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On the final definition of the causal boundary and its relation with the conformal boundary

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Abstract

The notion of *causal boundary* ∂M for a strongly causal spacetime M has been a controversial topic along last decades: on one hand, some attempted definitions were not fully consistent, on the other, there were simple examples where an open conformal embedding $i : M \hookrightarrow M_0$ could be defined, but the corresponding conformal boundary $\partial_i M$ disagreed drastically with the causal one. Nevertheless, the recent progress in this topic suggests that a final option for ∂M is available in most cases. Our study has two parts:

(I) To give general arguments on a boundary in order to ensure that it is *admissible* as a causal boundary at the three natural levels, i.e., as a point set, as a chronological space and as a topological space. Then, the essential uniqueness of our choice is stressed, and the relatively few admissible alternatives are discussed.

(II) To analyze the role of the conformal boundary $\partial_i M$. We show that, in general, $\partial_i M$ may present a very undesirable structure. Nevertheless,

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it is well behaved under certain general assumptions, and its accessible part $\partial_i^* M$ agrees with the causal boundary.

This study justifies both boundaries. On one hand, the conformal boundary $\partial_i^* M$, which cannot be defined for a general spacetime but is easily computed in particular examples, appears now as a special case of the causal boundary. On the other, the new redefinition of the causal boundary not only is free of inconsistencies and applicable to any strongly causal spacetime, but also recovers the expected structure in the cases where a natural simple conformal boundary is available. The cases of globally hyperbolic spacetimes and asymptotically conformally flat ends are especially studied.

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1 Presentation of the results

One of the major issues in Lorentzian geometry is to find a natural boundary for any spacetime, which would encode relevant information on it. In fact, a good number of boundaries have been defined, among them Geroch's geodesic boundary [16], Schmidt bundle boundary [36, 37] and Scott and Szekeres abstract boundary [32]. Two boundaries have a specially important role in general relativity, the conformal and the causal ones. The conformal boundary is clearly the most applied in mathematical relativity. Notions such as asymptotic flatness or tools as Penrose–Carter diagrams rely on the conformal boundary. Moreover, this boundary is also important in the framework of the AdS/CFT (anti de Sitter/conformal field theory) correspondence, as here one typically assumes that the field theory lives on the conformal boundary of the ambient spacetime. However, the conformal boundary has important limitations, as it is an *ad hoc* construction: no general formalism determines when the boundary of a reasonably general spacetime is definable, intrinsic, unique and containing useful information on the spacetime. This was also stressed by Marolf and Ross [26] in the framework of the AdS/CFT correspondence: it is trivial to consider a conformal boundary for anti-de Sitter spacetime but, for other backgrounds such as plane waves [4, 5], this boundary may no exist (or yield suspicious properties) so that the causal boundary (c-boundary, for short) must be used [13, 26].

One of the main motivations of the c-boundary is to fill this gap of the conformal one — as well as gaps of the other type of boundaries, see [18, 35]. The c-boundary is intrinsic to the spacetime and conformally invariant, according to the seminal construction by Geroch *et al.* [17]. As a difference with the conformal one, the c-boundary takes into account only timelike curves and directions. Thus, the naive expectation is that the part of the conformal boundary which is accessible by means of timelike directions of the original spacetime, must agree with the c-boundary. In this case, the (accessible) conformal boundary would be endowed with a natural intrinsic meaning. There are other motivations for the study of the c-boundary. As suggested in [35], this boundary yields an appealing extended causal picture of the spacetime. Indeed, the possible lack of global hyperbolicity of the spacetime is associated to the existence of a timelike point of the c-boundary (naked singularity); the non-timelike points generate naturally a future and a past infinity. From the viewpoint of the hyperbolic equations and the initial value problem, the solution on any neighborhood of the timelike points and the past (or future) infinity, would determine the solution. In simple cases, the boundary conditions for the equations might be imposed on the timelike points, and the initial conditions might be imposed in the past infinity, or regarded as a sort of limit on it.

Nevertheless, there has been a serious obstacle for the c-boundary until now. This is the so-called *identification problem* between future and past preboundary points, a problem of consistency which also affects the choice of a natural topology for the c-completion. Many authors have tried to solve or circumvent it [7, 17, 20, 21, 33, 38, 39] (see [15, 35] for detailed reviews). However, among other problems, the choices of identifications and topologies have been affected by a major drawback: the so-defined c-boundaries did not agree with the conformal one, even in some simple open subsets of Lorentz Minkowski — where a choice of the conformal boundary seemed obvious [23–25, 34, 35]. The unsatisfactory behavior appeared clearly on the side of the c-boundary, especially in its topology.[1]

[1]Kuang and Liang, who found quite a few of these examples, commented in the last one: "We are inclined to believe that the whole project of constructing a singular boundary has to be given up" [25].

However, in the last years, some new ideas have been introduced on these problems. First, Harris introduced the notion of chronological set [20], crucial to justify the universal character of the c-boundary approach, and suggested a limit operator on it [21]. Independently, Marolf and Ross [27], introduced a different viewpoint for the identification problem: no future and past preboundary point must be identified, but any c-boundary point must be regarded as a pair of two subsets of the spacetime (a future and a past one), which satisfy a certain binary relation (the *Szabados relation*, or simply, *S-relation*). Then, these authors introduced a reasonable c-boundary as a point set, endowed with a natural extended chronological relation. They also suggested a pair of possible topologies, with the hope that some topology between them were good enough. Taken into account this viewpoint of pairs, the first-named author introduced in [9] a very general notion of conformally invariant completion for spacetimes, or, in general, for Harris' chronological sets. In spite of its generality, a natural assumption on minimality for the completions made the S-relation to appear, providing in particular a good support for Marolf–Ross (MR) construction. Moreover, this author: (1) stretched to the limit the assumption on minimality, arriving to the *chronological completions*, which may differ from MR one, and (2) introduced a different topology, the *chronological (chr.) topology*, inspired in some ideas by Harris. These two fresh ideas, however, introduced more doubts on the existence of a unique satisfactory definition of c-boundary. Nevertheless, the critical review of the c-boundary by the third-named author [35], announced that, with all the previous ingredients at hand, there is a natural choice for the c-boundary in most cases. Moreover, this choice will be consistent with the conformal boundary in the natural cases, giving a good support to both boundaries. Summing up, the moral is the following. Even though our choice of the c-boundary may be revised and modified, if some mild hypotheses are satisfied, then it is both, unique and endowed with a good number of satisfactory properties. As these characteristics should be reproduced by any further redefinition, the c-boundary can be used safely in most cases, with independence of future redefinitions.

Our purpose here is to develop this idea in detail. The plan of work is the following. After some preliminaries in Section 2, the paper is divided into two parts. The first one is devoted to the c-boundary (Section 3), and the second to the conformal one (Section 4).

In the first part, our main aim is to find a definition of the c-boundary for any strongly causal spacetime which is supported by simple and general requirements. These requirements are widely discussed and compared with possible alternatives. Once the c-boundary is defined, its properties are analyzed. More precisely, in Section 3.1, we isolate which minimal conditions must be fulfilled, in order to have a satisfactory completion. So, starting at the general framework derived from Geroch *et al.* [17], Marolf and Ross [27] and Flores [9], we introduce some conditions on admissibility for the c-boundary at the three levels — point set, chronologically and topologically. At each level, we prove that there are few possible alternatives and, essentially, only one reasonable option. Concretely, we show that, as a point set, the admissible boundaries may lie between Flores' chronological completions (minimal options) and MR one (maximum one). All of them will coincide in the relevant cases. Our canonical choice here will be the univocally determined MR's one — even though the framework of chronological ones may be required further. As a chronological space, different arguments will confirm that the extended chronological relation, redefined by Marolf and Ross from the work by Szabados [38], is the undisputable admissible chronology. However, the question is subtler as a topological space. Here, the essential conditions of admissibility are two: (A1) chronological futures and pasts must be open, and (A2) the limits of the converging sequences in the completions and the point-set structure of the futures and pasts of the terms of the sequences, must satisfy a minimum compatibility. As a third condition (A3), we also impose to have the coarsest topology which fulfills the other requirements. This last condition not only avoids undesirable topologies, such as the discrete one, but it is also necessary to fix one topology. In fact, we choose Flores' chr. topology and show that, when it is of first order (Definition 3.10), then it is selected univocally — it is the unique sequential admissible topology, Theorem 3.22. As such a hypothesis is very mild (see Section 3.6, in particular Proposition 3.44), any other admissible topology would differ only in the rather pathological cases where it is not satisfied. So, our choices for the c-boundary ∂M and c-completion $\overline{M}$ are concluded here. In Section 3.2, we discuss MR topology in [27]. In fact, they introduced two topologies, the first one did not satisfy the condition of admissibility (A2), and the second one did not satisfy (A1). As (A2) is somewhat subtler, we analyze further both, (A2) and the first MR topology. Of course, this topology will also agree with the chr. topology in most cases. However, some properties which can be derived by using (A2) (as the T_1 character of the topology) allows to fix a single one and, so, more non-T_1 topologies similar to MR one should be conceivable. In Section 3.3, we show that our choice of c-boundary fulfills satisfactory intrinsic properties, (Theorem 3.27) — and they also hold for any admissible completion as a point set. In Section 3.4, we focus on the c-boundary of globally hyperbolic spacetimes, and provide a natural definition of (asymptotically) conformally flat end. Finally, in Section 3.5, we discuss the following question. The c-completion ensures that any timelike curve will have an endpoint either in the spacetime or in the boundary, at what extent does this remain true for causal curves? We show that, even though this property may not hold in some cases (and, eventually, the c-boundary might be redefined to ensure it), it holds for some relevant and general families of spacetimes (Theorem 3.35, Remark 3.36). Finally, in the discussion in the last section of this part, we explain some subtleties about the topologies determined by a limit operator, such as the chr and Harris ones, which turn out important to understand their possible pathologies.

The second part has a three-fold aim: first, to stress the difficulties for a useful definition of the conformal boundary (Sections 4.1, 4.2), second to provide general conditions so that the (accessible) conformal boundary agrees with the causal one (Section 4.3) and third to obtain easily computable conditions for the conformal boundary points which are C^1 (Section 4.4). More precisely, in Section 4.1, we provide the general notions of conformal envelopment $i : M \hookrightarrow M_0$, boundary $\partial_i M$ and completion $\overline{M}_i$ (Definition 4.1), discuss the possible chronological and causal relations definable in $\overline{M}_i$ (Remark 4.2), focus on the accessible parts $\partial_i^* M, \overline{M}_i^*$ (Definition 4.4), and impose chronological completeness (Definition 4.5). Under this assumption, there exist well-defined projections $\hat{\pi}$, $\check{\pi}$ from the past and future causal completions to the conformal one (Theorem 4.7). However, these projections induce a well-defined projection of the c-completion $\pi : \overline{M} \to \overline{M}_i^*$ only when some compatibility between $\hat{\pi}$ and $\check{\pi}$ holds (formula (4.3)). Of course, we would like not only that π were well defined, but also that it were an isomorphism at the three levels (point-set, chronology and topology) — so that one can identify $\overline{M} \equiv \overline{M}_i^*$. The difficulties are discussed in Section 4.2. Then, in Section 4.3, we give the general notion of *regular accessibility* for conformal boundary points (Definition 4.14), which ensures the required identification (see the main result Theorem 4.16). This notion comprises two properties, *timelike deformability* (Definition 4.10) and *timelike transitivity* (Definition 4.13). Even though such properties are not difficult to check in practice, in Section 4.4, we focus on C^1 boundary points, and find natural interpretations to ensure them. In fact, there are two natural chronological relations $\ll_i, \ll_i^{\mathrm{S}}$, and two natural causal relations $\leq_i, \leq_i^{\mathrm{S}}$ definable in $\overline{M}_i$. When these relations agree (i.e., the conformal completion is *chronologically tame*, $\ll_i = \ll_i^{\mathrm{S}}$, and *causally tame*, $\leq_i = \leq_i^{\mathrm{S}}$), then the identification of the two completions is ensured, $\overline{M} \equiv \overline{M}_i^*$ (Theorem 4.26). Moreover, in practice one has to check only the following property: if a point of the boundary $z \in \partial_i M$ is the endpoint of a future-directed (resp. past-directed) timelike curve γ in M, then γ can be chosen such that it is also smooth and timelike at z (*strong accessibility*, Definition 4.19, see Corollary 4.28 and Remark 4.29). We end this section by studying conformal boundaries which are C^1 at any accessible boundary point $z \in \partial_i^* M$. In this case, z is called *timelike* if so is its tangent hyperplane $T_z(\partial_i^* M)$. We prove that the absence of timelike points implies the equivalence of the boundaries $\overline{M} \equiv \overline{M}_i^*$ (Theorem 4.32). Moreover, among other results, we prove rigourously a frequently claimed assertion:

the absence of timelike points is equivalent to the global hyperbolicity of the spacetime (Corollary 4.34). So, for globally hyperbolic spacetimes with C^1 boundary the conformal and causal completions are fully equivalent ($\overline{M} \equiv \overline{M}_i^* \equiv \overline{M}_i$). Finally, quite a few examples about the differences between the causal and conformal boundaries are collected in Section 4.5. They are referred along all Section 4, but its independent reading may help to understand the process and choices we have carried out.

2 Preliminaries

Here, we introduce some basic concepts and terminology. However, some background and motivation on the c-boundary is assumed and, so, the survey [35] is recommended. For the basic elements of causality, we follow the conventions in [28,31]. So, a *spacetime* is a Lorentzian manifold $M \equiv (M, g)$ of dimension $N > 1$ where a time-orientation is implicitly assumed. A tangent vector $v \in T_pM, p \in M$ is *causal* if it is either timelike ($g(v, v) < 0$) or lightlike ($g(v, v) = 0, v \neq 0$).

2.1 Background on the c-boundary

The c-boundary of a spacetime was introduced by Geroch *et al.* [17] under an appealing philosophy: to add *ideal points* to the spacetime so that any timelike curve acquires some endpoint in the completed space. To formalize this GKP seminal idea, the following elements are basic.

A non-empty subset $P \subset M$ is called a *past set* if it coincides with its past; i.e., $P = I^-[P] := \{p \in M : p \ll q$ for some $q \in P\}$. The *common past* of $S \subset M$ is defined by $\downarrow S := I^-[\{p \in M : \; p \ll q \;\; \forall q \in S\}]$. A past set that cannot be written as the union of two proper subsets, both of which are also past sets, is called *indecomposable past* set, IP. An IP which does coincide with the past of some point of the spacetime $P = I^-(p)$, $p \in M$ is called *proper indecomposable past set*, PIP. Otherwise, $P = I^-[\gamma]$ for some inextensible future-directed timelike curve γ, and it is called *terminal indecomposable past set*, TIP. The dual notions of *future set*, *common future*, IF, TIF and PIF, are obtained just by interchanging the roles of past and future in previous definitions.

