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Hermitian–Einstein connections on polystable parabolic principal Higgs bundles

Indranil Biswas and Matthias Stemmler

School of Mathematics, Tata Institute of Fundamental Research,
Homi Bhabha Road, Bombay 400005, India
indranil@math.tifr.res.in, stemmler@math.tifr.res.in

Abstract

Given a smooth complex projective variety X and a smooth divisor D on X, we prove the existence of Hermitian–Einstein connections, with respect to a Poincaré-type metric on $X \setminus D$, on polystable parabolic principal Higgs bundles with parabolic structure over D, satisfying certain conditions on their restriction to D.

1 Introduction

The Hitchin–Kobayashi correspondence relating the stable vector bundles and the solutions of the Hermitian–Einstein equation has turned out to be extremely useful and important (see [11, 19, 18]). The Hitchin–Kobayashi correspondence has evolved into a general principle finding generalizations to numerous contexts. Here, we consider the parabolic Higgs G-bundles from this point of view.

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Parabolic vector bundles on curves were introduced by Seshadri [16]. This was generalized to higher dimensional varieties by Maruyama and Yokogawa [13]. Motivated by the characterization of principal bundles using Tannakian category theory given by Nori [14], in [3], parabolic principal bundles were defined. Later ramified principal bundles were defined in [4]; it turned out that there is a natural bijective correspondence between ramified principal bundles and parabolic principal bundles; cf. [4, 8]. Higgs fields on ramified principal bundles were defined in [9].

In [5], Biquard considered vector bundles on a compact Kähler manifold (X, ω_0), with parabolic structure over a smooth divisor D, equipped with a Higgs field that has a logarithmic singularity on D. He showed that these data induce certain Higgs bundles (in an adapted sense) on D, which he calls "spécialisés". In the case of Higgs fields with nilpotent residue on D, these are just the graded pieces of the parabolic filtration equipped with an induced Higgs structure. Given a stable parabolic Higgs bundle such that these induced bundles are polystable and satisfy an additional condition on their slope, he proves the existence of a Hermitian–Einstein metric on $X \setminus D$ with respect to a Poincaré-type Kähler metric. The Hermitian–Einstein metric is unique up to multiplication by a constant element of $\mathbb{R}^+$.

Our aim here is to extend Biquard's result to the case of parabolic principal Higgs G-bundles, where G is a connected reductive linear algebraic group defined over $\mathbb{C}$. Given such a bundle (E_G, θ), there is an adjoint parabolic Higgs vector bundle $(\mathrm{ad}(E_G), \mathrm{ad}(\theta))$. The Higgs field $\mathrm{ad}(\theta)$ has a nilpotent residue on D. This $\mathrm{ad}(\theta)$ induces Higgs fields on the graded pieces $\mathrm{Gr}_\alpha \,\mathrm{ad}(E_G)$ for the parabolic vector bundle $\mathrm{ad}(E_G)$. The Higgs field on $\mathrm{Gr}_\alpha \,\mathrm{ad}(E_G)$ induced by θ will be denoted by $\mathrm{ad}(\theta)_\alpha$.

Let $\psi : E_G \longrightarrow X$ be the natural projection. The restriction of ψ to $\psi^{-1}(D)$ will be denoted by $\widehat{\psi}$. Let $\mathcal{K}$ be the trivial vector bundle over $\psi^{-1}(D)$ with fiber $\mathrm{Lie}(G)$. The group G acts on $\mathcal{K}$ using the adjoint action of G on $\mathrm{Lie}(G)$. Define the invariant direct image

$$\mathcal{E} := (\widehat{\psi}_* \mathcal{K})^G,$$

which is a vector bundle over D. The Higgs field θ defines a Higgs field on $\mathcal{E}$, which will be denoted by θ'.

Fix a Kähler form ω_0 on X such that the corresponding class in $H^2(X, \mathbb{R})$ is integral.

We obtain the following (see Theorem 4.1 and Proposition 4.1):

Theorem 1.1. *Let (E_G, θ) be a parabolic Higgs G-bundle on X such that (E_G, θ) is polystable with respect to ω_0, and satisfies the following two conditions:*

- *the Higgs bundle $(\mathcal{E}, \theta')$ on D is polystable; and*
- *for the graded pieces $(\mathrm{Gr}_\alpha \operatorname{ad}(E_G), \operatorname{ad}(\theta)_\alpha)$ of $(\operatorname{ad}(E_G)|_D, \operatorname{ad}(\theta)|_D)$ the condition*

$$\mu(\mathrm{Gr}_\alpha \operatorname{ad}(E_G)) = -\alpha \deg(N)$$

holds, where degrees are computed using ω_0 and N is the normal bundle to D.

Then there is a Hermitian–Einstein connection on E_G over $X \setminus D$ with respect to the Poincaré-type metric.

Conversely, if there is such a Hermitian–Einstein connection satisfying the condition that the induced connection on the adjoint vector bundle $\operatorname{ad}(E_G)|_{X\setminus D}$ lies in the space $\mathcal{A}$ (see (3.3)), then (E_G, θ) is polystable with respect to ω_0.

2 Parabolic Higgs bundles

Let X be a connected smooth complex projective variety of complex dimension n, and let D be a smooth reduced effective divisor on X. We first recall the definition of a parabolic Higgs vector bundle on X with parabolic structure over D.

