the adoption of the curve definition he should have been at some pains to prove that the numerous series of the type $\sum_1^{\infty} f_n(x)$ scattered throughout the book give rise to curves with tangents, whereas he never even takes the trouble to prove that they are continuous functions of x in any sense of the term. No more damaging charge can be brought against any treatise laying claim to thoroughness than that of recklessness in the use of infinite series; and yet Mr. Edwards has everywhere laid himself open to this charge. One of the most difficult things to teach the beginner in mathematics is to give proper attention to the convergency of the series dealt with. All the more need, then, that a text book of this nature should set an example of consistent, even *aggressive* carefulness in this respect. We do, it is true, find an occasional mention of convergence (pp. 9, 81, 454, etc.), but as a rule it is ignored. Mr. Edwards rearranges the terms of infinite and doubly infinite series, applying the law of commutation without pointing out that his series are unconditionally convergent; he differentiates $f(x) = \sum_1^{\infty} f_n(x)$ term by term, and gets $f'(x) = \sum_1^{\infty} f'_n(x)$, implying that the process is universally valid (*e.g.* p. 84); or, at all events, giving no hint that there are cases in which the differential coefficient of the sum of a convergent series is different from the sum of the differential coefficients of the individual terms. We find no formal recognition of the importance of uniform convergence in modern analysis, nothing even to suggest that he has ever heard of the distinction between uniform and non-uniform convergence. We begin to suspect that he has never looked into Chrystal's Algebra.

The unreasoning mechanical facility thus acquired in performing operations unhampered by any doubts as to their legitimacy, naturally leads Mr. Edwards to view with favour "the analytical house of cards, composed of complicated and curious formulæ, which the academic tyro builds with such zest upon a slippery foundation," *—and to build up many a one. A curious and interesting specimen is

$$f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$$

to be continued to infinity. This expression has been examined by Seidel,† who points out that Eisenstein's paper in *Crelle*, vol. 28, requires correction. Before such an expression can be differentiated, a definite meaning must be assigned to it; but Seidel's conclusion is that, denoting x^x by x_1, x^{x_1} by x_2, x^{x_2} by x_3, and so on, then as x varies from 0 to $1/e^e$, $\underset{n=\infty}{L}\ x_{2n}$ increases from 0 to $1/e$, while $\underset{n=\infty}{L}\ x_{2n+1}$ decreases from 1 to $1/e$; beyond these limits for x, the case is different. In particular when $x > e^{1/e}$, the expression diverges. Our objection is not to the non-acceptance of Seidel's conclusions, but to the unnecessary use of a function of this doubtful character. Examples can be found to illustrate every point that ought to be brought up in an elementary treatise on the differential calculus without ranging over examination papers in search of striking novelties.

Feeling now somewhat familiar with Mr. Edwards' point of view, we examine his proofs of the ordinary expansions with a tolerably clear idea of what we are to expect. We find, of course, "the time-honoured short proof of the existence of the exponential limit, which proof is half the real proof plus a *suggestio falsi*"; we find in the chapter on expansions a general disregard of convergency considerations; we find throughout the book the assumption that

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* Professor CHRYSTAL, in *Nature*, June 25, 1891.

† *Abhandlungen der k. Ak. d. Wiss.* Bd. xi.

$\varphi(a) = \underset{x=a}{L}\, \varphi(x)$, and that $\varphi(0, 0) = \underset{x=0,\, y=0}{L} \varphi(x, y)$ *; we find the usual assumptions as to expansibility in series proceeding by integral powers, with disastrous results further on. We find the usual dread of the complex variable, though Mr. Edwards has given one or two examples involving it, without however explaining what is meant by $f(x + iy)$. We can hardly regard these examples, even with § 190, as a sufficient recognition of the complex variable in a treatise of this size. We must notice also the thoroughly faulty treatment of the inverse functions. For example, no explanation is given of the signs in $\frac{dy}{dx}$, when $y = \cos^{-1} x$ or $\sin^{-1} x$. Mr. Edwards' attitude towards many valued functions is simple enough; as a rule, he ignores the inconvenient superfluity of values. He does, it is true, give in § 54 a note, clear and correct, on this point; but he is very careful to confine this within the limits of the single section, and to indicate, by choice of type, that it is quite unimportant.