To construct the GKP future causal completion, first identify every event $p \in M$ with its PIP, $I^-(p)$ (this is consistent for distinguishing spacetimes, see definition below). Then, the *future c-boundary* $\hat{\partial}M$ of M is defined as the set of all the TIPs in M. Therefore, *the future causal completion* $\hat{M}$

becomes the set of all the IPs:

$$M \equiv \text{PIPs}, \quad \hat{\partial}M \equiv \text{TIPs}, \quad \hat{M} \equiv \text{IPs}.$$

Analogously, every event $p \in M$ can be identified with its PIF, $I^+(p)$. Then, the *past c-boundary* $\check{\partial}M$ of M is defined as the set of all the TIFs in M, and thus, *the past causal completion* $\check{M}$ is the set of all the IFs:

$$M \equiv \text{PIFs}, \quad \check{\partial}M \equiv \text{TIFs}, \quad \check{M} \equiv \text{IFs}.$$

In order to define the (total) causal completion, consider first the precompletion $M^\sharp = (\hat{M} \cup \check{M})/\sim$, equal to the space $\hat{M} \cup \check{M}$ with PIP's and PIF's identified in an obvious way ($I^-(p) \sim I^+(p)$ on $\hat{M} \cup \check{M}$ for all $p \in M$). However, some additional identifications between the preboundary points $\hat{\partial}M \cup \check{\partial}M$ seemed necessary in order to obtain a reasonably consistent definition. The study of this identification problem was already initiated in [17], and developed further in [7,33,38,39], but without fully satisfactory results. In the initial GKP idea the problem is linked to the choice of an appropriate topology: $M^\sharp$ is made a topological space, and $\sim$ yields the minimum identifications so that the quotient is Hausdorff. The so-constructed completion was expected to work for strongly causal spacetimes, although the existence of $\sim$ required stable causality [38]. But even in this case, some examples showed that the boundary was unsatisfactory [23,24]. Budic and Sachs [7] introduced a direct identification between IPs and IFs, later refined by Szabados [38], which is then called the S-relation $\sim_S$ (see (3.1), (3.2) below). However, the choices of topologies by these and other authors did not fulfill the expectations [23–25]. According to the new viewpoint developed by Marolf and Ross [27], the identifications of preboundary points are abandoned. Instead, one takes all the pairs (P, F) such that P is an IP, F is an IF and they are related by the S-relation. This pairing process is justified in [9] by general arguments involving a minimality condition.

Related important ingredients were introduced by Harris in [20,21]. On one hand, a limit operator, which lies in the core of the topology defined by Flores [9]. On the other, Harris defined the following notion, in order to describe simultaneously the properties of the spacetimes and their completions.

Definition 2.1. A *chronological set* $(X, \ll)$ is a set X endowed with a *chronology*, i.e., a binary relation $\ll$ which is transitive, anti-reflexive and satisfies

(i) it contains no isolates: each $x \in X$ satisfies $x \ll y$ or $y \ll x$ for some $y \in X$, and

(ii) it is *chronologically separable*, that is, there exists a countable set $\mathcal{S} \subset X$ which is *chronologically dense*: for all $x \ll y$ there exists $s \in \mathcal{S}$ such that $x \ll s \ll y$.

Notice that definitions for spacetimes on causality and GKP construction such as chronological futures and pasts $I^{\pm}(x)$, IPs, TIPs, etc., can be translated immediately to chronological sets. In particular, future-directed timelike curves are replaced by *future-directed (chronological) chains*, i.e., sequences $\varsigma = \{x_n\}_n$ such that $x_n \ll x_{n+1}$ for all n. Chronological separability ensures that IPs coincide with pasts of future-directed chains [20, Theorem 3]. A point $x \in X$ is *regular* if both, $I^-(x)$ is an IP and $I^+(x)$ an IF; notice that this happens in spacetimes (where they are PIP and PIF, resp.) but not in general, see figure 9. The chronological set will be *distinguishing* when $x \neq y \Rightarrow I^-(x) \neq I^-(y)$ and $I^+(x) \neq I^+(y)$, i.e., each point $x \in X$ is identifiable with both, $I^-(x)$ and $I^+(x)$.

Remark 2.2. Chronological sets constitute the general framework for c-boundaries and, trivially, our choice of c-boundary can be formulated on them. However, we will focus on minimally well-behaved spacetimes, being strong causality enough for our purposes. The reason is that we are looking for the admissible conditions which must be satisfied by a c-boundary. A chronological set does not have a simple distinguished topology — except the Alexandrov one, which is *not* appropriate for the completions of spacetimes, even the strongly causal ones. Although the chronological topology, which will be our final choice, can be formulated in chronological sets, the motivations for this topology become more transparent in spacetimes.

In what follows, all spacetimes will be strongly causal, except if otherwise is said explicitly.

2.2 Some technicalities related to the conformal boundary

Once the c-boundary is constructed, our aim is to test it by means of the conformal one. For this boundary, one embeds conformally the original spacetime M in a new (*aphysical*) spacetime M_0 of the same dimension, and regards the topological boundary of M in M_0 as the conformal boundary. We will discuss widely this idea below; here, we just point out some technicalities.

Remark 2.3. (1) In causality theory, the causal and timelike curves with two fixed endpoints are typically assumed *piecewise smooth*, i.e., they have a finite number of breaks. In principle, there is no problem to smooth such curves retaining its causal or timelike character. Nevertheless, a difficulty arises when the conformal boundary is studied. Here, we consider inextensible timelike curves γ in the original spacetime M which will have an endpoint in the aphysical spacetime M_0. In principle, one can allow that these curves have an infinite number of breaks, and all the breaks may be smoothed. However, the extended curve $\bar{\gamma}$ to M_0 may not be smooth at its endpoint in $M_0 \setminus M$ (figure 2). The difficulty to smooth it satisfactorily is inherent to the fact that the endpoint lies in the boundary but the curve γ must be smoothed in M.

(2) The most general space for causal curves is the one of, say, future-directed *continuous causal curves.* Such a curve ρ is a continuous one which satisfies $t_1 < t_2 \Rightarrow \rho(t_1) \leq \rho(t_2)$ (usually, this condition is assumed to hold for arbitrarily small neighborhoods, however, we can drop this requirement as our spacetimes will be always strongly causal). Future-directed continuous causal curves can be characterized as those locally absolutely continuous curves such that its velocity is almost everywhere future-directed causal [8] (in particular, if a continuous curve lies in the conformal closure of a spacetime, its possible causal character depends only on this closure). Continuous causal curves will play a role in the interplay between the causal relations in M and M_0.

Summing up, in what follows all the causal curves in M will be assumed smooth with no loss of generality and, if a causal curve in M_0 is regarded only as continuous, then it will be called explicitly continuous causal curve. Finally, as a technical result to be used frequently:[2]

Proposition 2.4. *Let M be a strongly causal spacetime and $\gamma : [a, b) \to M$ a future-directed timelike curve. The curve γ can be continuously extended to some $p \in M$ if either*

(i) for some sequence $\{t_n\}_n \nearrow b$ the sequence $\{\gamma(t_n)\}_n$ converges to p, or

(ii) $I^-[\gamma] = I^-(p)$.

Proof. For (i), notice that, otherwise, strong causality would be violated at p. For (ii), just recall that $I^-[\gamma]$ is a PIP and, thus, γ is continuously extensible [3, Proposition 6.14]. $\square$

[2] All the results will be stated for future-directed curves, TIPs, etc., and the dual versions for past-directed curves, TIFs or dual elements, will be used without further mention.

3 Admissible completions: uniqueness of the c-boundary

In this part, we analyze the admissible possibilities for a c-boundary. We retain the original idea that any inextensible future or past-directed timelike curve must converge to a boundary point. Of course, one can think that non-timelike curves should be also included in a full completion. This will be discussed for lightlike curves in Section 3.5. However, we will not try to include spacelike curves, as this would require a very different viewpoint — c-boundary is based on Causality, which is the global conformal invariant of a spacetime. At any case, if a bigger conformally invariant completion were developed, the c-boundary should be included in some way (see also Section 3.5).

3.1 Conditions on admissibility

These conditions will appear at the three levels: point-set, chronological and topological.

3.1.1 Admissible completions as point sets

We will identify M with the subset of $\hat{M} \times \check{M}$ formed by all the pairs $(I^-(p), I^+(p))$. Denote also $\hat{M}_\emptyset = \hat{M} \cup \{\emptyset\}$ (resp. $\check{M}_\emptyset = \check{M} \cup \{\emptyset\}$). The S-relation $\sim_{\rm S}$ is defined in $\hat{M}_\emptyset \times \check{M}_\emptyset$ as follows. First, for $(P, F) \in \hat{M} \times \check{M}$:

$$P \sim_S F \Longleftrightarrow \begin{cases} F & \text{is included and is a maximal IF in} \quad \uparrow P \\ P & \text{is included and is a maximal IP in} \quad \downarrow F. \end{cases} \tag{3.1}$$

(Recall that, as proved by Szabados [38], $I^-(p) \sim_{\rm S} I^+(p)$ for all $p \in M$, and these are the unique S-relations involving proper indecomposable sets, according to (3.1)). For $(P, F) \in \hat{M}_\emptyset \times \check{M}_\emptyset$, with $(P, F) \neq (\emptyset, \emptyset)$ we also put

$$P \sim_{\rm S} \emptyset, \quad (\text{resp. } \emptyset \sim_{\rm S} F) \tag{3.2}$$

if P (resp. F) is a (non-empty, necessarily terminal) indecomposable set and is not S-related by (3.1) to any other indecomposable set. Notice that $\emptyset$ is never S-related to itself.

Definition 3.1. A subset $\overline{M} \subset \hat{M}_\emptyset \times \check{M}_\emptyset$ is called an *admissible completion as a point set* for M if it contains M (i.e., $\{(I^-(p), I^+(p)) : p \in M\} \subset \overline{M}$) and satisfies the following conditions:

(1) Completeness: every TIP and TIF in M is the component of some pair in the *boundary of the completion*, i.e., in the subset $\partial M := \overline{M} \setminus M$,
(2) S-relation: if a pair $(P, F) \in \hat{M}_\emptyset \times \check{M}_\emptyset$ lies in $\overline{M}$ then $P \sim_S F$.

Remark 3.2. (Justification of the conditions on admissibility.)

(A) To regard $\overline{M}$ as a subset of $\overline{M} \subset \hat{M}_\emptyset \times \check{M}_\emptyset$ is just a general framework (introduced in [27]). The axiom (1) is necessary in order to formalize the idea that any timelike curve which is inextensible towards the future (or past) will have an *endpoint* in the boundary. This can be formalized more accurately in the general setting of chronological spaces by requiring that any chain has an *endpoint* (see [9, Section 3] for definitions).

In principle, it is possible to consider even a more general framework. The notion of *completion* introduced in [9, Definition 3.2] allows $\overline{M}$ to be included not only in $\hat{M}_\emptyset \times \check{M}_\emptyset$ but also in the larger set $M_P \times M_F$ which is formed by all the pairs composed by a future and a past set (see also [35, Definition 5.3, Proposition 5.2]). In this case, the notion of endpoint is a bit stronger than the one of limit point. However, the analysis of this a priori more general completions in [9, Theorem 7.4] (see also below), shows that the relevant completions will turn out subsets of $\hat{M}_\emptyset \times \check{M}_\emptyset$ (i.e., IPs and IFs of the original GKP idea are recovered) where axiom (2) holds. In this case, the notions of endpoint and limit point become equivalent for chains, and our simplified framework is sufficient.

(B) Condition (2) plus the following one,

(NR) *Non-redundancy*: if $(P, F_1), (P, F_2) \in \partial M$ and $F_1 \neq F_2$ (resp. (P_1, F), $(P_2, F) \in \partial M$ and $P_1 \neq P_2$) then F_i (resp. P_i), $i = 1, 2$, do not appear in another pair of ∂M,

characterize those completions which are *minimal* in a natural sense. This minimality condition was introduced in the general setting of completions in $M_P \times M_F$ for a chronological set [9, Definitions 7.3, 7.1]. In this reference, completions satisfying (NR) are called *chronological completions*, and it is proved that, when a chronological completion of a strongly causal spacetime is considered, it satisfies axiom (2) automatically. That is, the axiomatically imposed condition (2) can be deduced from a more fundamental viewpoint, supporting the appearance of the S-relation.

Remark 3.3. (Uniqueness of the admissible completions as a point set.) MR completion [27] is the one which contains all the possible pairs compatible with (2), that is, if $P \sim_S F$ then $(P, F) \in \overline{M}$. Notice that, in this case, condition (1) becomes superfluous, that is, MR completion is characterized by the following strengthening of (1) and (2):

(MR) A pair $(P, F) \in \hat{M}_\emptyset \times \check{M}_\emptyset$ lies in $\overline{M}$ if and only if $P \sim_{\mathrm{S}} F$.

So, as a point set, MR completion is characterized as the maximum admissible completion. In particular, MR completion is unique, as a difference with chronological completions. However, it may be redundant, even in cases where the chronological completion is unique, see [9, Example 10.6], [27, Appendix A].

Summing up:

1. The axioms for admissible completions, including the S-relation, are not only intuitively sound but also appear naturally from the general notion of chronological completions.
2. Then, MR completion (characterized by *(MR)* above) is selected here as the maximum admissible completion — in principle, the unique admissible completion canonically determined.
3. At any case, the differences between the different admissible completions, from the minimal (chronological) completions to the maximum (MR) one are essentially irrelevant from the practical viewpoint: in the natural physical examples all of them coincide (this includes, for example, the subtler case of minimally well-behaved wave-type spacetimes [13]).

   So, in order to construct more than one admissible completion for the same spacetime, one has to consider some rather sophisticated examples as those exhibited in [9], where some regions are artificially removed, or in [12], where standard stationary spacetimes with a very peculiar behavior close to its Busemann boundary are considered.

3.1.2 Admissible chronology

Definition 3.4. Let $\overline{M}$ be an admissible completion as a point set for some strongly causal spacetime M. A chronology $\tilde{\ll}$ on the completion $\overline{M}$ (Definition 2.1) is called

(i) An *extended chronology* if

$$p \in P \Rightarrow (I^-(p), I^+(p)) \tilde{\ll} (P, F), \tag{3.3}$$

$$q \in F \Rightarrow (P, F) \tilde{\ll} (I^-(q), I^+(q)). \tag{3.4}$$

for all $p, q \in M$ and $(P, F) \in \overline{M}$.

(ii) An *admissible chronology* if $I^{\pm}((P,F)) \subset \overline{M}$ computed with $\tilde{\ll}$ satisfies:

$$I^{-}((P,F)) \cap M = P \quad \text{and} \quad I^{+}((P,F)) \cap M = F \quad \forall (P,F) \in \overline{M}. \quad (3.5)$$

Remark 3.5. (A) When restricted to points in M, conditions (3.3), (3.4) only say

$$p \ll q \Rightarrow p \tilde{\ll} q \quad \forall p, q \in M.$$

When $(P,F) \in \partial M$, those conditions mean only that (P,F) is (future or past) chronologically related with the points in the TIP or TIF which define the pair. That is, an extended chronology contains both, the original chronology $\ll$ and the chronological relations which define the boundary points.[3]

(B) Of course, an admissible chronology is an extended one. In fact, condition (3.5) not only implies (3.3) and (3.4) but also says that no other relations are introduced between points in the spacetime or a point in the spacetime and one in the boundary. That is, the chronology $\tilde{\ll}$ is called admissible when it is an extended chronology which does not introduce new (spurious) relations between a point in M and any point in $\overline{M}$.

Remark 3.5 (B) suggests the possibility to single out an admissible chronology just by requiring that no new relations between two points in ∂M are introduced. This will be our choice of extended chronology, which was firstly suggested by Szabados [38], refined to make it consistent by Marolf and Ross [27] and also considered in [9, 35].

