ON THE DOUBLY INFINITE PRODUCTS.*

BY DR. THOMAS S. FISKE.

THE familiar singly infinite products for the sine and cosine, due to Euler,†

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\ldots,$$

$$\cos x = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{9\pi^2}\right)\left(1 - \frac{4x^2}{25\pi^2}\right)\ldots,$$

or in another form,

$$\sin x = x \mathop{\Pi}_{-\infty}^{+\infty}\left[1 - \frac{x}{m\pi}\right],$$

$$\cos x = \mathop{\Pi}_{-\infty}^{+\infty}\left[1 - \frac{x}{(m+\frac{1}{2})\pi}\right],$$

were first generalized by Abel. By a brilliant stroke of genius he obtained for the elementary doubly periodic functions the remarkable expressions ‡

$$\operatorname{sn} x = \frac{x\,\Pi\,\Pi\left[1 - \frac{x}{m\omega + m'\omega'}\right]}{\Pi\,\Pi\left[1 - \frac{x}{(m+\frac{1}{2})\omega + (m'+\frac{1}{2})\omega'}\right]},$$

$$\operatorname{cn} x = \frac{\Pi\,\Pi\left[1 - \frac{x}{(m+\frac{1}{2})\omega + m'\omega')}\right]}{\Pi\,\Pi\left[1 - \frac{x}{(m+\frac{1}{2})\omega + (m'+\frac{1}{2})\omega'}\right]},$$

$$\operatorname{dn} x = \frac{\Pi\,\Pi\left[1 - \frac{x}{m\omega + (m'+\frac{1}{2})\omega'}\right]}{\Pi\,\Pi\left[1 - \frac{x}{(m+\frac{1}{2})\omega + (m'+\frac{1}{2})\omega'}\right]},$$

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† *Introductio in Analysin Infinitorum* (1748), lib. I. cap. IX.

‡ *Œuvres, Nouvelle édition* (1881), t. I., p. 343.
Journal für die reine u. angewandte Math. (CRELLE), Bd. II., p. 172.

in which m and m' are independent of each other and assume successively all integral values from $-\infty$ to $+\infty$, the simultaneous system $m = m' = 0$ alone being excluded in the numerator of the first fraction. Abel, however, did not make a complete and rigorous investigation as to the convergency of these products nor as to their identity with the functions of Jacobi. Cayley made the four doubly infinite products contained in the above expressions the starting point of a series of investigations.* He found for them a complete theory, based in part upon a geometrical interpretation, and upon it he built up the whole theory of the elliptic functions. Almost immediately afterwards, Eisenstein † discussed in a very elaborate manner, and by purely analytic methods, the general doubly infinite product

$$\Pi\,\Pi\left[1 - \frac{x}{m\alpha + n\beta + \gamma}\right],$$

and arrived at results which, when supplemented by the more recent theory of primary factors, due to Weierstrass,‡ have given to the subject a permanent and classical form.

The path which the student naturally follows in the study of the periodic functions, leads him directly to the consideration of these products and, at the same time, indicates their paramount importance. A theorem of Jacobi § shows him that no more general periodic functions of a single variable are possible than the doubly periodic or elliptic functions. He learns that such functions are but the ratios of single valued functions of another class, the so-called theta-functions; and these, it is soon seen, are nothing more or less than doubly infinite products. There is no doubt that the theory of the theta-functions of a single variable forms the natural introduction to that of the elliptic functions.

Before taking up the general products, the limit of the single product ‖

$$u = x \prod_{-p}^{q}\left[1 - \frac{x}{m\pi}\right],$$

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* *Camb. Math. Journ.*, vol. IV., 1845, pp. 257–277.
Journ. des Math. (Liouville), t. X., 1845, pp. 385–420.
Collected Math. Papers, vol. I., nos. 24 and 26.