A *parabolic vector bundle* E_* on X with parabolic divisor D is a holomorphic vector bundle E on X together with a parabolic structure on it, which is given by a decreasing filtration $\{F_\alpha(E)\}_{0\leqslant\alpha\leqslant 1}$ of holomorphic subbundles of the restriction $E|_D$, which is continuous from the left, satisfying the conditions that $F_0(E) = E|_D$ and $F_1(E) = 0$. The *parabolic weights* of E_* are the numbers $0 \leqslant \alpha_1 < \cdots < \alpha_l < 1$ such that $F_{\alpha_i+\varepsilon}(E) \neq F_{\alpha_i}(E)$ for all $\varepsilon > 0$. For later use, we denote the graded pieces of this filtration as

$$\mathrm{Gr}_\alpha E := F_\alpha(E)/F_{\alpha+\varepsilon}(E), \quad \alpha \in \{\alpha_1, \dots, \alpha_l\}, \quad \varepsilon > 0 \text{ sufficiently small.} \tag{2.1}$$

Let $\mathrm{ParEnd}(E_*)$ be the sheaf of holomorphic sections of $\mathrm{End}(E) = E \otimes E^*$ which preserve the above filtration of $E|_D$. Let $\Omega^k_X(\log D)$ be the vector bundle on X defined by the sheaf of logarithmic k-forms. Note that there is a residue homomorphism

$$\mathrm{Res}_D : \mathrm{ParEnd}(E_*) \otimes \Omega^1_X(\log D) \longrightarrow \mathrm{ParEnd}(E_*)|_D$$

defined by the natural residue homomorphism $\Omega^1_X(\log D) \longrightarrow \mathcal{O}_D$.

Definition 2.1. A *parabolic Higgs vector bundle* with parabolic divisor D is a pair (E_*, θ) consisting of a parabolic vector bundle E_* on X with parabolic divisor D and a section

$$\theta \in H^0(X, \mathrm{ParEnd}(E_*) \otimes \Omega^1_X(\log D)),$$

called the *Higgs field*, such that the following two conditions are satisfied:

- $\theta \bigwedge \theta \in H^0(X, \mathrm{ParEnd}(E_*) \otimes \Omega^2_X(\log D))$ vanishes identically, where the multiplication is defined using the Lie algebra structure of the fibers of $\mathrm{End}(E)$, and the exterior product $\Omega^1_X(\log D) \otimes \Omega^1_X(\log D) \longrightarrow \Omega^2_X$ $(\log D)$, and
- the residue $\mathrm{Res}_D(\theta)$ is nilpotent with respect to the parabolic filtration in the sense that

$$\mathrm{Res}_D(\theta)(F_\alpha(E)) \subset F_{\alpha+\varepsilon}(E)$$

for some $\varepsilon > 0$.

In the following, we will omit the subscript "$*$" in E_*, and denote a parabolic vector bundle by the same symbol as its underlying bundle.

Now we will recall the definitions of ramified Higgs principal bundles and parabolic Higgs principal bundles. For this, let G be a connected reductive linear algebraic group defined over $\mathbb{C}$.

Definition 2.2. A *ramified G-bundle* over X with ramification over D is a smooth complex variety E_G equipped with an algebraic right action of G

$$f : E_G \times G \longrightarrow E_G$$

and a surjective algebraic map

$$\psi : E_G \longrightarrow X,$$

such that the following conditions are satisfied:

- $\psi \circ f = \psi \circ p_1$, where $p_1 : E_G \times G \to E_G$ denotes the natural projection;

- for each point $x \in X$, the action of G on the reduced fiber $\psi^{-1}(x)_{\mathrm{red}}$ is transitive;
- the restriction of ψ to $\psi^{-1}(X \setminus D)$ makes $\psi^{-1}(X \setminus D)$ a principal G-bundle over $X \setminus D$, meaning the map ψ is smooth over $\psi^{-1}(X \setminus D)$ and the map to the fiber product

$$\psi^{-1}(X \setminus D) \times G \longrightarrow \psi^{-1}(X \setminus D) \times_{X \setminus D} \psi^{-1}(X \setminus D)$$

given by $(z, g) \longmapsto \big(z, f(z, g)\big)$ is an isomorphism;
- the reduced inverse image $\psi^{-1}(D)_{\mathrm{red}}$ is a smooth divisor on E_G; and
- for each point $z \in \psi^{-1}(D)_{\mathrm{red}}$, the isotropy group $G_z \subset G$ for the action of G on E_G is a finite cyclic group acting faithfully on the quotient line $T_z E_G / T_z(\psi^{-1}(D)_{\mathrm{red}})$.

Parabolic principal G-bundles were defined in [3] as functors from the category of rational G-representations to the category of parabolic vector bundles, satisfying certain conditions; this definition was modeled on [14]. There is a natural bijective correspondence between the ramified principal G-bundles with ramification over D and parabolic principal G-bundles on X with D as the parabolic divisor [4, 8]. Let us briefly recall a construction of parabolic principal G-bundles from ramified principal G-bundles.

Let E_G be a ramified G-bundle on X with ramification over D. There is a finite (ramified) Galois covering

$$\eta : Y \longrightarrow X$$

such that the normalizer

$$F_G := \widetilde{E_G \times_X Y} \tag{2.2}$$

of the fiber product $E_G \times_X Y$ is smooth. Write $\Gamma := \mathrm{Gal}(\eta)$ for the Galois group of η. Let

$$h : \Gamma \longrightarrow \mathrm{Aut}(Y) \tag{2.3}$$

be the homomorphism giving the action of Γ on Y. The projection $F_G \longrightarrow Y$ yields a Γ-linearized principal G-bundle on Y in the following sense:

Definition 2.3. A Γ-*linearized principal G-bundle* on Y is a principal G-bundle

$$\psi : F'_G \longrightarrow Y$$

together with a left action of Γ on F'_G

$$\rho : \Gamma \times F'_G \longrightarrow F'_G$$

such that the following two conditions are satisfied:

- the actions of Γ and G on F'_G commute, and
- $\psi(\rho(\gamma, z)) = h(\gamma)(\psi(z))$ for all $(\gamma, z) \in \Gamma \times F'_G$, and h is defined in (2.3).