Definition 3.6. Let $\overline{M}$ be an admissible completion as a point set for some strongly causal spacetime M. The *extended chronological relation* $\overline{\ll}$ is the relation on $\overline{M}$ given by:

$$(P,F) \overline{\ll} (P',F') \;\;\Leftrightarrow\;\; F \cap P' \neq \emptyset, \quad \forall (P,F), (P',F') \in \overline{M}. \quad (3.6)$$

The name of this relation suggests that it will be the canonical one on $\overline{M}$. To justify this, it is easy to check first that $\overline{\ll}$ is an admissible chronology (see, for example, [9, Theorem 4.3]), as well as the following characterization.

[3] In the discussion at the end of Section 4.2, in the item (4), we will see that the conformal boundary may lead to some dilemma in the definition of a chronological relation for the conformal completion. In fact, from the viewpoint of this boundary, it seems consistent to admit $p \in P$ but $p \tilde{\not\ll} (P,F)$ for some $p \in M$ and (P,F) in the boundary. However, such a possibility appears only because of the non-intrinsic character of the conformal boundary, and would generate new problems. Moreover, if such a possibility were admitted in general, one could also speculate with very undesirable chronological relations on $\overline{M}$, as the one which maintains only the original chronological relations in M.

Theorem 3.7. *Let $\overline{M}$ be an admissible completion as a point set for some strongly causal spacetime M, and $\tilde{\ll}$ an extended chronology on $\overline{M}$. Then, the extended chronological relation $\overline{\ll}$ satisfies:*

$$(P, F)\overline{\ll}(P'F') \Rightarrow (P, F)\tilde{\ll}(P'F'), \quad \forall (P, F), (P', F') \in \overline{M}. \tag{3.7}$$

That is, the extended chronological relation $\overline{\ll}$ is the minimum extended chronology for an admissible completion as a point set $\overline{M}$ of a strongly causal spacetime.

Proof. Assume that $(P, F)\overline{\ll}(P', F')$ for some $(P, F), (P', F') \in \overline{M}$. From (3.6), there exists some $p \in F \cap P' \neq \emptyset$. From (3.3), (3.4), $(P, F)\tilde{\ll}(I^-(p), I^+(p))\tilde{\ll}(P', F')$. Hence, by the transitivity of $\tilde{\ll}$, we deduce $(P, F)\tilde{\ll}(P', F')$. □

Even though this result singles out the extended chronological relation, one could still speculate with the possibility of an admissible chronology such that additional chronological relations are introduced between the boundary points. However, the following result suggests that this is not reasonable — at least if the construction is intended to has a minimum compatibility with a reasonable topology.

Theorem 3.8. *Let $\overline{M}$ be an admissible completion as a point set for some strongly causal spacetime M, and $\tilde{\ll}$ an admissible chronology on $\overline{M}$. Assume that $\overline{M}$ is endowed with any topology such that:*[4] *(i) M is dense into $\overline{M}$, and (ii) the topology is compatible with $\tilde{\ll}$ in the sense that $I^{\pm}((P, F))$ is open in $\overline{M}$ for all $(P, F) \in \overline{M}$.*

Then, $\tilde{\ll}$ is equal to the extended chronological relation $\overline{\ll}$.

Proof. Taking into account Theorem 3.7, we have just to prove the converse of (3.7), under our additional hypotheses for $\tilde{\ll}$. Assume that $(P, F)\tilde{\ll}(P', F')$. Since $\tilde{\ll}$ is chronologically separable (Definition 2.1), there exists $(P'', F'') \in I^+((P, F)) \cap I^-((P', F'))$ (notice that $I^{\pm}$ are computed with $\tilde{\ll}$ in $\overline{M}$). Since $I^{\pm}(\cdot)$ are open and M is dense in $\overline{M}$, we can choose (P'', F'') equal to a point $p \in M$. As $\tilde{\ll}$ is admissible, $I^+((P, F)) \cap M = F$, $I^-((P', F')) \cap M = P'$, and $p \in F \cap P' \neq \emptyset$, i.e. $(P, F)\overline{\ll}(P', F')$, as required. □

[4] We anticipate here some properties of consistency for the topology, which will be developed below.

Summing up:

> The extended chronological relation $\overline{\ll}$ (Definition 3.6) is singled out as
> (1) The minimum extended chronology (Theorem 3.7).
> (2) The unique admissible chronology compatible with a minimally well-behaved topology (Theorem 3.8).
>
> Moreover, it is, in principle, the unique admissible chronology canonically defined. So, even though other chronologies have been defined on $\overline{M}$ (see [35, Remark 5.1, Subsection 3.3.1] for a discussion), the properties above make $\overline{\ll}$ the standard choice of chronology in $\overline{M}$.
>
> So, $\overline{\ll}$ will be taken in any admissible completion as a point set $\overline{M}$ along the remainder of this paper.

Convention 3.9. *In what follows, the points of the spacetime are denoted by lower case letters $p, q \in M$, and their chronological future and past in M by $I^+(p)$, $I^-(q)$. Points in the completion (eventually also in M) are denoted as pairs (P, F), and $I^+((P, F)), I^-((P, F)) \subset \overline{M}$ denote their chronological future and past in $\overline{M}$ for the extended chronological relation. In case of possibility of confusion, the notation $I^+(p, M)$, $I^+(p, \overline{M})$ is used for the chronological future of $p \in M$ in M and $\overline{M}$, resp. Consistently, a superscript c will denote the complementary of the corresponding subset; for example, $I^+((P, F))^{\mathrm{c}} = \overline{M} \setminus I^+((P, F))$.*

3.1.3 Admissible topologies

In the following, we are going to assume that $\overline{M}$ is an admissible completion as a point set (Definition 3.1) endowed with the extended chronological relation $\overline{\ll}$ (Definition 3.6). Our next goal is to determine when a topology on $\overline{M}$ can be admitted as admissible and, even more, if one such topology can be singled out. In principle, there are two good candidates, one of the two choices by Marolf and Ross in [27] (see Section 3.2) and the chronological topology introduced in [9]. First, we consider the latter in order to make clear its point-set viewpoint.

Definition 3.10. Let $\overline{M}$ be an admissible completion as a point set for some strongly causal spacetime M. The *(chr.) limit operator* L maps each sequence $\sigma = \{(P_n, F_n)\} \subset \overline{M}$ in the subset $L(\sigma) \subset \overline{M}$ defined as:

$$(P, F) \in L(\sigma) \quad \Leftrightarrow \quad \text{if } P \neq \emptyset,\ P \in \hat{L}(P_n) \quad \text{and} \quad \text{if } F \neq \emptyset,\ F \in \check{L}(F_n),$$

where

$$P \in \hat{L}(P_n) \Leftrightarrow \begin{cases} P \subset \mathrm{LI}(P_n) \\ P \text{ is a maximal } \mathit{IP} \text{ in } \mathrm{LS}(P_n) \end{cases}$$

$$F \in \check{L}(F_n) \Leftrightarrow \begin{cases} F \subset \mathrm{LI}(F_n) \\ F \text{ is a maximal } \mathit{IF} \text{ in } \mathrm{LS}(F_n) \end{cases}$$

and LS, LI denote the lim-sup and lim-inf operators in set theory.

The *chr. topology* is the topology in $\overline{M}$ such that $C \subset \overline{M}$ is defined as closed if and only if for any sequence σ in C, necessarily $L(\sigma) \subset C$. The limit operator and, then, the chr. topology is *of first order* if $\sigma \to (P, F)$ implies $(P, F) \in L(\sigma)$ (see Section 3.6 for subtleties on this definition).

The topology of M is generated by the sets type $I^{\pm}(p), p \in M$ (Alexandrov topology) as M is strongly causal. In principle, a first requirement for any admissible topology on $\overline{M}$ seems to maintain all the sets $I^{\pm}((P, F))$ as open. This property is characterized in the following lemma.

Lemma 3.11. *Let $\overline{M}$ be an admissible completion as a point set for a strongly causal spacetime M. If a topology on $\overline{M}$ satisfies that $I^{\pm}((P, F))$ are open for all $(P, F) \in \overline{M}$ then the following property holds: if $(P_n, F_n) \to (P, F)$ then $P \subset \mathrm{LI}(P_n)$, $F \subset \mathrm{LI}(F_n)$. Moreover, the converse is true if the topology is sequential* (*see Definition* 3.37 *and Remark* 3.38)*. In this case, $I^{\pm}(p)(= I^{\pm}(p, M))$ is also open in $\overline{M}$ for all $p \in M$.*

Proof. ($\Rightarrow$). If $p \in P$ then $(P, F) \in I^{+}(p, \overline{M})$, which is open. As $\{(P_n, F_n)\} \to (P, F)$, necessarily $(P_n, F_n) \in I^{+}(p, \overline{M})$ for all n big enough. Whence, $p \in P_n$ for big n, and thus, $p \in LI(P_n)$. In conclusion, $P \subset \mathrm{LI}(P_n)$. The inclusion $F \subset \mathrm{LI}(F_n)$ is analogous.

($\Leftarrow$). Let us prove that $I^{+}((P_0, F_0))^{\mathrm{c}}$ is sequentially closed for any $(P_0, F_0) \in \overline{M}$. Let (P, F) be in the limit of a sequence $\{(P_n, F_n)\} \subset I^{+}((P_0, F_0))^{\mathrm{c}}$ and so $F_0 \cap P_n = \emptyset$ for all n. By assumption, $P \subset \mathrm{LI}(P_n)$ and, therefore, $F_0 \cap P = \emptyset$. Thus, $(P, F) \in I^{+}((P_0, F_0))^{\mathrm{c}}$. (This last argument applied to points in M also proves the last assertion.) $\square$

Definition 3.12. The coarsest topology in $\overline{M}$ such that $I^{\pm}((P, F))$ is open for all $(P, F) \in \overline{M}$ will be called the *coarsely extended Alexandrov topology (CEAT).*

Remark 3.13. (A) CEAT is characterized by (see Remark 3.45(2)):

$$(P_n, F_n) \to (P, F) \quad \Leftrightarrow \quad P \subset \mathrm{LI}(P_n) \text{ and } F \subset \mathrm{LI}(F_n). \tag{3.8}$$

As pointed out by Geroch *et al.* [17] such a topology is not enough for the completion, see figure 1 (A) or (B). Thus, an admissible topology must also contain other open subsets, but they must be “as few as

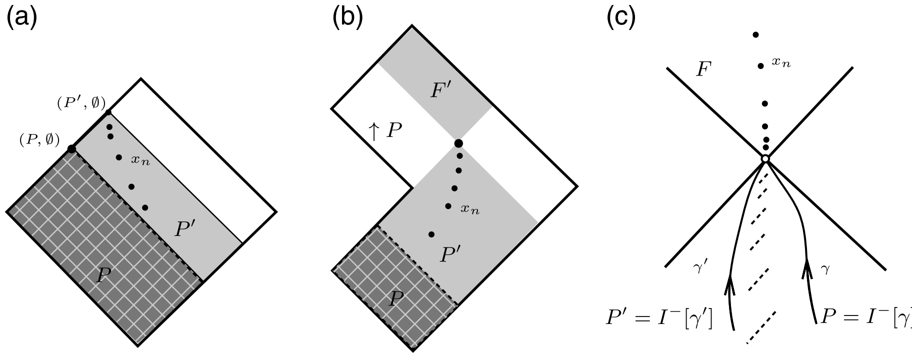

Figure 1: Some open subsets of Lorentz–Minkowski $\mathbb{L}^2$. In figure (a), the sequence $\{x_n\}_n$ converges to both $(P,\emptyset)$, $(P',\emptyset)$ with CEAT, but does not converge to $(P,\emptyset)$ with any topology satisfying (A1), (A2) in Definition 3.18 (requirement (A3) ensures the convergence to $(P',\emptyset)$). Analogously, in figure (b) the sequence $\{x_n\}_n$ converges to both $(P,\emptyset)$, (P',F') with CEAT, but does not converge to $(P,\emptyset)$ if (A1), (A2) are fulfilled. In figure (c), the sequence $\{x_n\}_n$ (as well as the sequence constantly equal to (P,F)) converges to both (P,F), $(P',\emptyset)$ with the MR topology, but only converges to (P,F) with the chr. topology.

possible" — otherwise, a topology such as the discrete one (which regards all the subsets as open) could be admitted, but this is not useful at all. More precisely, properties as the convergence of sequences in M or the density of M in $\overline{M}$, must be retained; so, not too many subsets must be admitted as open.

(B) The sets $I^{\pm}((P,F))$ are always open for the chr. topology. In fact, the chr. topology is sequential (see Proposition 3.39) and Lemma 3.11 is applicable (recall Remark 3.45(C)).

Indeed, the following results will show that, for the points (P,F) with $P \neq \emptyset \neq F$, the CEAT will be enough. Recall previously the following general property (one only needs the first countability of CEAT to prove it, see Remark 3.45(1)):

Proposition 3.14. *For any topology τ finer than CEAT, they are equivalent:*

(1) *τ and CEAT admit a common local basis of neighborhoods of (P,F).*
(2) *If $(P_n,F_n) \not\to (P,F)$ with τ then $(P_n,F_n) \not\to (P,F)$ with CEAT.*

(Notice that the converse of (2) always holds by the hypotheses on τ.)

Lemma 3.15. *Let $\{(P_n, F_n)\}$ be a sequence of pairs in $\overline{M}$ and assume that $P \sim_S F$ with $P \neq \emptyset \neq F$. If $P \subset \mathrm{LI}(P_n)$, $F \subset \mathrm{LI}(F_n)$ then P and F are maximal into $\mathrm{LS}(P_n)$ and $\mathrm{LS}(F_n)$, resp.*

Proof. Assume by contradiction the existence of, say, some P' satisfying $P \subsetneq P' \subset \mathrm{LS}(P_n)$. Then, any point of P' will lie in the past of infinitely many pairs (P_n, F_n) and, by transitivity, will lie also in the past of any point of F (recall that $F \subset \mathrm{LI}(F_n)$). Therefore, P' will be included in $\downarrow F$, in contradiction with $P \sim_S F$. $\square$

Now, we have

Proposition 3.16. *Let $(P, F) \in \overline{M}$ with $P \neq \emptyset \neq F$. Then $\{(P_n, F_n)\}$ converges to (P, F) with CEAT iff it converges with the chr. topology, i.e., $P \in \hat{L}(P_n)$ and $F \in \check{L}(F_n)$.*

Proof. The implication to the right follows from Lemma 3.15 (see Definition 3.10 and (3.8)), and to the left from Remark 3.13 (B). $\square$

Remark 3.17. (CEAT/chr. topology is the natural one around any (P, F) with $P \neq \emptyset \neq F$). Previous proposition means that, around any pair (P, F) with $P \neq \emptyset \neq F$, CEAT has so many open sets (in the sense of Proposition 3.14) as the chronological topology. As this topology will not suffer the pathologies found in figure 1 (a) (see below), it is not necessary to add new open subsets to CEAT around such a pair (P, F). So, this topology (or, equivalently, the chr. one) becomes the coarsest admissible possibility. Moreover, as the CEAT is second countable (recall Remark 3.45(1)), each pair (P, F) has a countable neighborhood basis.

In particular, the topology induced on M from the chr. topology agrees with the manifold topology, as M is strongly causal and, thus, endowed with Alexandrov topology.