† *Mathematische Abhandlungen*, Berlin, 1847, pp. 213–334.
Journal für die reine u. angewandte Math. (Crelle), Bd. XXXV., 1847, pp. 153–247.

‡ *Abhandlungen der Königl. Akad. der Wissenschaften zu Berlin vom Jahre* 1876.
Abhandlungen aus der Functionenlehre, von Karl Weierstrass. Berlin, 1886, pp. 1–52.

§ *Gesammelte Werke*, Bd. I., p. 262.

‖ Cf. Hermite, *Cours à la Sorbonne*, *Quatrième édition*, p. 89.

when p and q both become infinitely great, should be considered. It will be found to be indeterminate. In fact, if we have in the limit

$$\log \frac{p}{q} = a,$$

a being a given constant, then

$$u = e^{ax} \sin x.$$

A similar result holds for the infinite product representing $\cos x$.

In the investigations of Cayley corresponding results were developed in connection with the double products, for example

$$u = x\, \Pi\, \Pi \left[1 - \frac{x}{m\omega + m'\omega'} \right],$$

by the introduction of an auxiliary geometrical construction. The periods ω and ω' being always assumed respectively real and imaginary, a pair of rectangular axes were drawn, and corresponding to every factor in the product a point was set down, the coefficient of the real period being the abscissa and that of the imaginary period the ordinate. The entire finite portion of the plane was thus covered with a series of points forming the vertices of a net-work of squares constructed on the linear unit. These points were all enclosed within a contour of infinite dimensions, the form of which depended upon the relations between the infinite limits of the products. The value of the product was shown to depend upon the form of the contour, and in Cayley's memoirs the bounding contour is regarded successively as a square, a circle, and an infinite horizontal ribbon, and an infinite vertical ribbon.

By the application of logarithms one obtains

$$\log u = -x \Sigma\Sigma \frac{1}{m\omega + m'\omega'} - \frac{x^2}{2} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^2}$$
$$- \frac{x^3}{3} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^3} - \dots .$$

Making the contour symmetrical with respect to the origin the terms containing odd powers vanish, or

$$\log u = - \frac{x^2}{2} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^2} - \frac{x^4}{4} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^4} - \dots .$$

For another contour, similarly

$$\log u' = - \frac{x^2}{2} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^2} - \frac{x^4}{4} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^4} - \dots .$$

Hence

$$\log \frac{u}{u'} = \frac{x^2}{2} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^2} + \frac{x^4}{4} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^4} + \ldots,$$

the sums extending to the region enclosed between the two contours. All the terms except the first being infinitely small, we have

$$u' = ue^{-Bx^2}$$

where

$$B = \tfrac{1}{2} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^2} = \tfrac{1}{2} \iint \frac{dm\, dm'}{(m\omega + m'\omega')^2},$$

from which is readily seen the relation between two different systems of theta-functions. The system of theta-functions corresponding to the infinite horizontal ribbon is identical with that given by Jacobi.*

In Eisenstein's researches we have

$$u = \Pi\Pi \left[1 - \frac{x}{m\alpha + n\beta + \gamma}\right],$$

whence

$$\log u = -x \Sigma\Sigma \frac{1}{m\alpha + n\beta + \gamma} - \frac{x^2}{2} \Sigma\Sigma \frac{1}{(m\alpha + n\beta + \gamma)^2}$$

$$- \frac{x^3}{3} \Sigma\Sigma \frac{1}{(m\alpha + n\beta + \gamma)^3} - \ldots.$$

The whole theory is thus dependent upon that of the very general series.

$$\Sigma\Sigma \frac{1}{(m\alpha + n\beta + \gamma)^{\mu}}.$$

Eisenstein's elegant investigation as to the convergency of this series has been recognized as fundamental and has found its way into the text-books.† He deduced as the necessary condition for convergence

$$\mu > 2.$$

It follows that in the expansion of $\log u$ the coefficients of all the powers of x except the first two, have fixed sums indepen-