We pass on now to the second part, applications to plane curves; and here we must object emphatically to the introduction of so many detached and disconnected propositions relating to the theory of higher plane curves. From Mr. Edwards' point of view this is doubtless justified; we are quite ready to acknowledge that we know of no book that would enable a candidate to answer more questions on subjects of whose theory he is totally ignorant. The deficiency of a curve, *e.g.*, is a conception entirely independent of the differential calculus; but probably this single page will obtain many marks for candidates in the Mathematical Tripos; these we should not grudge if we thought an equivalent would be lost by a reproduction of Mr. Edwards' treatment of cusps. Our spirits rose when we remarked the italicised phrase on p. 224, that there is "*in general a cusp*" when the tangents are coincident. But three pages further on we find that the exception here indicated is simply our old friend, the conjugate point, whose special exclusion from the class in which it appears must be a perpetual puzzle to a thoughtful student with no better guidance than a book of this kind. Such a student, probably already familiar with projection, knows that the real can be projected into the imaginary, and the imaginary into the real. If then the acnode, appearing as a cusp, has to be specially excluded, why not the crunode? But here Mr. Edwards reproduces the now well established

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* See *e.g.* p. 122; and on this page note also the assumption that the relation between h, k, while $x + h$, $y + k$, tend to the limits x, y exerts no influence on the result.

error, calling tacnodes, formed by the contact of real branches, double cusps of the first and second species, and excluding those formed by the contact of imaginary branches; he even goes further astray, introducing Cramer's osculinflexion as a cusp that changes its species.

This matter of double cusps is a fundamentally serious one, and not a mere question of nomenclature. This persistent misnaming effectually disguises the essential characteristic of the cusp. It is *not the coincidence of the tangents* that makes a cusp. From the geometrical point of view it is the turning back of the (real) tracing point, expressed by the French and German names, {*point de rebroussement, Rückkehrpunkt*}; from the point of view of algebraical expansions (of y in terms of x, $y = 0$ being the tangent) the essential characteristic of a single cusp is that at some stage in the expansion there shall be a fractional exponent with an even denominator, so that the branch changes from real to imaginary *along its tangent;* from the point of view of the function theory, which is really equivalent to the last, the simple cusp is characterised by the presence of a *Verzweigungspunkt* combined with a double point.. The simple cusp, that is, presents itself as an evanescent loop. A double cusp, then, in the sense in which Mr. Edwards uses the term, does not exist. There cannot be two consecutive cusps, vertex to vertex; for the branch if supposed continued through the cusp, changes from real to imaginary; and two *distinct* cusps, brought together to give a point of this appearance, produce a quadruple point.

While on this subject, we must mention Mr. Edwards' rule for finding the nature of a cusp. Find the two values of $\frac{d^2y}{dx^2}$; these by their signs determine the direction of convexity (§ 296). How does this apply *e.g.* to $y^2 = x^3$?

This confusion regarding cusps is made worse by the assumption already noticed that when $f(x, y) = 0$ is the equation of the curve, y can be expanded in a series of integral powers of x. This error is repeated on p. 258, where to obtain the branches at the origin, this being a double point, we are told to expand y by means of the assumption $y = px + \frac{qx^2}{2!} +$ etc. The whole exposition of this theory of expansion is most inadequate. In § 382 there is no hint that the terms obtained are the beginning of an infinite series, giving the expansion of (say) y in powers, not necessarily integral, of x; there is no hint what to do when the first terms of the expansion are found; there is no suggestion of the interpretation of the result when two expansions begin with the same terms. A thoughtful student *may* by a happy comparison of scattered examples (p. 200, and ex. 3, p. 230) arrive at the correct theory; but he surely deserves better guidance.