Previous remark covers points with $P \neq \emptyset \neq F$ and, now, we state conditions for topological admissibility which cover all the points.

Definition 3.18. Let $\overline{M}$ be an admissible completion as a point set for some strongly causal spacetime M. The *conditions of admissibility* for a topology τ on $\overline{M}$ are:

(A1) τ is finer than CEAT, i.e., $I^{\pm}((P, F))$ is open for all $(P, F) \in \overline{M}$.

(A2) The limits for τ are compatible with the empty set, i.e., if $\{(P_n, F_n)\}_n \to (P, \emptyset)$ (resp. $(\emptyset, F)$) and it happened $P \subset P' \subset \mathrm{LI}(P_n)$ (resp. $F \subset F' \subset LI(F_n)$) for some $(P', F') \in \overline{M}$, then $(P', F') = (P, \emptyset)$ (resp. $(P', F') = (\emptyset, F)$).

(A3) τ is maximally coarse among the topologies satisfying the other conditions, i.e., no topology satisfying (A1) and (A2) (or eventually alternative — or additional — conditions of admissibility) is strictly coarser than τ.

Remark 3.19. (Discussion on the role of the conditions for admissibility.) Conditions (A1) and (A3) have been widely justified above, so, we focus on condition (A2). This seems the obvious choice from the viewpoint of set convergence in figures 1 (a) and (b). However, it can be also understood from the following more general viewpoint.

First, notice that, under the hypotheses in the formulation of (A2), any topology which satisfies a minimum compatibility with the set-point limit operator L must ensure that the constant sequence $\{(P', F')\}_n$ converges to $(P, \emptyset)$. In fact, by hypothesis, $(P_n, F_n) \to (P, \emptyset)$. But P' is closer to P that all P_n (and also P' is "better adapted" as a point set limit to P_n than P) because $P \subset P' \subset LI(P_n)$. Moreover, this inclusion also implies that F' is closer to $\uparrow P$ than all F_n (as $F' \subset \uparrow P' \subset \uparrow P$). In conclusion, the convergence $\{(P', F')\}_n \to (P, \emptyset)$ is compelling from the set-point approach.

But, now, it is also natural to impose $(P', F') = (P, \emptyset)$, since, otherwise, we would admit an (apparently artificial) pair of non-T_1 separated points. However, as MR topology is not T_1, we will come back to this point in the next section (Proposition 3.25, Remark 3.26).

Moreover, under (A1) the hypothesis (A2) has also a very simple characterization, which can be also regarded as an argument in favor of it.

Proposition 3.20. *Let τ be any topology which satisfies the condition of admissability (A1). Then, condition (A2) is equivalent to*

(*) *If $\{(P_n, F_n)\}_n \to (P, \emptyset)$ (resp.$(\emptyset, F)$) then $P \in \hat{L}(P_n)$(resp. $F \in \check{L}(F_n)$).*

Proof. To check that (*) implies (A2), assume that $\{(P_n, F_n)\}_n \to (P, \emptyset)$, and so, $P \in \hat{L}(P_n)$. If $P \subset P' \subset \mathrm{LI}(P_n)$ for some $(P', F') \in \overline{V}$, then $P \subset P' \subset \mathrm{LS}(P_n)$. Therefore, the maximality of P into $\mathrm{LS}(P_n)$ ensures $P = P'$, and thus, $F' = \emptyset$.

For the converse, assume again $\{(P_n, F_n)\}_n \to (P, \emptyset)$. By (A1), $P \subset \mathrm{LI}(P_n)$. Assume that $P \subset P' \subset \mathrm{LS}(P_n)$. Take some chain $\{p'_k\}_k$ generating P', and choose some sequence $\{n_k\}_k$ so that $p'_k \in P_{n_k}$ for all k. Then,

$p'_k \in P_{n_{k'}}$ for all $k' \geq k$, and so, $P \subset P' \subset \mathrm{LI}(P_{n_k})$. Moreover, notice that $\{(P_{n_k}, F_{n_k})\}_k \to (P, \emptyset)$. Hence, by (A2), $P = P'$, $F' = \emptyset$. In conclusion, P is maximal into $\mathrm{LS}(P_n)$, and thus, $P \in \hat{L}(P_n)$. $\square$

The following result shows that the T_1 character of the topology is obtained by grant under the hypotheses (A1), (A2).

Proposition 3.21. *Any topology τ on $\overline{M}$ which satisfies the admissibility conditions* (A1) *and* (A2) *is T_1.*

Proof. Assume the existence of $(P, F), (P', F') \in \overline{M}, (P', F') \neq (P, F)$ such that $\{(P', F')\}_n \to (P, F)$. By condition (A1) and Lemma 3.11, $P \subset P'$, $F \subset F'$. Assume, for example, that the first inclusion is strict. In this case, either $F = \emptyset$ or $P = \emptyset$, since, otherwise, P' would violate the maximality of P in $\downarrow F$, in contradiction with $P \sim_{\mathrm{S}} F$. Therefore, the conclusion follows by applying condition (A2). $\square$

Finally, we show how these conditions do single out a topology.

Theorem 3.22. *Let $\overline{M}$ be an admissible completion as a point set for a strongly causal spacetime M, and assume that the chr. topology on $\overline{M}$ is of first order. Then, it is the unique topology among the sequential ones which satisfies the conditions of admissibility* (A1), (A2) *and* (A3) *in Definition* 3.18.

Proof. As pointed out in Remark 3.13 (B), the chr. topology satisfies condition (A1). In order to prove (A2), assume that $\{(P_n, F_n)\} \to (P, \emptyset)$. By the first-order property, $(P, \emptyset) \in L((P_n, F_n))$. Then, P is a maximal IP in $\mathrm{LS}(P_n)$, and so, if $(P', F') \in \overline{M}$ satisfies $P \subset P' \subset \mathrm{LI}(P_n)$, necessarily $P = P'$ (and therefore $F' = \emptyset$).

Let τ be any sequential topology satisfying (A1) and (A2). Condition (A3), plus the uniqueness, will hold if any sequentially closed set C for the chr. topology is also sequentially closed for τ. Thus, it is enough to check that, whenever $\{(P_n, F_n)\}_n \to (P, F)$ with τ, the limit also holds with the chr. topology. In the case $P \neq \emptyset \neq F$, the result follows from Proposition 3.16. Otherwise, assume, for example, $P \neq \emptyset$, $F = \emptyset$. From Lemma 3.11, $P \subset \mathrm{LI}(P_n)$. By contradiction, assume $P \subsetneq P' \subset \mathrm{LS}(P_n)$. There exists a subsequence $\{(P_{n_k}, F_{n_k})\}$ such that $P \subsetneq P' \subset \mathrm{LI}(P_{n_k})$ (for every k take n_k such that $p'_k \in P_{n_k}$, being $\{p'_k\}$ a future chain generating P'). From the completeness condition in Definition 3.1, there exists some F' such that $(P', F') \in \overline{M}$. As the subsequence also converges to $(P, \emptyset)$, the hypothesis (A2) yields the contradiction $P = P'$. $\square$

Summing up:

> Three conditions on admissibility for a topology have been stated:
> Condition (A1) is just a minimum natural requirement on compatibility with the chronology (even though it disregards some topologies in the literature, see Remark 3.28 below).
> Condition (A2) becomes natural for set point convergence (Remark 3.19) and ensures the T_1 character of the topology. However, it is not fulfilled by MR topology (this is discussed further in Section 3.2).
> Condition (A3) is a minimality requirement which, on one hand, is necessary to ensure that not too many subsets are regarded as open (Remark 3.13 (A)) and, on the other, can be claimed to ensure uniqueness of the topology.
> In fact, when the chr. topology is of first order (recall Remark 3.40(2)), it becomes the unique admissible sequential topology (Theorem 3.22), and so, it is canonically selected.

3.1.4 Summary on possible completions and our choice

Taking into account the stated conditions on admissibility plus the uniqueness of the elements singled out by these conditions, the following definitions are distinguished.

Definition 3.23. Let M be a strongly causal spacetime. Then

(A) An *admissible completion* $\overline{M}$ of M is any admissible completion as a point set (according to Definition 3.1) endowed with the extended chronological relation $\overline{\ll}$ (Definition 3.6) and an admissible topology (Definition 3.18).

(B) The *MR completion* (introduced by these authors in [27]) is the set $\overline{M}$ of all the S-related pairs (P, F) endowed with the extended chronological relation $\overline{\ll}$ and the MR topology (Definition 3.24).

(C) A *chronological completion* (introduced by Flores in [9]) is any completion $\overline{M}$ which is minimal as a point set (in the sense of [9, Definitions 7.3, 7.1]), endowed with the extended chronological relation $\overline{\ll}$ and the chr. topology.

(D) The *c-completion* of M (singled out by Sánchez in [35]) is the completion $\overline{M}$ composed by all the S-related pairs (P, F), endowed with the extended chronological relation $\overline{\ll}$ and the chr. topology. A *standard completion* is any generalization of the c-completion obtained by using an admissible completion as a point set, instead of all the S-related pairs (P, F).

Admissible completions (A) constitute the general framework. They may vary, as a point set, between the minimal option (C) and the maximum and canonically determined one in (B) and (D) (notion of c-completion). Its topology may differ of the chr. topology only if this topology is not of the first order or if a non-sequential admissible topology were chosen — such possibilities will be analyzed in future research on this topic.

MR completion (B) is discussed below. It satisfies many of the nice properties of these boundaries, even though its failure in the condition of admissibility (A2) suggests, in particular, that this completion is a non specially privileged one among other conceivable possibilities.

The notion of c-completion in (D) avoid the lack of uniqueness of the minimal completions for a single spacetime in (C). However, the standard completions defined in (D) include both, the c-completion and all the chronological completions in (C), as well as any intermediate possibility (whenever it is admissible as a point set). Even though the c-boundary will be our choice to be used in Section 4, we consider the general framework of standard completions along the remainder of Section 3. This stresses that most properties remain valid for all of them. Moreover, this viewpoint may be also useful for future developments, in order to consider the inclusion of further boundary points (see Section 3.5).

3.2 MR topology versus chr. topology

Marolf and Ross [27] introduced two reasonable topologies for their completion. One of them does not fulfill the condition of admissibility (A1) (see [35, Section 5.4, footnote 9]) and will not be taken into account here. The other one fulfills (A1), but does not fulfill (A2) (for example, it may be non-T_1). Of course this topology agrees the chr. one in most cases. Nevertheless, we will stress here the differences. MR topology can be rewritten as follows [9]:

Definition 3.24. Let $\overline{M}$ be an admissible completion as a point set for some strongly causal spacetime M. The *MR topology* on $\overline{M}$ is generated by using $L^+(S)$, $L^-(S)$, for any subset $S \subset \overline{M}$, as closed subsets of $\overline{M}$, where

$$L^+(S) = Cl_{FB}[S \cup L^+_{IF}(S)]$$
$$L^-(S) = Cl_{PB}[S \cup L^-_{IP}(S)]$$

with

$$
\begin{aligned}
Cl_{FB}(S) = S \cup \{(P,\emptyset) \in \overline{M} : P = I^-(\mathrm{LI}(P_n)) \\
\text{for some sequence } (P_n, F_n) \in S\} \\
Cl_{PB}(S) = S \cup \{(\emptyset, F) \in \overline{M} : F = I^+(\mathrm{LI}(F_n)) \\
\text{for some sequence } (P_n, F_n) \in S\}
\end{aligned}
$$

and

$$
\begin{aligned}
L^+_{IF}(S) = \{(P,F) \in \overline{M} : F \neq \emptyset, F \subset \cup_{(P',F')\in S} F'\} \\
L^-_{IP}(S) = \{(P,F) \in \overline{M} : P \neq \emptyset, P \subset \cup_{(P',F')\in S} P'\}.
\end{aligned}
$$

Some caution must be taken into account, for example, the equality $Cl_{FB}((Cl_{FB}(S))) = Cl_{FB}(S)$ must be checked. Assuming that it holds (eventually, under favorable hypotheses in the spirit of Proposition 3.44), one can find properties such as:

(a) The MR topology on any admissible completion $\overline{M}$ satisfies (A1) (i.e., it includes CEAT).
(b) Let $(P,F) \in \overline{M}$ with $P \neq \emptyset \neq F$. A sequence $\{(P_n, F_n)\}_n \subset \overline{M}$ converges to (P,F) with the MR topology iff it converges with the chr. topology (and thus iff it converges with CEAT).

Therefore, the differences between both topologies appear when condition $P \neq \emptyset \neq F$ does not hold. This is related to the possible violation of (A2). In order to analyze it, notice the following result, which is straightforward from Proposition 3.21.

Proposition 3.25. *Let $\overline{M}$ be an admissible completion as a point set of a strongly causal spacetime M endowed with a topology τ which satisfies (A1). Then, (A2) is equivalent to the following pair:*

(A2)' *If $\{(P_n, F_n)\}_n \to (P,\emptyset)$ (resp.$(\emptyset, F)$) and it happened $P \subset P' \subset \mathrm{LI}(P_n)$ (resp. $F \subset F' \subset \mathrm{LI}(F_n)$) for some $(P', F') \in \overline{M}$, then the constant sequence $\{(P', F')\}_n$ also converges to $(P,\emptyset)$ (resp. $(\emptyset, F)$)*

(A2)" *τ is T_1.*

Remark 3.26. Our conclusion on MR topology in relation to (A2) is the following:

(i) The arguments in Remark 3.19 show that condition (A2)' is compelling for any useful topology. Nevertheless, the cases when MR topology satisfies (A2)' do not seem clear.

(ii) MR topology may not satisfy condition (A2)," as Marolf and Ross argued that the example in figure 1 (c) cannot be T_1. Recall that this property not only seems harmless and desirable but also allows to fix univocally a topology. That is, even if (A2)' were satisfied, the absence of (A2)" makes unclear that a topology could be univocally selected by a requirement of coarseness such as[5] (A3) in Theorem 3.22.

Summing up, one can state a version of Theorem 3.22 by considering only conditions (A1), (A2)' and (A3), which seem to include MR topology at least in favorable cases. However, even in this case, T_1 separability must be imposed to fix a topology, i.e., the full condition (A2) seems unavoidable.

3.3 Properties of the standard completions

The properties fulfilled by the completions of a chronological space X were studied in [9]. Some of them were proved in a more general setting of completions (Remark 3.2 (A)) and others for the chronological completions of strongly causal spacetimes. At any case, they hold for any admissible completion and, then, for the c-completion. So, we can prove:[6]

Theorem 3.27. *For any standard completion $\overline{M}$ of a strongly causal spacetime M:*

- (i) *Any chain in the completion has a limit. Moreover, any inextensible timelike curve in M has a limit in ∂M.*
- (ii) *The inclusion $\mathbf{i}: M \to \overline{M}$ is a topological embedding, and $\mathbf{i}(M)$ is dense in $\overline{M}$.*
- (iii) *The boundary ∂M is a closed subset of $\overline{M}$.*
- (iv) *$I^{\pm}((P,F))$ is open for any $(P,F) \in \overline{M}$.*
- (v) *Each single point is closed, i.e., the topology is T_1.*

Proof. Statement (i) follows directly from the *completeness* and *S-relation* conditions satisfied by any admissible completion as a point set (Definition 3.1 (1), (2)). In particular, notice that any future-directed chain ς admits as a limit any $(P,F) \in \overline{M}$ with $P = I^-[\varsigma]$ (Definition 3.10, Proposition 3.39(1)).