* JACOBI, *Fundamenta Nova* (1829), cap. 61.
† Cf. JORDAN, *Cours d'Analyse*, t. I., p. 165.

dent of the arrangement of their elements. Since however the first two coefficients may alter their values with a change in the arrangement of the factors of u, two functions which are related to each other in this way will be connected by an equation of the form

$$u' = u\, e^{-px - \frac{1}{2}qx^2}.$$

One finds in Eisenstein's memoir a very elaborate investigation as to the nature and value of the quantities p and q, and the results are applied to a general theory of the elliptic functions. In spite of the great interest of these further developments, it is unnecessary for our present purpose to enter into details upon them on account of the wonderful simplification brought about through Weierstrass's theory of primary factors.*

This theory enables us to express any continuous function which does not become infinite for finite values of the variable in a factorized form. It shows us, however, that the simplest factors of such a transcendental function, should differ from the linear factors of a rational entire algebraic function, in that each should have an exponential associated with it. Thus we find, according to this theory,

$$\sin x = x \prod_{-\infty}^{+\infty} \left[1 - \frac{x}{m}\right] e^{\frac{x}{m}}$$

an expression from which every element of indetermination has been eliminated. Now it is evident, after the investigations of Eisenstein, that we can remove the indetermination from the product

$$u = \Pi\,\Pi \left[1 - \frac{x}{m\alpha + n\beta + \gamma}\right]$$

by introducing the exponential factor

$$e^{x\Sigma\Sigma \frac{1}{m\alpha + n\beta + \gamma} + \frac{x^2}{2}\Sigma\Sigma \frac{1}{(m\alpha + n\beta + \gamma)^2}}$$

The result may be exhibited as a product of the form

$$\Pi\,\Pi \left[1 - \frac{x}{m\alpha + n\beta + \gamma}\right] e^{\frac{x}{m\alpha + n\beta + \gamma} + \frac{1}{2}\frac{x^2}{(m\alpha + n\beta + \gamma)^2}}$$

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* WEIERSTRASS, *loc. cit.*
Cf. also JORDAN, *Cours d'Analyse*, t. II., pp. 315-317.

This product consequently denotes a function of unique character possessing all the essential properties of an ordinary theta-function.

The special case given by the formula *

$$u = x\, \Pi\, \Pi \left(1 - \frac{x}{w}\right) e^{\frac{x}{w} + \frac{1}{2}\left(\frac{x}{w}\right)^2},$$

in which

$$w = 2\mu\omega + 2\mu'\omega',$$

has been called by Weierstrass the sigma-function $\sigma(x)$, and is the basis of his beautiful theory of elliptic functions.

EARLY HISTORY OF THE POTENTIAL.

BY PROF. A. S. HATHAWAY.

THE object of the present article is to correct an error that occurs in Todhunter's "History of the Theories of Attraction" (vol. II., arts. 789, 1007, and 1138), and that is repeated, doubtless on Todhunter's authority, in various encyclopædias. This error consists in assigning to Laplace, instead of Lagrange, the honor of the introduction of the Potential into dynamics, an honor that the Encyclopædia Britannica makes the basis of a eulogy to Laplace (art. *Laplace*) in the words: "The researches of Laplace and Legendre on the subject of attractions derive additional interest and importance from having introduced two powerful engines of analysis for the treatment of physical problems, Laplace's Coefficients and the Potential Function. The expressions for the attraction of an ellipsoid involved integrations which presented insuperable difficulties; it was, therefore, with pardonable exultation that Laplace announced his discovery that the attracting force in any direction could be obtained by the direct process of differentiating a single function. He thereby translated the forces of nature into the language of analysis and laid the foundations of the mathematical sciences of heat, electricity, and magnetism."

The announcement here referred to was made by Laplace

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* BIERMANN, *Theorie der analytischen Functionen*, Leipzig, 1887, p. 334.

SCHWARZ. *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen*, Göttingen, 1885.