One or two more points must be noticed. The theory of asymptotes, when two directions to infinity coincide, cannot be satisfactorily developed without assuming a knowledge of double points; and the only way of giving the true geometrical significance is to introduce the conception of the line infinity, and to consider the nature of the intersections of the curve by this line. A tangent lying entirely at infinity does *not* "count as one of the n theoretical asymptotes"; if counted among the asymptotes at all, it has to be counted as the equivalent of two out of the n. This is one of the strongest arguments against including the line infinity in enumerating the asymptotes. The various expressions for the radius of curvature involve an ambiguity in sign; what is the meaning of this? The omission of this explanation causes obscurity, notably in § 330. The equation of a curve, referred to oblique axes, being $\varphi(x, y) = 0$, what is the condition for an inflexion? As a matter of fact it is the same as in the case of rectangular axes, given on p. 264; but as this is obtained from a formula for the radius of curvature, the investigation is not applicable. Throughout Mr. Edwards displays an almost exclusive preference for rectangular axes, and seems to regard the metric properties so obtained as of equal importance with descriptive properties. For instance, in the case of an ordinary double point (p. 224) instead of the *three* cases usually distinguished, we have *four*, the additional one being that of perpendicular tangents.

In the third part we notice that in the chapter on "undetermined forms" there is no discussion of the case of two variables, though it is on this that we have to rely for a rigorous proof of the theorem $\dfrac{\delta^2\varphi}{\delta x\delta y} = \dfrac{\delta^2\varphi}{\delta y\delta x}$. We recognize an old friend, the discussion of the limit of ∞/∞, in which it is first assumed, and then proved, that the limit exists. The statement of ex. 17, p. 457, is somewhat misleading; the formula there given for the expansion of $(x + a)^m$ is true when m is a positive integer; but when $m = -1$, it is evidently not true for $x = -b, -2b$, etc.* The treatment of maxima and minima of functions of two variables (§§ 497–501) is incomplete and incorrect. The geometrical illustration, as given on p. 424, omits the case of a section with a cusp, which is the simplest case that can occur when $rt = s^2$; of the more complicated cases Mr. Edwards attempts no discrimination; he does not even state correctly the principles that must guide us in this discrimination. The inexactness of the ordinary

* LAURENT, *Traité d'Analyse*, iii., 386.

criteria (given in § 498) appears at once from the example $u = (y^2 - 2px)(y^2 - 2qx)$ [Peano]. The origin is a point satisfying the preliminary conditions; taking then for x, y, small quantities h, k, the terms of the second degree are positive for all values except $h = 0$; when $h = 0$, the terms of the third degree vanish, and the terms of the fourth degree are positive; nevertheless the point does not give a minimum, which it should do by the test of § 498. For we can travel away from O in between the two parabolas, so coming to an adjacent point at which u has a small negative value, while for points inside or outside both parabolas the value of u is positive. The truth is, the nature of the value a of the function u at a point (x_0, y_0) at which $\frac{\delta \varphi}{\delta x}$ and $\frac{\delta \varphi}{\delta y}$ vanish, depends on the nature of the singularity of the curve $u = a$ at this point. If this curve has at (x_0, y_0) an isolated point of any degree of multiplicity, we have a true maximum or minimum of u; but if through (x_0, y_0) pass any number of real non-repeated branches of the curve, we have not a maximum or minimum; in Peano's example the branches coincide in the immediate neighbourhood of the origin, but then they separate, and therefore we have not a minimum value for u.

We object, then, to Mr. Edwards' treatise on the Differential Calculus because in it, notwithstanding a specious show of rigour, he repeats old errors and faulty methods of proof, and introduces new errors; and because its tendency is to encourage the practice of cramming "short proofs" and detached propositions for examination purposes.

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BRYN MAWR, PA., *May* 18, 1892.

NOTE ON RESULTANTS.

BY PROF. M. W. HASKELL.

ON page 151 of Prof. Gordan's lectures on determinants* is to be found the theorem

$$R_{f,\phi} = R_{f+\phi.\psi,\phi}$$

where $R_{f,\phi}$ denotes the resultant of two functions f and ϕ of a single variable x of degree m and n respectively. This

* *Vorlesungen über Invariantentheorie*, herausgegeben von KERSCHENSTEINER. Erster Band. Leipzig, 1885.