[5] MR topology may be non coarser than the chr. one, in contraposition to what was stated in [9, Theorem 8.1]. In fact, in the proof of this result the following property was wrongly assumed: if $\{P_n\}_n$, $\{P_k^n\}_k$ are sequences of IPs such that $P \in \hat{L}(P_n)$ and $P_n \in \hat{L}(P_k^n)$ for all n, then there exists some subsequence $\{k_n\}_n \subset \{n\}_n$ such that $P \in \hat{L}(P_{k_n}^n)$.

[6] In [9, Theorem 7.9] the following property was also claimed: *if two points (P,F), $(P',F') \in \overline{M}$ are non-Hausdorff related then both lie in ∂M.* However, this result is not true in general (it was obtained as a consequence of the also uncorrect result [9, Proposition 7.8]), see [11, Section 2.3] for an explicit counterexample.

This also proves the second assertion in statement (ii). For the first one, recall Remark 3.17. This remark also proves that $i(M)$ is an open subset and, so, statement (iii) (see also [9, Theorem 7.6], which is valid for any admissible completion, non necessarily minimal). Assertion (iv) has been imposed for any admissible completion, and have been proved for the chr. one in Remark 3.13 (B). For (v), just apply Definition 3.10. $\square$

Remark 3.28. It is not clear the cases when (v) is fulfilled by MR topology, and (iv) is not fulfilled neither by Harris' topology (who necessarily uses a different notion of chronological relation) nor by the alternative MR topology (as pointed out in [35, Section 5.4, Figure 10]).

3.4 Globally hyperbolic case and conformally flat ends

Put $I(p,q) := I^+(p) \cap I^-(q)$, $J(p,q) := J^+(p) \cap J^-(q)$. Next, we focus our attention on the case when M is globally hyperbolic, i.e., M is causal with compact $J(p,q)$ [6]. First, following the proof of [9, Theorem 9.1] we have:

Theorem 3.29. *Let ∂M be any admissible boundary of a strongly causal spacetime M. The following properties are equivalent:*

(i) *M is globally hyperbolic;*
(ii) *if $(P,F) \in \partial M$ then either $P = \emptyset$ or $F = \emptyset$;*
(iii) *if $(P,F) \in \partial M$ then either $\uparrow P = \emptyset$ or $\downarrow F = \emptyset$.*

In particular, the c-completion is the unique standard completion.

Proof. $(i) \Rightarrow (ii)$ Assume the existence of $(P,F) \in \partial M$ with $P \neq \emptyset \neq F$. Choose points $p \in P$, $q \in F$, and a chain $\varsigma \subset M$ generating P, and thus, with endpoint (P,F). Then, ς is eventually contained in $I(p,q)$. Moreover, no subsequence of ς converges in M, since, otherwise, we would contradict the terminal character of P. Whence, $J(p,q) \subset M$ is not compact, in contradiction with (i).

$(ii) \Rightarrow (iii)$ Assume the existence of $(P,\emptyset) \in \partial M$ with $\uparrow P \neq \emptyset$. Let $\emptyset \neq F$ be a maximal IF into $\uparrow P$. Notice that F must be terminal, since, otherwise, $F = I^+(p)$ and any chain ς generating P would satisfy $\varsigma \to p$ with the topology of the manifold (in contradiction with the terminal character of P). Let $\emptyset \neq P \subset P'$ be some maximal (thus, necessarily terminal) IP into $\downarrow F$. Then, $P' \sim_S F$, in contradiction with (ii).

$(iii) \Rightarrow (i)$ By [31, p. 409, Lemma 14], it suffices to show that $J(p,q)$ is included in some compact set $K \subset M$. By contradiction, assume that

$K := \overline{I(p', q')}$ (which includes $J(p,q)$ if $p' \ll p$, $q \ll q'$) is not compact for some points $p', q' \in M$. Then, there exists some $\sigma \subset I(p', q')$ with no subsequences converging in M. From [10, Theorem 5.11] applied to $\hat{M}$, there exists some subsequence $\sigma' \subset \sigma$ and some TIP $P \in \hat{L}(\sigma')$ such that $\emptyset \neq I^-(p') \subset P$. Moreover, P constitutes some pair (P, F) (recall the completeness of $\overline{M}$, Definition 3.1 (1)) and $\uparrow P \supset I^+(q') \neq \emptyset$, in contradiction with (iii). $\square$

Next, our aim is to study the concept of asymptotic conformal flatness. The following result will be useful in order to discuss this notion, but it may have interest in its own right.

Proposition 3.30. *Any point of a standard completion $\overline{M}$ of a globally hyperbolic spacetime M, admits a neighborhood which is sequentially compact.*

Proof. We have to prove that any $(P, F) \in \overline{M}$ admits a neighborhood V such that any sequence in V admits a converging subsequence. Since M is locally compact and it is open in $\overline{M}$, we can restrict our attention to some $(P_0, \emptyset) \in \partial M$. For some $p_0 \in P_0$ take $V = \{(P, F) \in \overline{M} : I^-(p_0) \subset P\}$. Notice that V is a neighborhood of $(P_0, \emptyset)$ because it contains $I^+(p_0, \overline{M})$. So, it suffices to show that V is sequentially compact. Let σ be any sequence inside V. From [10, Theorem 5.11], there exists some subsequence σ^∞ with $\hat{L}(\sigma^\infty) \neq \emptyset$ and choose $P' \in \hat{L}(\sigma^\infty)$, so that $I^-(p_0) \subset P'$. Now, if P' is a TIP then $(P', \emptyset) \in \partial M$ (recall Theorem 3.29), and so, $(P', \emptyset) \in L(\sigma^\infty) \cap V$. Otherwise, $P' = I^-(p)$, for some $p \in M$, we can assume without restriction that $\sigma^\infty \subset M$ (recall that ∂M is closed), and so, from [21, Theorem 2.3] or [9, Proposition 2.5], we have $\sigma^\infty \to p \in V$. $\square$

The following example shows that the hypothesis of global hyperbolicity in previous result is necessary.

Example 3.31. The c-completion $\overline{M}$ of the spacetime M in figure 2 does not satisfy the property in Proposition 3.30. In fact, the spacetime M is defined by removing from $\mathbb{L}^2$ in coordinates (x, t) the region $x \geq 0$ and the interior regions of the isosceles triangles T_n^+, T_n^- whose bases are determined, resp., by the pair of vertexes $(0, 1/n)$, $(0, 1/(n+1))$ and $(0, -1/n)$, $(0, -1/(n+1))$ for all n, and the angle between their sides and the t axis are $\pm 45°$. Notice that no vertex on the t-axis correspond with a point in the c-boundary, but $(0,0)$ does correspond, concretely, with (P, F) where $P = \{(x,t) \in M : t < x < 0\}$ and $F = \{(x,t) \in M : -t < x < 0\}$. Let $\{p_n = (x_n, -1/n)\}$, $\{q_n = (x_n', 1/n)\}$ be chains in M generating P, F, resp. It suffices to observe (recall Remark 3.17) that any open neighborhood $U_n = I^+(p_n, \overline{M}) \cap I^-(q_n, \overline{M}) \subset$

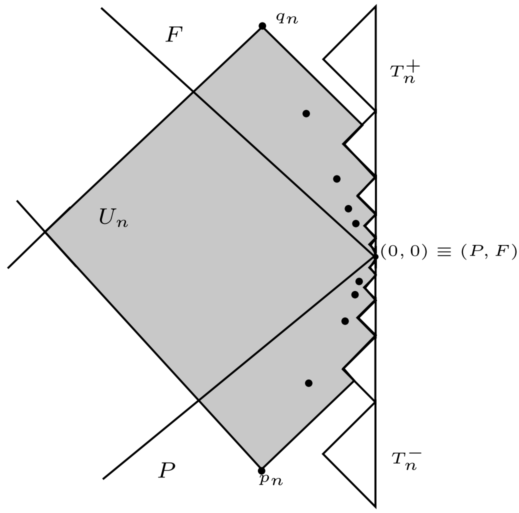

Figure 2: The origin corresponds with a pair $(P,F) \in \partial M$ which does not have a sequentially compact neighborhood (see Example 3.31). Any open neighborhood $U_n = I^+(p_n, \overline{M}) \cap I^-(q_n, \overline{M}) \subset \overline{M}$ of (P,F) contains a sequence with no limit in $\overline{M}$.

$\overline{M}$ of (P,F) contains a sequence $\{x_n^m\}_m \subset U_n$ composed by points with t components constantly equal to some $1/n'$, $n' > n$ and approaching to the t-axis; in particular, $\{x_n^m\}_m$ has no limit in $\overline{M}$.

Next, we are going to apply the c-boundary approach to formulate the notions of *conformally flat end* and *asymptotically conformally flat end.*

In Riemannian Geometry, there is a natural way to say that a (connected) Riemannian manifold (M, g_{R}) has a *flat* or *asymptotically flat end.* Namely, a flat end appears when, for some closed ball $\overline{B(p,R)} \subset M$ of sufficiently big radius R, a connected component of $M \setminus \overline{B(p,R)}$ is isometric to $\mathbb{R}^n$ minus some compact subset. Here $\mathbb{R}^n$ is endowed with its natural Euclidean metric g_{E}, but if we allow a metric equal to g_{E} plus terms which decrease with some power of $1/r$, (where r is the g_{E}-distance to the origin) then an asymptotically flat end appears. Notice that, if one considers the classical Cauchy completion M_{C} of (M, g_{R}) (defined by using classes of equivalence of Cauchy sequences), the expression "some closed ball $\overline{B(p,R)} \subset M$ of sufficiently big radius R" in previous definition can be substituted by "some compact subset of the Cauchy completion[7] M_{C}."

[7] A subtlety appears here with the topology. One can consider M_{C} endowed with the metric topology associated to the Riemannian distance on M extended to M_{C}. However, this topology may be non-locally compact and, so, the Cauchy completion of a bounded

Our aim is to point out that the viewpoint "a flat/asymptotically flat end appears when, up to a compact subset of the completion, the manifold is isomorphic to some model space," can be transplanted directly to c-completions of spacetimes, even though some small technicalities must be noticed.

Notice that the c-completion of $\mathbb{L}^4$ is equal to the usual conformal one obtained from the natural Penrose embedding in Einstein static universe $\mathbb{L}^1 \times \mathbb{S}^3$, but the so-called *spacelike infinity* $i^0 \in \overline{\mathbb{L}^4}$ (see more details in Remark 4.29 (1) below). Assume that, for some compact subset K of the c-completion $\overline{M}$ of a spacetime, a connected component E of $M \setminus K$ is conformal to $\mathbb{L}^4 \setminus \mathcal{K}$, where $\mathcal{K} \subset \overline{\mathbb{L}^4}$ is also compact (say, $\mathcal{K}$ eventually contains the future and past timelike infinities $i^\pm$). To regard this connected component E as a *conformally flat end* of the spacetime is then natural. If $\mathbb{L}^4$ is replaced by an ambient spacetime which behaves as an asymptotically flat one (say, which satisfies the axioms in [40, Chapter 11]) then E can be regarded as an *asymptotically conformally flat end*. From a formal viewpoint, it is also convenient to give a more abstract notion of end, because of the dependence of the previous one with the "concrete details" of the removed K. Summing up, let $\overline{M}$ be the c-completion of a strongly causal spacetime M, and recall the following definitions (for simplicity, they are stated just for conformal flatness):

(1) Let $K \subset \overline{M}$ compact. A connected component E of $M \setminus K$ is a *conformally flat end* if it is conformal to $\mathbb{L}^4 \setminus \mathcal{K}$, where $\mathcal{K}$ is some compact subset of the c-completion[8] $\overline{\mathbb{L}^4}$ which includes $i^\pm$.
(2) Two conformally flat ends E, E' are *equivalent*, $E \sim E'$, if there exists some compact $\tilde{K} \subset \overline{M}$ such that[9] $E \setminus \tilde{K} = E' \setminus \tilde{K}$. Then, each class of the relation of equivalence $\sim$ is called an *abstract conformally flat end*.

Notice that in non-locally compact cases as in figure 2, there are no asymptotically flat ends according to previous definition. Nevertheless, one might like that the vertexes of each triangle $T_n^\pm$ in the t-axis corresponded to an

---

region may be non-compact. However, M_{C} can be identified as a point set with a subset of the Gromov completion (which becomes the natural completion for c-boundaries, see [12]). For Gromov's topology the previous undesirable property cannot hold and, so, it can be used rigourously along the present discussion.

[8]Notice that this map is then extensible to a unique continuous map from $\overline{M} \setminus K$ to $\overline{\mathbb{L}^4} \setminus \mathcal{K}$, which can be also called *conformal* as it preserves the chronological relations.

[9]Alternatively, if $(E \cap E') \setminus \tilde{K}$ is an end. Recall that even in this version some subset $\tilde{K}$ must be removed because, even though E and E' are connected, it might happen that $E \cap E'$ is not.

asymptotic end. One can enlarge our definition, say, admitting the existence of ends when, in our previous definition, $J^+(K) \cup J^-(K)$ is removed, instead of K. In principle, we will not worry about this possibility, as it seems important only when the c-boundary is not locally sequentially compact, and this seems of limited interest (recall Proposition 3.30).

Analogous definitions and considerations can be stated for the notion of asymptotically conformally flat spacetimes — which, obviously, include all the classical asymptotically flat ones, such as $\mathbb{L}^4$, Schwarzschild, the stationary part of Kerr (all with a unique end), Kruskal (two ends) or Reissner–Nordstrom (infinitely many ends).

3.5 Endpoints for causal curves

Up to now, we have dealt only with chronological relations, timelike curves etc. However, one can wonder what would happen if we also consider causal elements such as extended causal relations, endpoints of causal curves, etc. To this aim, recall first:

Proposition 3.32. *Let $\gamma : [a, b) \to M$ be a future-directed inextensible causal curve. Then*

(i) $I^-[\gamma]$ is a TIP.

(ii) The inclusion $\uparrow \gamma \subset \uparrow I^-[\gamma]$ always holds, and the equality holds if $\gamma \subset I^-[\gamma]$.

Proof. (i) To check that $I^-[\gamma]$ is an IP, notice that from its definition it is enough to prove that, for any two points $p_1, p_2 \in I^-[\gamma]$ there exists another point $p_3 \in I^-[\gamma] \cap I^+(p_1) \cap I^+(p_2)$. As $p_i \ll \gamma(t_i)$ $i = 1, 2$ for some t_i, it is enough $p_3 = \gamma(t_3)$ for any $t_3 \in (t_1, b) \cap (t_2, b)$. The terminal character of $I^-[\gamma]$ follows from the inextensibility of γ by the same argument as in the timelike case.

(ii) The first assertion follows directly from the definitions and the second one from the fact that $\gamma \subset I^-[\gamma]$ implies $\uparrow \gamma \supset \uparrow I^-[\gamma]$. □

The part (i) means that no more generality is obtained if chronological past and futures of causal curves are considered instead of TIP's and TIF's — and, then, for all the construction of the boundary, which depends only on these sets. Nevertheless, the part (ii) suggests that inextensible causal curves may have no endpoint in the c-boundary.