Consider F_G constructed in (2.2). Given a finite-dimensional complex G-module V, there is the associated Γ-linearized vector bundle $F_G(V) = F_G \times^G V$ on Y with fibers isomorphic to V. This $F_G(V)$ in turn corresponds to a parabolic vector bundle on X with D as the parabolic divisor, cf. [7]; this parabolic vector bundle will be denoted by $E_G(V)$.

The earlier mentioned functor, from the category of rational G-representations to the category of parabolic vector bundles, associated to the ramified G-bundle E_G sends any G-module V to the parabolic vector bundle $E_G(V)$ constructed above.

In the following, we will identify the notions of parabolic and ramified G-bundles.

Let $\mathfrak{g}$ be the Lie algebra of G; it is equipped with the adjoint action of G. Setting $V = \mathfrak{g}$, the parabolic vector bundle $E_G(\mathfrak{g})$ constructed as above is called the adjoint parabolic vector bundle of E_G, and it is denoted by $\mathrm{ad}(E_G)$.

Let E_G be a ramified G-bundle over X with ramification over D. Let

$$\mathcal{K} \subset TE_G \tag{2.4}$$

be the holomorphic subbundle defined by the tangent space of the orbits of the action of G on E_G; since all the isotropies, for the action of G on E_G, are finite groups, $\mathcal{K}$ is indeed a subbundle. Note that $\mathcal{K}$ is identified with the trivial vector bundle over E_G with fiber $\mathfrak{g}$. Let

$$\mathcal{Q} := TE_G/\mathcal{K}$$

be the quotient vector bundle. The action of G on E_G induces an action of G on the tangent bundle TE_G, which preserves the subbundle $\mathcal{K}$. Therefore, there is an induced action of G on the quotient bundle $\mathcal{Q}$. These actions in turn induce a linear action of G on $H^0(E_G, \mathcal{K} \otimes \mathcal{Q}^*)$. Combining the exterior algebra structure of $\Lambda\mathcal{Q}^*$ and the Lie algebra structure on the fibers of $\mathcal{K} = E_G \times \mathfrak{g}$, one obtains a homomorphism

$$\tau : (\mathcal{K} \otimes \mathcal{Q}^*) \otimes (\mathcal{K} \otimes \mathcal{Q}^*) \longrightarrow \mathcal{K} \otimes \Lambda^2\mathcal{Q}^*.$$

For $y \in E_G$, and $a, b \in (\mathcal{K} \otimes \mathcal{Q}^*)_y$, the image $\tau(a \otimes b)$ will also be denoted by $a \bigwedge b$.

Definition 2.4. (1) A *Higgs field* on E_G is a section

$$\theta \in H^0(E_G, \mathcal{K} \otimes \mathcal{Q}^*)$$

such that

- θ is invariant under the action of G on $H^0(E_G, \mathcal{K} \otimes \mathcal{Q}^*)$, and
- $\theta \bigwedge \theta = 0$.

(2) A *parabolic Higgs G-bundle* is a pair (E_G, θ) consisting of a parabolic G-bundle E_G and a Higgs field θ on E_G.

Now let $H \subset G$ be a Zariski closed subgroup, and let $U \subset X$ be a Zariski open subset. The inverse image $\psi^{-1}(U) \subset E_G$ will be denoted by $E_G|_U$; as before, ψ is the projection of E_G to X.

Definition 2.5. A *reduction of structure group of E_G to H over U* is a subvariety

$$E_H \subset E_G|_U$$

satisfying the following conditions:

- E_H is preserved by the action of H on E_G;
- for each point $x \in U$, the action of H on $\psi^{-1}(x) \bigcap E_H$ is transitive; and
- for each point $z \in E_H$, the isotropy subgroup G_z, for the action of G on E_G, is contained in H.

Clearly, such an E_H is a ramified H-bundle over U. Let

$$\iota : E_H \longrightarrow E_G|_U \tag{2.5}$$

be a reduction of structure group of E_G to H over U. Define the bundles $\mathcal{K}_H$ and $\mathcal{Q}_H$ as before with respect to E_H (in place of E_G). Then by [9, (3.8)],

$$\mathrm{Hom}(\mathcal{Q}_H, \mathcal{K}_H) \subset \iota^* \mathrm{Hom}(\mathcal{Q}, \mathcal{K})\,.$$

Let $\theta \in H^0(E_G, \mathrm{Hom}(\mathcal{Q}, \mathcal{K}))$ be a Higgs field on E_G.

Definition 2.6. The reduction E_H in (2.5) is said to be *compatible* with the Higgs field θ if

$$\theta|_{E_H} \in H^0(E_H, \mathrm{Hom}(\mathcal{Q}_H, \mathcal{K}_H)) \subset H^0(E_H, \iota^* \mathrm{Hom}(\mathcal{Q}, \mathcal{K})).$$

Fix a very ample line bundle ζ on X. Define the degree $\deg \mathcal{F}$ (respectively, the parabolic degree par-deg E_*) of a torsion-free coherent sheaf $\mathcal{F}$ (respectively, a parabolic vector bundle E_*) on X with respect to this polarization ζ.

Fix a basis of $H^0(X, \zeta)$. Using this basis we get an embedding of X in $\mathbb{CP}^{N-1}$, where $N = \dim H^0(X, \zeta)$. Let ω_0 be the restriction to X of the Fubini–Study metric on $\mathbb{CP}^{N-1}$.

Let H be a parabolic subgroup of G. Then G/H is a complete variety, and the quotient map $G \longrightarrow G/H$ defines a principal H-bundle over G/H. For any character χ of H, let

$$L_\chi \longrightarrow G/H$$

be the line bundle associated to this principal H-bundle for the character χ. Let $R_u(H)$ be the unipotent radical of H (it is the unique maximal normal unipotent subgroup). The group $H/R_u(H)$ is called the *Levi quotient* of H. There are subgroups $L(H) \subset H$ such that the composition $L(H) \hookrightarrow H \longrightarrow H/R_u(H)$ is an isomorphism. Such a subgroup $L(H)$ is called a *Levi subgroup* of H. Any two Levi subgroups of H are conjugate by some element of H.