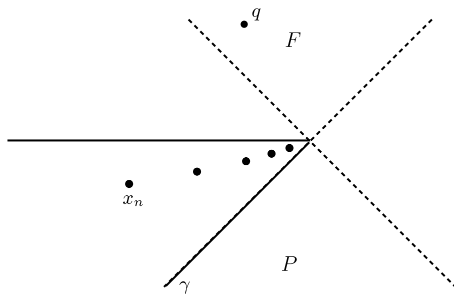

Figure 3: Non-properly causal completion for any admissible boundary.

Definition 3.33. A standard completion is *properly causal* if any inextensible causal curve has an endpoint.

The following example shows a non-properly causal c-completion.

Example 3.34. Let $M = \mathbb{L}^2 \setminus \{(x,0) : x \leq 0\}$ and put $\gamma(s) = (s,s)$ for $s \in [-1,0)$ (see figure 3). Then $\uparrow \gamma \subsetneq \uparrow I^-[\gamma]$ and γ has no endpoint in the c-completion $\overline{M}$. This example (emphasized in [35]) stresses our main choices for the c-boundary: (i) $I^-(q) \cap F \neq \emptyset$, $I^-(q) \cap I^+(x_n) = \emptyset$, so $(P,F)\overline{\ll}q$, but $x_n \not\ll q$; (ii) as we imposed that $I^-(q)$ is open, this implies $x_n \not\to (P,F)$ (in fact, $F \notin \check{L}(I^+(x_n))$; (iii) analogously, the causal curve γ will not have an endpoint. So, in order to ensure the convergence of causal curves, one should include new boundary points (namely, $(P,\emptyset)$, $(\emptyset,F)$, but recall that their components are not S-related). Properties (i), (ii) and (iii) also happen in any admissible topology as well as in the MR topology (Definition 3.24).

If one is interested in defining a properly c-boundary for any spacetime, the framework of the already constructed admissible boundaries is useful. As suggested in previous example, the c-boundary would be enlarged in order to obtain a properly causal one. Nevertheless, in order to get this larger boundary as a full general construction, one must restart all the process above and, in particular, the causal relation should be extended to the completion. Some difficulties of this extension were pointed out by Marolf and Ross in [27, Section 3.2], and some of the choices may be non-indisputable at this moment.

Therefore, instead of considering such a general construction, we give some simple and quite general criteria which ensures that the c-boundary is properly causal. Presumably, they hold in most cases of interest.

Theorem 3.35. *A standard completion $\overline{M}$ is properly causal if one of the following properties hold:*

(i) $\overline{M}$ is compact.

(ii) M is globally hyperbolic.

(iii) For any future-directed (resp. past-directed) inextensible lightlike pregeodesic $\rho : [a, b) \to M$ with no cut points[10] *it is $\uparrow \rho \supset \uparrow I^-[\rho]$ (resp. $\downarrow \rho \supset \downarrow I^+[\rho]$).*

Proof. For case (i), take some inextensible future-directed causal curve γ. By the compactness of $\overline{M}$, there exists some sequence $\varsigma_0 = \{\gamma(t_n)\}_n$ exhausting γ such that $\varsigma_0 \to (P, F)$ for some $(P, F) \in \overline{M}$. As $L(\varsigma_0) \cup \varsigma_0$ is closed (any $(P', F') \in L(\varsigma_0)$ satisfies $P' = I^-(\gamma)$; then, F' is maximal in $\uparrow I^-(\gamma)$ and $L(L(\varsigma_0) \cup \varsigma_0) = L(\varsigma_0) \cup \varsigma_0$), necessarily $(P, F) \in L(\varsigma_0)$, i.e.,

$$\begin{cases} P \subset \mathrm{LI}(I^-(\gamma(t_n))) \\ P \ \text{ maximal } \ IP \ \text{ in } \ \mathrm{LS}(I^-(\gamma(t_n))), \end{cases}$$
$$\begin{cases} F \subset \mathrm{LI}(I^+(\gamma(t_n))) \\ F \ \text{ maximal } \ IF \ \text{ in } \ \mathrm{LS}(I^+(\gamma(t_n))), \end{cases} \tag{3.9}$$

(the second lines required only if $P \neq \emptyset$ or $F \neq \emptyset$). Moreover, as the points on γ are causally related, conditions (3.9) also hold for any other sequence exhausting γ. Hence, (P, F) is an endpoint of γ.

For case (ii), just observe that Proposition 3.32 (i) and Theorem 3.29 (ii) ensure that any inextensible future-directed causal curve γ has endpoint $(I^-[\gamma], \emptyset) \in \overline{M}$.

Finally, for case (iii) let $\gamma : [a, b) \to M$ be some inextensible future-directed causal curve. Assume first that some restriction $\rho = \gamma|_{[c,b)}$ lies under condition (iii). From this condition, ρ has endpoint any $(P, F) \in \overline{M}$ with $P = I^-[\gamma]$, and so has γ. Now, assume that previous hypothesis does not hold for any restriction of γ. Then, there exists a sequence of points $\{t_n\} \to b$ such that each $\gamma(t_{n+1})$ is the cut point of $\gamma|_{[t_n,b)}$. Connecting the even consecutive points $\gamma(t_{2n})$ by timelike curves (and eventually smoothing it), an inextensible timelike curve ρ through all $\gamma(t_{2n})$ is constructed. Then, the curve ρ has some endpoint, and so has the curve γ. □

Remark 3.36. The criteria in Theorem 3.35 can be combined. So, if (i) holds up to a globally hyperbolic subset, its c-boundary becomes also properly causal. In fact, in this case, either the inextensible causal curve γ

[10] Recall that these points, as well as conjugate points, are conformally invariant for lightlike geodesics.

crosses infinitely many times a compact subset K (thus, one can apply (i)) or a sequence of points exhausting γ remains in some globally hyperbolic connected component of $M \setminus K$ (and thus, one can apply (ii)). In particular, this observation is applicable to spacetimes with asymptotically flat ends (recall Section 3.4).

3.6 Discussion: L-operator and convergence in chr. topology

Notice that the chr. topology, as well as Harris' one [21], is defined by starting at a limit operator L, which defines the closed sets (Definition 3.10). It is easy to check that these sets satisfy the axioms of a topology, just by taking into account that, if $\tilde{\sigma}$ is a subsequence of σ, then $L(\sigma) \subset L(\tilde{\sigma})$. Nevertheless, the name "limit operator" suggests that L should map each sequence σ in the set composed by all the points to which σ converges, and this property may fail for the operator L in the chr. topology. However, this is a rather pathological property, and the requirement of being "of first order" according to Definition 3.10, prevents it.

Let us see this question from a more general viewpoint (see also [21, p. 562]). Let X be any set, $\mathcal{S}(X)$ the set of all the sequences in X and $\mathcal{P}(X)$ the set of parts of X. We will mean by a limit operator any map $\mathcal{L} : \mathcal{S}(X) \to \mathcal{P}(X)$ such that if $\sigma \in \mathcal{S}(X)$ and σ' is a subsequence of σ then $\mathcal{L}(\sigma) \subset \mathcal{L}(\sigma')$. Its *derived topology* $\tau_{\mathcal{L}}$ is the one such that a subset $C \subset X$ is closed if and only if $\mathcal{L}(\sigma) \subset C$ for any sequence σ of C. The limit operator is *of first order* if the convergence of σ to $x \in \mathcal{P}(X)$ with the $\tau_{\mathcal{L}}$ topology implies that $x \in \mathcal{L}(\sigma)$. For example, if $\mathcal{L}$ is the limit operator which maps all $\mathcal{S}(X)$ in the empty set, then $\tau_{\mathcal{L}}$ is the discrete topology and $\mathcal{L}$ *is not* of first order.

Definition 3.37. A subset C of a topological space X is *sequentially closed* if for any sequence in C which converges to some point $x \in X$, then $x \in C$. The topological space X is a *sequential space* if every sequentially closed subset of X is closed.

Remark 3.38. In general, closed subsets are sequentially closed, but the converse does not hold. Observe that, in a sequential space, any non-closed subset A contains a sequence converging to a point in $\overline{A} \setminus A$. If, in addition, for any point $x \in \overline{A} \setminus A$ there exists a sequence in A which converges to x, then the topological space X is called Fréchet–Urysohn. Any first countable space is Fréchet–Urysohn and, then, sequential (see [19] for a systematic study).

In the case of a topology defined by a limit operator $\mathcal{L}$, the following result holds.

Proposition 3.39. *Let X be a set endowed with a topology $\tau_{\mathcal{L}}$ derived from a limit operator $\mathcal{L}$. Then, the following assertions hold:*

(i) *If $x \in \mathcal{L}(\sigma)$ with $\sigma = \{x_n\} \subset X$, then σ converges to x.*
(ii) *X is a sequential space.*

Proof. (i) By contradiction, assume that σ does not converge to x with the topology $\tau_{\mathcal{L}}$. Then, there exists some neighborhood U of x and some subsequence $\tilde{\sigma} = \{x_{n_k}\}_k$ such that $x_{n_k} \notin U$ for any k. Since $x \in \mathcal{L}(\sigma)$, it is also $x \in \mathcal{L}(\tilde{\sigma})$, and thus, x must belong to any closed set containing $\tilde{\sigma}$. In particular, x belongs to the closed set U^{c}, in contradiction to the initial hypothesis.

(ii) Let C be a sequentially closed subset of X. Let $\sigma \subset C$ be a sequence, and assume $x \in \mathcal{L}(\sigma)$. From (i), σ converges to x. Since C is sequentially closed, necessarily $x \in C$. Thus, C is closed, and so, X is a sequential space. $\square$

Remark 3.40. (1) This proposition proves, in particular, that *the chr. topology is sequential*. Nevertheless, there exist examples showing that it may not satisfy the first axiom of countability, i.e., some points may not admit a countable topological basis.

In fact, the pair $(P', \emptyset)$ in the example illustrated by figure 1(c) shows this situation. To see it, consider the natural coordinates (x, t) (such that the central removed point is the origin of $\mathbb{L}^2$) and the sequence of points $\{(-1/n, 1/n)\}$, which converges to the pair $(P', \emptyset)$, but does not converge to (P, F). By contradiction, assume that $(P', \emptyset)$ admits a countable basis of neighborhoods U_k. For each k, the points $(-1/n, 1/n)$ must enter in U_k for all n big enough. In particular, $(-1/n_k, 1/n_k)$ belongs to U_k for some n_k. Thus, one can construct a sequence of points (y_k, t_k) in the spacetime such that $t_k > |y_k|$, with y_k, t_k converging to 0, and (y_k, t_k) belonging to U_k for each k. In particular, (y_k, t_k) converges to (P, F), and does not converge to $(P', \emptyset)$, in contradiction to the fact that U_k is a basis of neighborhoods. Notice also that F, regarded as a subset of $\overline{M}$, also shows explicitly that, in general, the closure of a set does not coincide with the set of limits of sequences there, i.e., the chr. topology is not Fréchet–Urysohn.

(2) However, the possibilities for the chr. topology of being non-first countable, or non-first order, are very pathological (these possibilities will be studied in a forthcoming paper). In fact, both possibilities are forbidden under mild hypotheses, as the property of being *separating* explained next.

Lemma 3.41. *For any two pairs (P, F), $(P', \emptyset)$, in the chr. completion with $P' \subsetneq P$, the set $P \setminus \overline{P'}$ is not empty.*

Proof. Assume, by contradiction, $P \subset \overline{P'}$ under $P' \subsetneq P$. Then, any chain $\{p_n\}$ generating P satisfies $\{p_n\} \subset \overline{P'}$ but it is not included in P' for n big enough. So, for such a n, the open subset $I^+(p_n) \cap I^-(p_{n+1})$ (included in P, and, thus, in $\overline{P'}$), does not intersect P', a contradiction. □

Definition 3.42. The chr. topology of a standard completion is *separating* if the following property for pairs type $(P', \emptyset)$, plus the analogous one for $(\emptyset, F')$, holds: for any two pairs (P, F), $(P', \emptyset)$, with $P' \subsetneq P$, there exists some $x \in P \setminus \overline{P'}$ such that the open sets $I^+(x, \overline{M})$, $\overline{I^+(x, \overline{M})}^{\mathrm{c}}$ separate them (i.e., $(P, F) \in I^+(x, \overline{M})$, $(P', \emptyset) \in \overline{I^+(x, \overline{M})}^{\mathrm{c}}$).

Example 3.43. The chr. topology of the example in figure 1(c) is not separating. A simple case where the separating property holds non-trivially, is $M = \mathbb{L}^2 \setminus \{(x, t) : -x \leq t \leq 0\}$ (think, for example, in P, F, P' as the following subsets computed on $\mathbb{L}^2$: $P = I^-((0,0)), F = I^+((0,0)), P' = I^-((1,-1))$).

Proposition 3.44. *Assume that the chr. topology of a standard completion $\overline{M}$ is separating. Then, the chr. topology is of first order and second countable. Moreover, for any $D = \{x_n\} \subset M$ which is dense in M (and thus in $\overline{M}$) and chronologically dense in M, the set*

$$\begin{aligned}\Omega = \{&I^+(x_m, \overline{M}) \cap I^-(x_n, \overline{M}) : n, m \in \mathbb{N}\} \\ &\cup \{I^\pm(x_n, \overline{M}) \cap \overline{\cup_{i=1}^k I^\pm(x_{j_i}, \overline{M})}^{\mathrm{c}} : n, k, j_i \in \mathbb{N}\},\end{aligned}$$

is a topological basis.

Proof. First, let us prove that Ω is a topological basis and, thus, the chr. topology is second countable. Clearly, we can restrict our attention to $(P', F') \in \partial M$.

First, assume $P' \neq \emptyset \neq F'$, and let $\{p'_n\}, \{q'_n\} \in D$ be chains generating P', F', resp. Consider the nested neighborhoods $U_n \in \Omega$ of (P', F') given by $U_n = I^+(p'_n, \overline{M}) \cap I^-(q'_n, \overline{M}) \subset \overline{M}$. Let U be an arbitrary neighborhood of (P', F') in $\overline{M}$. We have to prove that $U_n \subset U$ for some (and then all the following) n. Assume by contradiction the existence of $r_n \in U_n \setminus U$. Since each $r_n \in U_n$, necessarily $P' \subset \mathrm{LI}(I^-(r_n))$, $F' \subset \mathrm{LI}(I^+(r_n))$. Moreover, since $P' \sim_S F'$, Lemma 3.15 implies that P' is a maximal IP into $\mathrm{LS}(I^-(r_n))$, and F' is a maximal IF into $\mathrm{LS}(I^+(r_n))$, i.e., this completes $(P', F') \in L(r_n)$. Then, $r_n \to (P, F)$ with the chr. topology, in contradiction with $r_n \notin U$ for any n.

Next, assume that $(P', F') \in \partial M$ satisfies, say, $P' \neq \emptyset$, $F' = \emptyset$, and fix a chain $\{p'_n\} \subset D$ generating P'. Let $\{s_i\} \subset D$ be the (countable) subset of

$\overline{P'}^{\rm c}$ composed by those points in D which separate $(P',\emptyset)$ and (P,F), for some $(P,F)\in\overline{M}$. Recall about this subset that, from the separation property assumed in the hypotheses, for any (P,F) as above there exists some $x\in P\setminus\overline{P'}$ which separates $(P',\emptyset)$ and (P,F). Thus, any $s\in D$ satisfying $x\ll s\in P$ provides an element of $\{s_i\}$ associated to (P,F), and the full $\{s_i\}$ is constructed by taking all such s for all the pairs (P,F). In particular, the chronological density of D ensures that, if at least one (P,F) as above exists, then $\{s_i\}$ is infinite (otherwise, $\{s_i\}$ is empty). Now, define the open subsets $U_n\in\Omega$ of $(P',\emptyset)$ as $U_n=I^+(p'_n,\overline{M})\cap A_n$, where $A_n=\overline{\cup_{i=1}^n I^+(s_i,\overline{M})}^{\rm c}$ (if $\{s_i\}$ is empty, put $A_n=\overline{M}$), for all n. Since $(P',\emptyset)\not\in\overline{I^+(s_i,\overline{M})}$ for all i, the open subsets U_n contain $(P',\emptyset)$. Let U be an arbitrary neighborhood of $(P',\emptyset)$ in $\overline{M}$, and let us check $U_n\subset U$ for some n. Assuming by contradiction the existence of $r_n\in U_n\setminus U$ for all n as above, one deduces again $P'\subset {\rm LI}(I^-(r_n))$. Moreover, by the property defining the subset $\{s_i\}$, if $P'\subsetneq P\subset {\rm LS}(I^-(r_n))$, then $P\setminus\overline{P'}$ must contain some $s_{i_0}\in\{s_i\}$. Hence $(P,F)\in I^+(s_{i_0},\overline{M})$, and thus $s_{i_0}\in P\subset {\rm LS}(I^-(r_n))$. In conclusion, $s_{i_0}\in I^-(r_n)$ for infinitely many n, in contradiction with the hypothesis $r_n\in U_n$ (which implies $r_n\not\in I^+(s_{i_0})$ for all n big enough). Therefore, P' is maximal in ${\rm LS}(I^-(r_n))$, and so, $(P',\emptyset)\in L(r_n)$. Thus, $r_n\rightarrow(P',\emptyset)$ with the chr. topology, in contradiction with $r_n\not\in U$ for any n.

To check the first-order property, assume that $\sigma=\{(P_n,F_n)\}$ converges to (P',F') with the chr. topology. Then, it is not a restriction to assume that $(P_n,F_n)\in U_n$ for every n, being $\{U_n\}_n$ a countable basis of neighborhoods for (P,F) as above. The proof above shows that any sequence $\{r_n\}$ such that $r_n\in U_n$ satisfies $(P',F')\in L(\{r_n\})$ and, thus, the required inclusion $(P',F')\in L(\sigma)$. $\square$

Remark 3.45. (1) CEAT (Definition 3.12) is always second countable, as it admits as a topological subbasis: $\{I^{\pm}(x_n,\overline{M}):n\in\mathbb{N}\}$, with $D=\{x_n\}$ as in Proposition 3.44. In fact, for any (P,F) and any (P_0,F_0) in, say, $I^+(P,F)$, the density of D implies the existence of some $x_n\in F\cap P_0$ and, thus, $(P_0,F_0)\in I^+(x_n,\overline{M})\subset I^+(P,F)$.

(2) CEAT can be characterized in terms of the following limit operator L^0: for any sequence $\sigma=\{(P_n,F_n)\}$, we define $(P,F)\in L^0(\sigma)$ iff $P\subset{\rm LI}(P_n)$ and $F\subset{\rm LI}(F_n)$. In fact, denote by τ_{L^0} the topology derived from L^0. CEAT is finer than τ_{L^0} from the implication to the right of Lemma 3.11 (recall that, as CEAT is second countable, the continuity of the inclusion $(M,\text{CEAT})\hookrightarrow(M,\tau_{L_0})$ is characterizable by sequences). Analogously, τ_{L^0} is finer than CEAT from the implication to the left of Lemma 3.11, taking into account that, if $\{(P_n,F_n)\}$ converges to (P,F) with τ_{L^0}, then $P\subset{\rm LI}(P_n)$ and $F\subset{\rm LI}(F_n)$. To check this last property just argue by contradiction: if, say,

there exists some $p_0 \in P$ with $p_0 \notin P_n$ for infinitely many n, then the set of pairs (P', F') with $p_0 \notin P'$ is closed for τ_{L^0} and does not contain (P, F), an absurd. Recall that this property is equivalent to the first-order character of L^0, because of the particular expression of this operator.

(C) The same argument shows that if $\{(P_n, F_n)\}$ converges to (P, F) with the chr. topology, then $P \subset \mathrm{LI}(P_n)$ and $F \subset \mathrm{LI}(F_n)$. However, this does not imply that the chr. operator L is of first order.

4 Conformal boundary versus c-boundary

4.1 First definitions on the conformal boundary

As above, M will be a strongly causal spacetime and ∂M and $\overline{M}$ will denote the c-boundary and completion, resp. Recall that a smooth map $i : M \hookrightarrow M_0$ between two Lorentzian manifolds $(M, g), (M_0, g_0)$ is called an *open embedding* if $i(M)$ is an open subset in M_0 and i is a diffeomorphism between M and its image $i(M)$. The map i is *conformal* if the pullback metric i^*g_0 satisfies $i^*g_0 = \Omega g$ for some function $\Omega > 0$ on M; for spacetimes, we also assume that the time-orientations are preserved. Eventually, $\{\Omega(p_n)\}_n$ may tend to 0 or ∞ when the sequence $\{i(p_n)\}_n$ converges to the topological boundary of $i(M)$ in M_0. However, we will not care on this possibility, as we will consider only properties which are conformally invariant. So, once i is defined, one is free to re-scale g so that i becomes an isometry.

Definition 4.1 (Envelopment). A (conformal) envelopment of M is an open conformal embedding $i : M \hookrightarrow M_0$ in some ("*aphysical*") strongly causal spacetime M_0.

Then, the *conformal completion* of M (w.r.t. i) is the closure $\overline{M}_i := \overline{i(M)} \subset M_0$, and the *conformal boundary* is the topological one $\partial_i M := \overline{i(M)} \setminus i(M)$.

Remark 4.2. This definition considers the conformal completion as a point set, but it is natural to endow it with a topology and a chronology too. The problems to define these elements will be discussed later, but we point out now the natural choices. The topology on $\overline{M}_i$ is just the induced one from M_0. For the chronological relation, there are some candidates. As natural options, we can consider the following two definitions:

(W) $p \ll_i^{\mathrm{W}} q$ (or just $p \ll_i q$) iff there exists some continuous curve $\gamma : [a, b] \to \overline{M}_i$ with $\gamma(a) = p, \gamma(b) = q$ such that $\gamma\,|_{(a,b)}$ is future-directed smooth timelike and contained in M.

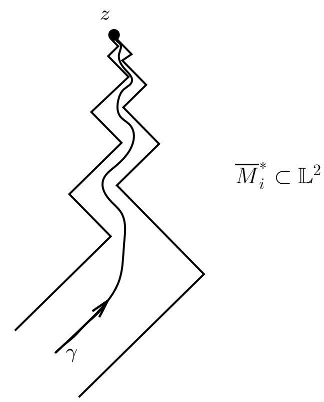

Figure 4: Because of the infinity breaks of $\partial_i^* M$, the causal curve γ, which is smooth in M, cannot be smoothly extended to $z \in \partial_i^* M$. The extended curve is causal continuous because there are close smooth causal curves in M_0 which connect any point of γ with z. Nevertheless, no such curves are contained in $\overline{M}_i^*$. Any point p on γ satisfies $p \ll_i^{\mathrm{W}} z$, but $p \not\ll_i^{\mathrm{S}} z$.

(S) $p \ll_i^{\mathrm{S}} q$ iff there exists some curve $\gamma : [a,b] \to \overline{M}_i$ which is smooth and future-directed timelike in M_0, with $\gamma(a) = p, \gamma(b) = q$ and $\gamma\mid_{(a,b)}$ contained in M.

These binary relations represent the weakest and the strongest reasonable choices for the chronology in the completion $\overline{M}_i$. They are different in general (figures 4 and 13) and may be non-transitive (figures 7 and 8). Notice that the chronological relation $\ll_i^{\mathrm{W}}$ is intrinsic to $\overline{M}_i$ and, so, it will be the natural candidate to agree with the relation $\overline{\ll}$ in the c-completion (as the c-boundary only has a topology but not a differentiable or metric structure). For this reason, $\ll_i^{\mathrm{W}}$ will be preferred to $\ll_i^{\mathrm{S}}$. Analogously, the following causal relations are defined:

(W) $p \leq_i^{\mathrm{W}} q$ (or just $p \leq_i q$) iff either $p = q$ or there exists some continuous curve $\gamma : [a,b] \to \overline{M}_i$ with $\gamma(a) = p, \gamma(b) = q$ such that γ is future-directed and continuous-causal (Remark 2.3) when regarded as a curve in M_0.

(S) $p \leq_i^{\mathrm{S}} q$ iff there exists some curve $\gamma : [a,b] \to \overline{M}_i$ which is smooth and future-directed causal in M_0, with $\gamma(a) = p, \gamma(b) = q$.

About the possible causal relations in $\overline{M}_i$, recall that the connecting causal curves may naturally go along the boundary. As this boundary may be non-smooth, the curve γ which is considered for $\leq_i^{\mathrm{W}}$ is only continuous-causal (as a curve in M_0). Nevertheless, even if γ is a smooth curve in M with just an endpoint in the boundary $\partial_i M$, the causal relation in M_0 becomes essential (see figure 4). So, both causal relations $\leq_i^{\mathrm{W}}, \leq_i^{\mathrm{S}}$ are extrinsic, i.e., they require the causality in M_0 and, in principle, depend on the conformal embedding. But $\leq_i^{\mathrm{W}}$ uses less structure on the boundary (which may be non-smooth) and so, it will be preferred below.

Convention 4.3. *When there is no possibility of confusion, the conformal embedding i will be dropped, and M will be regarded as an open subset of M_0. However, the subscript i is retained in $\partial_i M, \overline{M}_i$ –and so these spaces are distinguished from the causal ones $\partial M, \overline{M}$.*

The preferred relations $\ll_i^{\mathrm{W}}, \leq_i^{\mathrm{W}}$ will be denoted just $\ll_i, \leq_i$.

The elements in M such as the chronological futures or past will be denoted as above (say, $I^{\pm}(p) := I^{\pm}(p, M)$), and we will add the subscript 0 if these elements are taken in M_0 ($I_0^{\pm}(p) := I^{\pm}(p, M_0)$). Other elements will be written more explicitly. For example, $I^+(p, \overline{M}_i)(:= I^+(p, \overline{M}_i, \ll_i))$ denotes the chronological future of p in the completion $\overline{M}_i$ computed with the preferred relation $\ll_i$.

As an abuse of notation, if $z \in \partial_i M$ then $I^{\pm}(z)$ will represent $I^{\pm}(z, \overline{M}_i) \cap M$, which turns out the set of points in M which can be connected to z by means of a continuous curve $\gamma : [a, b] \to \overline{M}_i$, $\gamma(b) = z$, which is smooth, future or past-directed timelike, and contained in M, on $[a, b)$.

Definition 4.4 (**Accessibility**)**.** A future-directed (resp. past-directed) timelike curve $\gamma : [a, b) \to M$ has an *i-endpoint* $p \in \overline{M}_i$ if $p = \lim_{t\to b} \gamma(t)$.

If $p \in \overline{M}_i$ is the i-endpoint of a future-directed (resp. past-directed) timelike curve, then p is said *future (resp. past) accessible*, and p is called *accessible* if it is either future or past accessible. The set of all the accessible points of the conformal boundary $\partial_i M$ will be denoted by $\partial_i^* M$ and the set of all the accessible points of the conformal completion by $\overline{M}_i^* (= M \cup \partial_i^* M)$.

In order to have a useful envelopment we must ensure that it yields a true completion, that is, no part which should correspond to the boundary is missing. This is easy to formalize if one only cares on Causality.

Definition 4.5 (**Chr. completeness**)**.** An envelopment $i : M \hookrightarrow M_0$ is *chronologically complete* if any timelike curve $\gamma : [a, b) \to M$ which is inextensible in M (and, thus, generates a TIP or a TIF) has an i-endpoint p in the conformal boundary.

That is, chronological completeness means that the accessible boundary $\partial_i^* M$ contains i-endpoints for all inextensible timelike curves in M. For example, if M is the half-space of $\mathbb{L}^N$ obtained as $t > 0$, its inclusion in

$M_0 = \mathbb{L}^N$ is an envelopment which is *not* chronologically complete, as no inextensible future-directed curves in M will have an i-endpoint in M_0. Before going further, notice the following technicality which will be frequently claimed:

Lemma 4.6. *Let P be a TIP of M and γ_1, γ_2 two future-directed timelike curves which generate them, i.e., $P = I^-[\gamma_1] = I^-[\gamma_2]$. If γ_1 admits $z \in \partial_i^* M$ as its i-endpoint, then z is also the i-endpoint of γ_2.*

Proof. Under these hypotheses, $\gamma_1 \subset I^-[\gamma_2]$ and $\gamma_2 \subset I^-[\gamma_1]$, thus, $I_0^-[\gamma_1] = I_0^-[\gamma_2]$. As z is the i-endpoint of γ_1, $I_0^-[\gamma_1]$ is a PIP in M_0, i.e., $I_0^-(z) = I_0^-[\gamma_1] = I_0^-[\gamma_2]$ and, by Proposition 2.4 (ii) applied to M_0, z is also the i-endpoint of γ_2. □

So, one trivially has the following characterization of chronological completeness, which also defines the projection maps $\hat{\pi}, \check{\pi}$.

Theorem 4.7. *An envelopment is chronologically complete if and only if the natural projections of the future and past causal preboundaries, $\hat{\partial} M \to \partial_i^* M, \check{\partial} M \to \partial_i^* M$, which map each TIP $P = I^-[\gamma]$ or TIF $F = I^+[\gamma]$ in the i-endpoint of γ, are well defined on all $\hat{\partial} M$ and $\check{\partial} M$. In this case, the natural projections of the precompletions*

$$\hat{\pi} : \hat{M} \to \overline{M}_i, \quad \check{\pi} : \check{M} \to \overline{M}_i \tag{4.1}$$

are obviously defined.

When, say, a TIP P satisfies $\hat{\pi}(P) = z \in \partial_i^* M$, we say that P projects or *is associated* to z.

The simplest criterion to ensure chronological completeness is compactness (even though this may be too restrictive and the boundary may include non-accessible points).