Let $Z_0(G) \subset G$ be the connected component, containing the identity element, of the center of G. It is known that $Z_0(G) \subset H$. A character χ of H, which is trivial on $Z_0(G)$, is called *strictly antidominant* if the corresponding line bundle L_χ over G/H (defined above) is ample.

Definition 2.7. A parabolic Higgs G-bundle (E_G, θ) is called *stable* if for every quadruple (H, χ, U, E_H), where

- $H \subset G$ is a proper parabolic subgroup;
- χ is a strictly antidominant character of H;
- $U \subset X$ is a non-empty Zariski open subset such that the codimension of $X \setminus U$ is at least two; and
- $E_H \subset E_G|_U$ is a reduction of structure group of E_G to H over U compatible with θ,

the following holds:

$$\text{par-deg}(E_H(\chi)) > 0,$$

where $E_H(\chi)$ is the parabolic line bundle over U associated to the parabolic H-bundle E_H for the one-dimensional representation χ of H.

Let E_G be a parabolic G-bundle over X. A reduction of structure group $E_H \subset E_G$ to some parabolic subgroup $H \subset G$ is called *admissible* if for each character χ of H which is trivial on $Z_0(G)$, the associated parabolic line bundle $E_H(\chi)$ over X satisfies the following condition:

$$\text{par-deg}(E_H(\chi)) = 0.$$

Definition 2.8. A parabolic Higgs G-bundle (E_G, θ) is called *polystable* if either (E_G, θ) is stable, or there is a proper parabolic subgroup $H \subset G$ and a reduction of structure group

$$E_{L(H)} \subset E_G$$

of E_G to a Levi subgroup $L(H) \subset H$ over X such that the following conditions are satisfied:

- the reduction $E_{L(H)} \subset E_G$ is compatible with θ;
- the parabolic Higgs $L(H)$-bundle $(E_{L(H)}, \theta|_{E_{L(H)}})$ is stable (from the first condition it follows that $\theta|_{E_{L(H)}}$ is a Higgs field on $E_{L(H)}$); and
- the reduction of structure group of E_G to H, obtained by extending the structure group of $E_{L(H)}$ using the inclusion of $L(H)$ in H, is admissible.

3 Hermitian–Einstein connection on a parabolic Higgs *G*-bundle

Let E_G be a parabolic G-bundle over X. Let

$$0 \longrightarrow \text{ad}(E_G) \longrightarrow \text{At}(E_G) \longrightarrow TX \longrightarrow 0 \tag{3.1}$$

be the Atiyah exact sequence for the G-bundle E_G over $X \setminus D$. Recall that a *complex connection* on E_G over $X \setminus D$ is a C^∞ splitting of this exact sequence. Fix a maximal compact subgroup $K \subset G$. A complex connection on E_G over $X \setminus D$ is called *unitary* if it is induced by a connection on a smooth reduction of structure group E_K of E_G to K over $X \setminus D$. Note that (3.1) is a short exact sequence of sheaves of Lie algebras. For a complex unitary connection ∇ on E_G over $X \setminus D$, its *curvature form*

$$F \in H^0(X \setminus D, \Lambda^{1,1}T^*X \otimes \text{ad}(E_G))$$

measures the obstruction of the splitting of (3.1) defining ∇ to be Lie algebra structure preserving; see [2] for the details.

For a parabolic Higgs G-bundle (E_G, θ) on X, its restriction to $X \setminus D$ is a Higgs G-bundle in the usual sense. Given a smooth reduction of structure group E_K of E_G to a maximal compact subgroup $K \subset G$ over $X \setminus D$, the Cartan involution of $\mathfrak{g}$ with respect to K induces an involution of the adjoint vector bundle $\mathrm{ad}(E_G)$ over $X \setminus D$; this involution of $\mathrm{ad}(E_G)$ will be denoted by ϕ. Writing $\theta = \sum_i \theta_i dz^i$ in local holomorphic coordinates $z^1, \ldots, z^n$ on X around a point $x \in X \setminus D$, define

$$\theta^* := -\sum_i \phi(\theta_i) d\bar{z}^i.$$

This definition is clearly independent of the choice of local coordinates.

Let $\mathfrak{z}$ be the center of the Lie algebra $\mathfrak{g}$ of G. Since the adjoint action of G on $\mathfrak{z}$ is trivial, an element $\lambda \in \mathfrak{z}$ defines a smooth section of $\mathrm{ad}(E_G)$ over $X \setminus D$, which will also be denoted by λ.

Definition 3.1. Let (E_G, θ) be a parabolic Higgs G-bundle on X. A complex unitary connection on E_G over $X \setminus D$ is called a *Hermitian–Einstein connection* with respect to a Kähler metric ω on $X \setminus D$ and the Higgs field θ, if its curvature form F satisfies the equation

$$\Lambda_\omega(F + [\theta, \theta^*]) = \lambda$$

for some $\lambda \in \mathfrak{z}$, where the operation $[\cdot, \cdot]$ is defined using the exterior product on forms and the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$.

Note that λ in Definition 3.1 lies in $\mathfrak{z} \bigcap \mathrm{Lie}(K)$.

In [5], Biquard introduces a Poincaré-type metric on $X \setminus D$ as follows: let σ be the canonical section of the line bundle $\mathcal{O}_X(D)$ on X associated to the divisor D, meaning D is the zero divisor of σ. Let ω_0 be the Kähler form on X that we fixed earlier. Choose a Hermitian metric $\|\cdot\|$ on the fibers of $\mathcal{O}_X(D)$. Then

$$\omega := T\omega_0 - \sqrt{-1}\partial\bar{\partial} \log \log^2 \|\sigma\|^2 \tag{3.2}$$

defines a Kähler metric on $X \setminus D$ for $T \in \mathbb{R}^+$ large enough.

In [5], Biquard proves the existence of Hermitian–Einstein metrics on stable parabolic Higgs vector bundles under certain additional conditions (see [5, Théorème 8.1]). In his definition of parabolic Higgs vector bundles he does not require the residue of the Higgs field to be nilpotent.

Let (E, θ) be a parabolic Higgs vector bundle. Consider the graded pieces $\mathrm{Gr}_\alpha E$ in (2.1). Let

$$\theta_\alpha := \theta|_D : \mathrm{Gr}_\alpha E \longrightarrow (\mathrm{Gr}_\alpha E) \otimes (\Omega^1_X(\log D)|_D)$$

be the homomorphism given by θ. Since the residue of θ is nilpotent with respect to the quasi-parabolic filtration of $E|_D$, the composition

$$\mathrm{Gr}_\alpha E \xrightarrow{\theta_\alpha} (\mathrm{Gr}_\alpha E) \otimes (\Omega^1_X(\log D)|_D) \xrightarrow{\mathrm{id}\otimes\mathrm{Res}} \mathrm{Gr}_\alpha E \otimes \mathcal{O}_D = \mathrm{Gr}_\alpha E$$

vanishes identically. Therefore, $\theta_\alpha \in H^0(D, \mathrm{End}(\mathrm{Gr}_\alpha E) \otimes \Omega^1_D)$. The integrability condition $\theta \bigwedge \theta = 0$ immediately implies that $\theta_\alpha \bigwedge \theta_\alpha = 0$. Therefore, $(\mathrm{Gr}_\alpha E, \theta_\alpha)$ is a Higgs vector bundle on D.

In [5, pp. 47–48], Biquard uses the parabolic structure of E to construct a background metric on E over $X \setminus D$. Let ∇ be the corresponding Chern connection. He then restricts his attention to connections lying in the space

$$\mathcal{A} := \{\nabla + a : a \in \widehat{C}^{1+\vartheta}_\delta(\Omega^1_X \otimes \mathrm{End}(E))\} \tag{3.3}$$

(see [5, p. 58 and p. 70]), where the Hölder space $\widehat{C}^{1+\vartheta}_\delta(\Omega^1_X \otimes \mathrm{End}(E))$ is defined in [5, pp. 53–54]. Let

$$N \longrightarrow D$$

be the normal line bundle of the divisor D.

With these definitions, Biquard's theorem can be formulated as follows:

Theorem 3.1. *Let (E, θ) be a stable parabolic Higgs vector bundle on X with parabolic divisor D. Assume that all the graded Higgs bundles $(\mathrm{Gr}_\alpha E, \theta_\alpha)$ are polystable and satisfy the condition*

$$\mu(\mathrm{Gr}_\alpha E) = \text{par-}\mu(E) - \alpha \deg(N) \tag{3.4}$$

with respect to ω_0. Then there is a Hermitian metric h on E over $X \setminus D$, with Chern connection in $\mathcal{A}$, which is Hermitian–Einstein with respect to the Poincaré-type metric ω, meaning its Chern curvature form F satisfies

$$\sqrt{-1}\Lambda_\omega(F + [\theta, \theta^*]) = \lambda \cdot \mathrm{id}_E$$

for some $\lambda \in \mathbb{R}$.

Such a Hermitian metric is unique up to a constant scalar multiple.

4 Existence of Hermitian–Einstein connection

Let (E_G, θ) be a ramified Higgs G-bundle. Let

$$\psi : E_G \longrightarrow X$$

be the natural projection. The reduced divisor $\psi^{-1}(D)_{\mathrm{red}}$ will be denoted by $\widetilde{D}$. Let

$$\widehat{\psi} := \psi|_{\widetilde{D}} : \widetilde{D} \longrightarrow D$$

be the restriction. Consider the subbundle $\mathcal{K}$ defined in (2.4). The action of the group G on $\widetilde{D}$ produces an action of G on the direct image $\widehat{\psi}_*\mathcal{K} \longrightarrow D$. Define the invariant part

$$\mathcal{E} := (\widehat{\psi}_*\mathcal{K})^G \longrightarrow D; \tag{4.1}$$

it is a vector bundle over D.

We will give an explicit description of the vector bundle $\mathcal{E}$. As before, the isotropy subgroup of any $z \in \widetilde{D}$, for the action of G on $\widetilde{D}$, will be denoted by G_z. Let

$$\mathfrak{g}_z := \mathfrak{g}^{G_z} \subset \mathfrak{g}$$

be the space of invariants for the adjoint action of G_z. This $\mathfrak{g}_z$ is clearly a subalgebra of $\mathfrak{g}$. The elements of G_z are semisimple because G_z is a finite group. Since G_z is cyclic, the Lie subalgebra $\mathfrak{g}_z$ is reductive (see [12, p. 26, Theorem]). Let $\mathcal{S}$ be the subbundle of the trivial vector bundle $\widetilde{D} \times \mathfrak{g} \longrightarrow \widetilde{D}$ whose fiber over any $z \in \widetilde{D}$ is the subalgebra $\mathfrak{g}_z$. The action of G on $\widetilde{D}$ and the adjoint action of G on $\mathfrak{g}$ combine together to define an action of G on $\widetilde{D} \times \mathfrak{g}$; the identification between $\mathcal{K}|_{\widetilde{D}}$ and $\widetilde{D} \times \mathfrak{g}$ commutes with the actions of G . The action of G on $\widetilde{D} \times \mathfrak{g}$ clearly preserves the subbundle $\mathcal{S}$. We have

$$D = \widetilde{D}/G \quad \text{and} \quad \mathcal{E} = \mathcal{S}/G. \tag{4.2}$$

That $\mathcal{S}/G$ is a vector bundle over $\widetilde{D}/G$ follows from the fact that the isotropy subgroups act trivially on the fibers of $\mathcal{S}$.

Let h be any G-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$. The restriction of h to the centralizer, in $\mathfrak{g}$, of any semisimple element of G is known to be nondegenerate. From this it follows that the bilinear form induced by h on the vector bundle $\mathcal{S}$ in (4.2) is nondegenerate. Since h is G-invariant, from (4.2) we conclude that this nondegenerate bilinear form on $\mathcal{S}$ descends to a nondegenerate bilinear form on $\mathcal{E}$. This implies that $\mathcal{E}^* = \mathcal{E}$, in particular, $\deg(\mathcal{E}) = 0$ with respect to any polarization on D.

Recall that the fibers of $\mathcal{K}$ are identified with $\mathfrak{g}$. Using this Lie algebra structure of the fibers of $\mathcal{K}$, the Higgs field θ defines a homomorphism

$$\mathcal{K} \longrightarrow \mathcal{K} \otimes \mathcal{Q}^* \longrightarrow 0$$

of vector bundles. On the other hand, over $\widetilde{D}$, we have a natural restriction homomorphism

$$\mathcal{Q}^*|_{\widetilde{D}} \longrightarrow \Omega^1_{\widetilde{D}}$$

of vector bundles. Combining these two homomorphisms, we have a homomorphism of vector bundles

$$\beta : \mathcal{K}|_{\widetilde{D}} \longrightarrow \mathcal{K}|_{\widetilde{D}} \otimes \Omega^1_{\widetilde{D}}.$$

The group G acts on both $\mathcal{K}|_{\widetilde{D}}$ and $\Omega^1_{\widetilde{D}}$. The above homomorphism β commutes with the actions of G. Therefore, β produces a homomorphism

$$\theta' : \mathcal{E} = (\widehat{\psi}_* \mathcal{K})^G \longrightarrow (\mathcal{K}|_{\widetilde{D}} \otimes \Omega^1_{\widetilde{D}})^G = \mathcal{E} \otimes \Omega^1_D, \tag{4.3}$$

where $\mathcal{E}$ is defined in (4.1). From the condition $\theta \bigwedge \theta = 0$ (see Definition 2.4) it follows that θ' is a Higgs field on the vector bundle $\mathcal{E}$.

Consider the adjoint parabolic vector bundle $\mathrm{ad}(E_G)$ for the ramified G-bundle E_G. The Higgs field θ produces a Higgs field on the parabolic vector bundle $\mathrm{ad}(E_G)$. This induced Higgs field on $\mathrm{ad}(E_G)$ will be denoted by $\mathrm{ad}(\theta)$.

Theorem 4.1. *Let (E_G, θ) be a parabolic Higgs G-bundle on X such that (E_G, θ) is polystable with respect to the Kähler form ω_0 (see* (3.2)*), and satisfies the following two conditions:*

- *the Higgs bundle $(\mathcal{E}, \theta')$ constructed in* (4.1) *and* (4.3) *is polystable, and*
- *for the graded pieces $(\mathrm{Gr}_\alpha \mathrm{ad}(E_G), \mathrm{ad}(\theta)_\alpha)$ of $(\mathrm{ad}(E_G)|_D, \mathrm{ad}(\theta)|_D)$, the condition*
$$\mu(\mathrm{Gr}_\alpha \mathrm{ad}(E_G)) = -\alpha \deg(N) \tag{4.4}$$
holds, where degrees are computed using ω_0 and N is the normal bundle of D.

Then there is a Hermitian–Einstein connection on E_G over $X \setminus D$ with respect to the Poincaré-type metric described in Section 3.

Proof. We first note that it is enough to prove the theorem under the stronger assumption that the parabolic Higgs G-bundle (E_G, θ) is stable. Indeed, a polystable parabolic Higgs G-bundle (E_G, θ) admits a reduction of structure group $E_{L(P)} \subset E_G$ to a Levi subgroup $L(P)$ of some parabolic subgroup P of G such that the corresponding parabolic Higgs $L(P)$-bundle $(E_{L(P)}, \theta)$ is stable (see Definition 2.8). The connection on E_G induced by a Hermitian–Einstein connection on $E_{L(P)}$ is again Hermitian–Einstein. Hence it suffices to prove the theorem for (E_G, θ) stable.

Henceforth, in the proof we assume that (E_G, θ) is stable.

We will now show that it is enough to prove the theorem under the assumption that G is semisimple.

As before, $Z_0(G) \subset G$ is the connected component, containing the identity element, of the center of G. The normal subgroup $[G, G] \subset G$ is semisimple, because G is reductive. We have natural homomorphisms

$$Z_0(G) \times [G, G] \longrightarrow G \longrightarrow (G/Z_0(G)) \times (G/[G, G]).$$

Both the homomorphisms are surjective with finite kernel. In particular, both the homomorphisms of Lie algebras are isomorphisms. Let $\rho : A \longrightarrow B$ be a homomorphism of Lie groups such that the corresponding homomorphism of Lie algebras is an isomorphism, let E_A be a principal A-bundle, and let $E_B := E_A \times^{\rho} B$ be the principal B-bundle obtained by extending the structure group of E_A using ρ. Then there is a natural bijective correspondence between the connections on E_A and the connections on E_B. The curvature of a connection on E_B is given by the curvature of the corresponding connection on E_A using the homomorphism of Lie algebras associated to ρ. Therefore, to prove the theorem for G, it is enough to prove it for $G/Z_0(G)$ and $G/[G, G]$ separately. But $G/[G, G]$ is a product of copies of $\mathbb{C}^*$, hence in this case the theorem follows immediately from Theorem 3.1. The group $G/Z_0(G)$ is semisimple. Hence, it is enough to prove the theorem under the assumption that G is semisimple.

Henceforth, in the proof we assume that G is semisimple.

Denote by $\eta : Y \longrightarrow X$ the Galois covering with Galois group $\Gamma := \mathrm{Gal}(\eta)$ and by F_G the Γ-linearized G-bundle on Y corresponding to E_G as described in Section 2. According to [9, Proposition 4.1], the Higgs field θ on E_G corresponds to a Γ-invariant Higgs field $\widetilde{\theta}$ on F_G. This induces a Γ-invariant Higgs field $\mathrm{ad}(\widetilde{\theta})$ on the Γ-linearized vector bundle $\mathrm{ad}(F_G)$. By [6, Theorem 5.5], this in turn corresponds to a Higgs field $\mathrm{ad}(\theta)$ on the parabolic vector bundle $\mathrm{ad}(E_G)$. This way we construct the parabolic Higgs vector bundle $(\mathrm{ad}(E_G), \mathrm{ad}(\theta))$ on X defined earlier.

The strategy of the proof is to show that the hypotheses of Biquard's Theorem 3.1 are satisfied for $(\mathrm{ad}(E_G), \mathrm{ad}(\theta))$ and that the resulting

Hermitian–Einstein connection on $\mathrm{ad}(E_G)|_{X\setminus D}$ is induced by a Hermitian–Einstein connection on $E_G|_{X\setminus D}$.

First, we show that $(\mathrm{ad}(E_G), \mathrm{ad}(\theta))$ is parabolic polystable. Since (E_G, θ) is stable by hypothesis, it follows as in [9, Lemma 4.2] that $(F_G, \widetilde{\theta})$ is Γ-stable. In [1] it was shown that if a principal G-bundle E_G^1 is stable, then its adjoint vector bundle $\mathrm{ad}(E_G^1)$ is polystable (see [1, p. 212, Theorem 2.6]). The proof in [1] goes through if (E_G^1, θ^1) is Γ-stable, and gives that $(\mathrm{ad}(E_G^1), \mathrm{ad}(\theta^1))$ is Γ-polystable. Since the proof goes through verbatim with obvious modifications due to the Higgs field, we refrain from repeating the proof. Therefore, we have $(\mathrm{ad}(F_G), \mathrm{ad}(\widetilde{\theta}))$ to be Γ-polystable.

Since $(\mathrm{ad}(F_G), \mathrm{ad}(\widetilde{\theta}))$ is Γ-polystable, the parabolic Higgs vector bundle

$$(\mathrm{ad}(E_G), \mathrm{ad}(\theta))$$

is parabolic polystable (see [6, p. 611, Theorem 5.5]).

Let M be a reductive complex linear algebraic group. The connected component, containing the identity element, of the center of M will be denoted by $Z_0(M)$. Let (E_M, θ_M) be a polystable principal Higgs M-bundle on a connected complex projective manifold. If V is a complex M-module such that $Z_0(M)$ acts on V as scalar multiplications through a character of $Z_0(M)$, then it is known that the associated Higgs vector bundle $(E_M \times^M V, \theta_V)$ is polystable, where θ_V is the Higgs field on the associated vector bundle $E_M \times^M V$ defined by θ_M. Indeed, this follows immediately from the fact that (E_M, θ_M) has a Hermitian–Einstein connection; note that the connection on $(E_M \times^M V, \theta_V)$ induced by a Hermitian–Einstein connection on (E_M, θ_M) is also Hermitian–Einstein, provided the above condition for the action of $Z_0(M)$ on V holds. (See [1, p. 227, Theorem 4.10] for the Hermitian–Einstein connection on (E_M, θ_M).)

Since the Higgs bundle $(\mathcal{E}, \theta')$ constructed in (4.1) and (4.3) is given to be polystable, from the above observation it follows that each of the graded pieces $(\mathrm{Gr}_\alpha \mathrm{ad}(E_G), \mathrm{ad}(\theta)_\alpha)$ of $(\mathrm{ad}(E_G), \mathrm{ad}(\theta))$ is polystable.

Since G is semisimple, the Killing form on its Lie algebra $\mathfrak{g}$ is nondegenerate and thus induces an isomorphism $\mathrm{ad}(F_G) \simeq \mathrm{ad}(F_G)^*$. This implies that $\deg(\mathrm{ad}(F_G)) = 0$. By [7, p. 318, (3.12)], we have

$$\#\Gamma \cdot \text{par-deg}(\mathrm{ad}(E_G)) = \deg(\mathrm{ad}(F_G)),$$

and thus $\text{par-deg}(\mathrm{ad}(E_G)) = 0$, or equivalently, $\text{par-}\mu(\mathrm{ad}(E_G)) = 0$. Consequently, the hypothesis (4.4) on the slopes of the graded pieces implies that the condition (3.4) in Theorem 3.1 holds for the bundle $\mathrm{ad}(E_G)$. Therefore, we obtain from Theorem 3.1 a Hermitian–Einstein metric on $\mathrm{ad}(E_G)$ over $X \setminus D$ with respect to the Poincaré-type metric.

Finally, we have to show that the corresponding Hermitian–Einstein connection on $\mathrm{ad}(E_G)$ is induced by a connection on the principal Higgs G-bundle $E_G|_{X\setminus D}$; we note that if ∇ is a connection on $E_G|_{X\setminus D}$ inducing the Hermitian–Einstein connection on $\mathrm{ad}(E_G)$, then ∇ is automatically Hermitian–Einstein.

Let

$$\Phi \in H^0(X \setminus D,\, (\mathrm{ad}(E_G) \otimes \mathrm{ad}(E_G))^* \otimes \mathrm{ad}(E_G))$$

be the section defining the Lie bracket operation on $\mathrm{ad}(E_G)$. It can be shown that a connection ∇_{ad} on $\mathrm{ad}(E_G)|_{X\setminus D}$ is induced by a connection of $E_G|_{X\setminus D}$ if and only if Φ is parallel with respect to the connection on $(\mathrm{ad}(E_G) \otimes \mathrm{ad}(E_G))^* \otimes \mathrm{ad}(E_G)$ induced by ∇_{ad}. Indeed, this follows from the fact that G being semisimple the Lie algebra of the group of Lie algebra preserving automorphisms of $\mathfrak{g}$ coincides with $\mathfrak{g}$ (see proof of Theorem 3.7 of [1]).

Therefore, to complete the proof of the theorem it suffices to show that Φ is parallel with respect to the connection on $(\mathrm{ad}(E_G) \otimes \mathrm{ad}(E_G))^* \otimes \mathrm{ad}(E_G)$ induced by a Hermitian–Einstein connection on $\mathrm{ad}(E_G)$.

Since $\text{par-deg}(\mathrm{ad}(E_G)) = 0$, it follows that

$$\text{par-deg}((\mathrm{ad}(E_G) \otimes \mathrm{ad}(E_G))^* \otimes \mathrm{ad}(E_G)) = 0.$$

The connection on $(\mathrm{ad}(E_G) \otimes \mathrm{ad}(E_G))^* \otimes \mathrm{ad}(E_G)$ induced by the Hermitian–Einstein connection on $\mathrm{ad}(E_G)$ is also a Hermitian–Einstein connection. Since the Higgs field $\mathrm{ad}(\theta)$ is induced by the Higgs field θ on E_G, it follows that Φ is annihilated by the induced Higgs field on $(\mathrm{ad}(E_G) \otimes \mathrm{ad}(E_G))^* \otimes \mathrm{ad}(E_G)$. Thus the proof of Theorem 4.1 is completed by Lemma 4.1. $\square$

Lemma 4.1. *Let (E, θ) be a parabolic Higgs vector bundle on $X \setminus D$ admitting a Hermitian–Einstein connection ∇ with respect to the Poincaré-type metric. Assume that (E, θ) is polystable, and* $\text{par-deg}\, E = 0$. *Let s be a holomorphic section of E such that $\theta(s) = 0$. Then s is parallel with respect to ∇.*

Proof. Fix a Galois covering $\eta : Y \longrightarrow X$ such that there is a Γ-linearized Higgs vector bundle (V, φ) on Y that corresponds to (E, θ), where $\Gamma = \mathrm{Gal}(\eta)$. Fix the polarization $\eta^*\zeta$ on Y, where ζ is the polarization on X.

We know that (V, φ) is Γ-polystable because (E, θ) is polystable. Therefore, (V, φ) admits a Hermitian–Einstein connection [17, p. 978, Theorem 1].

Let $\tilde{s}$ be the holomorphic section of V over Y given by s. We note that $\varphi(\tilde{s}) = 0$ because $\theta(s) = 0$. We have $\deg V = 0$ because $\text{par-deg}\, E = 0$ [7, p. 318, (3.12)]. Since (V, φ) admits a Hermitian–Einstein connection with $\deg V = 0$, and $\varphi(\tilde{s}) = 0$, it follows that the holomorphic section $\tilde{s}$ is flat with respect to the Hermitian–Einstein connection on (V, φ) [10, p. 548, Lemma 3.4].

If s vanishes identically, then the lemma is obvious. Assume that s does not vanish identically. Since $\tilde{s}$ is flat with respect to the Hermitian–Einstein connection on (V, φ), the section $\tilde{s}$ does not vanish at any point of Y. Let $L^{\tilde{s}} \subset V$ be the holomorphic line subbundle generated by $\tilde{s}$. The action of Γ on V clearly preserves $L^{\tilde{s}}$. Since (V, φ) is Γ-polystable, this implies that there is a Γ-polystable Higgs vector bundle (V', φ') such that

$$(V, \varphi) = (V', \varphi') \oplus (L^{\tilde{s}}, 0)$$

as Γ-linearized Higgs vector bundles.

The above decomposition of the Γ-linearized Higgs vector bundle (V, φ) produces a decomposition

$$(E, \theta) = (E', \theta') \oplus (L^s, 0)$$

of the parabolic Higgs vector bundle; the line subbundle L^s of E is generated by s.

The direct sum of the Hermitian–Einstein connections on (E', θ') and $(L^s, 0)$ is a Hermitian–Einstein connection on (E, θ). Therefore, from the uniqueness of the Hermitian–Einstein connection (see the second part of Theorem 3.1) it follows immediately that s is parallel with respect to the Hermitian–Einstein connection ∇. $\square$

There is also a converse to Theorem 4.1:

Proposition 4.1. *Let (E_G, θ) be a parabolic Higgs G-bundle on X. Suppose there is a Hermitian–Einstein connection on E_G over $X \setminus D$ with respect to the Poincaré-type metric ω such that the induced connection on the adjoint vector bundle $\mathrm{ad}(E_G)|_{X \setminus D}$ lies in the space $\mathcal{A}$ (see (3.3)). Then (E_G, θ) is polystable with respect to ω_0.*

Proof. By [5, Proposition 7.2] we know that the parabolic degree of a parabolic sheaf on X with respect to ω_0 coincides with the degree of its restriction to $X \setminus D$ with respect to ω, computed using a Hermitian metric with Chern connection in $\mathcal{A}$. Thus, the proof in [15, pp. 28–29] of the proposition for ordinary principal bundles generalizes to our situation of parabolic Higgs G-bundles. □

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