Proposition 4.8. *An envelopment is chronologically complete if the conformal completion $\overline{M}_i$ is compact.*

Proof. Take $P \in \hat{\partial} M$ and let $\gamma : [a, b) \to M$ be some inextensible future-directed timelike curve in M such that $I^-[\gamma] = P$. Since $\overline{M}_i$ is compact, there exists some sequence $t_n \nearrow b$ such that $\{\gamma(t_n)\}_n$ converges to some $p_0 \in \overline{M}_i$. As M_0 is strongly causal, γ is continuously extensible to p_0 (Proposition 2.4(i)) and, as γ is inextensible in M, $p_0 \in \partial_i^* M$. □

Along all our study, we will assume that the envelopment is chronologically complete. Moreover, we will focus on the accessible part $\partial_i^* M$ of the conformal boundary $\partial_i M$. About these hypotheses notice

(1) *Chr. completeness.* If it were dropped, one may think that the completion could be "still completed." Moreover, this property allows to relate the conformal boundary with the c-boundary (Theorem 4.7).
(2) *Restriction to $\partial_i^* M$.* One might think that the accessible part of $\partial_i M$ may not be "complete enough" as one would like to consider also "completions of spacelike directions" — say, as in the case of the point i^0 in the standard compactification of $\mathbb{L}^4$. Nevertheless, we do not worry on non-accessible points because of the following:
   - One seems to be forced to impose compactness for $\overline{M}_i$ if spacelike directions were also completed in some sense. In fact, spacelike directions behave in a rather uncontrolled way, and we are considering a conformally invariant construction. Even in the (positive-definite) Riemannian case, a conformally invariant boundary cannot rely on the canonically associated distance, as all non-compact Riemannian manifolds are conformal to an incomplete one [30]. So, a Riemannian conformal completion becomes a compactification, as in the case of the Riemann sphere completing the complex plane. Nevertheless, the requirement to have a compact boundary may be very restrictive and, even if this is fulfilled, the properties of the points in $\partial_i M \setminus \partial_i^* M$ may be very irregular (recall examples such as figure 10, the point $(0, 1/2)$ in figure 11, or the case of i^0 itself, emphasized in Remark 4.18(2)).
   - Even an envelopment with compact $\overline{M}_i$ may have "non conformally invariant properties," as shown in figures 8 and 10. That is, a compactification by itself will not be enough, as boundary points in a compactification may "disappear" in a different one.

4.2 Requirements to relate the conformal and c-boundary

One can try to define a natural correspondence between the causal and conformal completions by using the projections (4.1) as follows:

$$\pi : \overline{M} \to \overline{M}_i^*, \quad \pi((P,F)) = \begin{cases} \hat{\pi}(P) & \text{if } P \neq \emptyset, \\ \check{\pi}(F) & \text{if } F \neq \emptyset. \end{cases} \tag{4.2}$$

Nevertheless, this map is well defined only if the following consistency holds:

$$(P,F) \in \partial M \ \text{ and } \ P \neq \emptyset \neq F \ \Rightarrow \ \hat{\pi}(P) = \check{\pi}(F). \tag{4.3}$$

Now, remark:

(1) Condition (4.3) does not hold in general. In fact, it is not only possible to find counterexamples for a general envelopment (figure 11), but also for an envelopment with a C^0 boundary (figure 12).
(2) Assume that (4.3) holds and, so, the projection π is well defined. As we are considering only the accessible part $\partial_i^* M$, the map π is automatically onto. It would be desirable that π were also one to one. However, these do not happen in general. This is not surprising for general envelopments, as arbitrary identifications can be introduced (figures 7 and 8). Nevertheless, other examples show that this is inherent to the particular approach. For example, when a "non-regular" point (in the sense of chronological sets, below Definition 2.1) for $\ll_i$ appears as in figure 9, we have two non-Hausdorf-related points of the c-boundary which project necessarily onto a single point of $\partial_i^* M$.
(3) Assume that the projection π is well defined and bijective. As emphasized in Remark 4.2, the completion $\overline{M}_i$ must be endowed with a topology and a chronology, and it would be desirable that π were a homeomorphism and a chronological isomorphism (the latter meaning that two points in $\overline{M}$ are chronologically related if and only if so are their images in $\overline{M}_i^*$). Nevertheless, figure 10 shows[11] that the topology of the conformal boundary may depend strongly on the conformal embedding (even if there are no artificial identifications as those in figures 7 and 8).
(4) Subtler problems appear for the chronology and its interplay with the topology. For example, figure 13 shows an open domain of $\mathbb{L}^2$ with C^∞ boundary and the following undesirable property: the chronological past of a point q (computed in $\overline{M}_i$ with $\ll_i$) is not open (with the induced topology from M_0).

Summing up, previous discussion shows that some additional conditions must be imposed on the envelopments in order to obtain a satisfactory conformal boundary. The smoothability of the boundary will play an important role (and will be discussed specifically in Section 4.4). Nevertheless, the examples cited above show that even this condition is not enough. Moreover, it may be also too restrictive and, so, more general conditions will be introduced first.

Remark 4.9. As a more speculative issue, one could try to define a more general notion of conformal boundary by considering a conformal embedding

[11]In this example π is not bijective. Nevertheless, this does not affect the essence of the example: if all the removed vertical segments (but the limit one) are widened a bit, a bijective example is obtained.

$i : M \hookrightarrow M_0$ not necessarily open (i.e., possibly $\dim M < \dim M_0$). Many of the notions and results for conformal envelopments could be extended directly to this more general case. In this case, to ensure the uniqueness of the conformal boundary would be more complicated, as there are more possibilities for the embedding. However, the existence for all stably causal spacetimes is guaranteed by a result in [29]. In this reference, any stably causal spacetime is shown to admit a conformal embedding in Lorentz–Minkowski $\mathbb{L}^{N_0+1}$, for some big N_0. By composing this embedding with the natural open conformal embedding of $\mathbb{L}^{N_0+1}$ in Einstein static universe $\mathbb{L}^1 \times \mathbb{S}^{N_0}$, one finds a conformal embedding $i : M \hookrightarrow \mathbb{L}^1 \times \mathbb{S}^{N_0}$ with compact closure and, then, chronologically complete. However, we focus only on the classical case of conformal boundary through open embeddings.

4.3 Main result

Next, let us state some general conditions so that the boundary will have a well-defined π with satisfactory properties. Of course, such conditions will not be intrinsic to M, but depends on how M lies in M_0.

Definition 4.10. Let $\gamma : [a, b] \to \overline{M}_i$ be a continuous curve such that $\gamma\mid_{[a,b)}$ is a future-directed (resp. past-directed) smooth timelike curve contained in M and $\gamma(b) \in \overline{M}_i$. Then, γ is future (resp. past) *deformably timelike (in its i-endpoint)* if there exists a neighborhood $U = U_0 \cap \overline{M}_i$ of $\gamma(b)$ (where U_0 is an open set of M_0) such that $\gamma(a) \ll_i w$ (resp. $w \ll_i \gamma(a)$) for all $w \in U$ (figure 5).

A point $z \in \partial_i^* M$ is *timelike deformable* if all the TIPs and TIFs associated to z (i.e., TIPs and TIFs defined by timelike curves with i-endpoint z) can be constructed as the chronological past or future of timelike deformable curves.

Trivially, timelike curves with i-endpoint in the spacetime M are timelike deformable. The role of timelike deformability is stressed by the following result.

Proposition 4.11. (1) *Let $z \in \partial_i^* M$ be timelike deformable. Then, all the future-directed timelike curves in M with i-endpoint z have the same chronological past in M, namely, $I^-(z)$ (according to Convention* 4.3).

(2) *All the points in $\partial_i^* M$ are timelike deformable if and only if the open sets in $\overline{M}_i$ for the CEAT associated to $\ll_i$ are open sets for the topology induced from M_0. In this case, any curve which is smooth and timelike in M and has an endpoint in $\partial_i M$ is timelike deformable.*

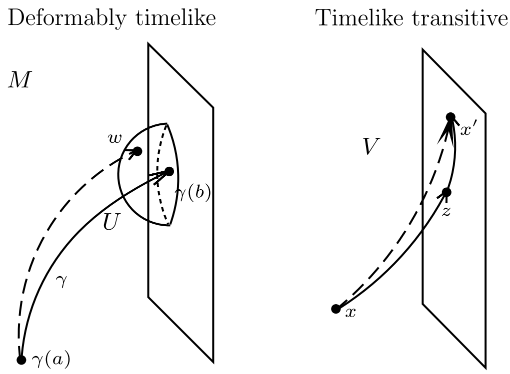

Figure 5: Examples of deformably timelike curve (left) and timelike transitive point (right).

Proof. (1) Let $\gamma, \tilde{\gamma}$ be two future-directed timelike curves with i-endpoint z, and define $P = I^-[\gamma], \tilde{P} = I^-[\tilde{\gamma}]$. Now, consider two future-directed timelike deformable curves $\sigma, \tilde{\sigma}$ such that $P = I^-[\sigma]$, $\tilde{P} = I^-[\tilde{\sigma}]$. By Lemma 4.6, $\sigma, \tilde{\sigma}$ converge to the same point that $\gamma, \tilde{\gamma}$, resp., that is, they converge to z. By timelike deformability, $\sigma \subset I^-(\tilde{\sigma})$ and viceversa and, thus, $P = \tilde{P}$.

(2) For the implication to the right, consider an open set U for the CEAT and suppose without restriction that $U := I^+(p, \overline{M}_i)$. As the chronological relation $\ll$ is open in M, for any point in $U \cap M$ there exists an open set for the induced topology containing that point and contained in U. On the other hand, if $z \in U \cap \partial_i^* M$, there exists a future-directed deformably timelike curve from p to z (just consider the TIP associated to z, which contains p, and apply that z is timelike deformable). Therefore, there exists an open set V for the induced topology such that $z \in V \subset U$.

For the implication to the left, observe that any timelike curve $\gamma : [a, b] \to \overline{M}_i^*$ with endpoint $z \in \partial_i^* M$ is timelike deformable. In fact, by hypothesis, $U := I^+(\gamma(a), \overline{M}_i)$ is an open set for the induced topology which contains z, and so, γ is timelike deformable. $\square$

Remark 4.12. If z is timelike deformable for all $z \in \partial_i^* M$, the CEAT is coarser than the topology on $\overline{M}_i$ induced from M_0. In particular, the condition of admissibility (A1) (see Definition 3.18) holds for this induced topology.

Definition 4.13. A point $z \in \partial_i^* M$ is (locally) timelike transitive if it admits a neighborhood $V = \overline{M}_i \cap V_0$ (V_0 open in M_0) such that for any $x, x' \in V$ (figure 5):

- $x \ll_i z \leq_i x' \Rightarrow x \ll_i x'$.
- $x \leq_i z \ll_i x' \Rightarrow x \ll_i x'$.

Definition 4.14. Let $i : M \hookrightarrow M_0$ be an envelopment. A point $z \in \partial_i^* M$ is *regularly accessible* if it is both timelike deformable and timelike transitive.

The boundary is *regularly accessible* if all its points in $\partial_i^* M$ are regularly accessible.

The following proposition will be useful:

Lemma 4.15. *Consider a point $p \in \overline{M}_i$ and suppose that there exists a sequence of future-directed causal curves $\gamma_n : [0, a_n] \to M$ such that p is an accumulation point of $\{\gamma_n(0)\}_n$. Moreover, suppose that the sequence $\{\gamma_n(a_n)\}_n$ does not converge to p. Then, there exists a future-directed causal curve $\gamma : [0, A] \to \overline{M}_i$, $A > 0$, which is limit curve*[12] *of $\{\gamma_n\}_n$, continuous causal in M_0 and satisfying $\gamma(0) = p$.*

Proof. It is a consequence of the proof of [3, Proposition 3.31] by taking into account that γ must lie in $\overline{M}_i$ as $\{\gamma_n\}_n \subset M$. □

Now, we are in conditions to establish the main result of this section.

Theorem 4.16. *Let $i : M \hookrightarrow M_0$ be a chronologically complete envelopment. If its boundary is regularly accessible then the conformal and c-completion are equivalent, in the sense:*

(a) *The map $\pi : \overline{M} \to \overline{M_i}^*$ is well defined and bijective.*
(b) *π is an homeomorphism.*
(c) *π is a chronological isomorphism.*

Proof. (a) Consider a pair $(P, F) \in \partial M$, i.e., $P \sim_S F$, and let $\{p_n\}_n, \{q_n\}_n$ be future and past-directed chains generating P, F resp. By contradiction, suppose that $p = \hat{\pi}(P) \neq \check{\pi}(F) = q$. As $P \subset \downarrow F$, up to a subsequence, there exists a future-directed timelike curve γ_n joining each p_n with q_n. As p, q are accumulation points of γ_n, Lemma 4.15 implies the existence of a limit continuous causal curve in M_0 such that $\gamma : [0, A] \to \overline{M}_i$, $A > 0$, with $\gamma(0) = p$.

As $\partial_i^* M$ is regularly accessible, $P = I^-[\alpha]$ for some future-directed timelike curve, necessarily timelike deformable (Proposition 4.11). Now, let $r \in Im(\gamma)\backslash\{p\}$ close enough to p, i.e., such that $r \in V$, where V is the neighborhood associated to the timelike transitivity of p. Then, r is reachable by a future-directed timelike deformable curve c starting at $\alpha(0)$, and $I^-[c] = I^-(r)$ is an IP by Proposition 4.11. Now,

[12]See [3, Section 3] for details.

- $P \subsetneq I^-[c]$. In fact, choose any p_n. By transitivity in V, there exists a timelike curve $\tilde{c_n}$ from p_n to r and, by Proposition 4.11, $p_n \in I^-[\tilde{c}_n] = I^-(r) = I^-[c]$, which shows the required inclusion. Moreover, $P \neq I^-[c]$ because otherwise Lemma 4.6 would contradict $p \neq r$.
- $I^-[c] \subset \downarrow F$. Let $r_m := c(t_m)$ be a timelike chain generating $I^-[c]$. By using that c is deformably timelike, for every r_m there exists a neighborhood U_m of r, such that $r_m \ll_i w$ for all $w \in U_m$. Moreover, because of the convergence of $\{\gamma_n\}$, there exist a subsequence $\{s_{n_k}\}$ such that $\gamma_{n_k}(s_{n_k}) \in U_m$ for all k. This last property implies $r_m \ll_i \gamma_{n_k}(s_{n_k}) \ll_i q_{n_k}$, and thus, $r_m \ll_i q_n$ for all large n. As m is arbitrary, the required inclusion follows.

This contradicts the maximality of P in $\downarrow F$ and, necessarily, $\hat{\pi}(P) = \check{\pi}(F)$.

As the map π is onto (its codomain was restricted to $\partial_i^* M$), we only have to check the injectivity. Notice that if $\pi(P,F) = \pi(P',F')$ and $P \neq \emptyset \neq F$ then implies $(P',F') = (P,F)$ (if, say, $P' \neq \emptyset$ Proposition 4.11(1) implies $P' = P$, thus $P' \not\sim \emptyset$ and analogously $F' = F$). So, reasoning by contradiction, it is necessary to check only the case when $(\emptyset, F), (P, \emptyset) \in \partial M$ are projected onto the same point $z \in \partial_i^* M$. Let γ, ρ be deformably timelike curves such that $P = I^-(\gamma)$ and $F = I^+(\rho)$. As γ is timelike deformable and γ, ρ converge necessarily to z, any point of the curve γ can be joined with a point of ρ sufficiently close to z, so $P \subset \downarrow F$ (and analogously $F \subset \uparrow P$). But we are assuming $P \not\sim_S F$ in M and, then, there exists P', F' with $P \subsetneq P'$, $F \subsetneq F'$ and $P' \sim_S F'$ (notice that if, say, $P \subsetneq P'$, then $P' \not\sim_S F$ as $\emptyset \sim_S F$). Then $z' := \pi(P',F')$ is well defined and, as $P' \neq \emptyset \neq F'$ necessarily $z \neq z'$. But this violates the strong causality of M_0, as the following almost closed timelike curves can be constructed. Let P, F, P', F' be generated by, resp. the chains $\{x_n\}_n, \{y_n\}_n, \{x'_n\}_n, \{y'_n\}_n$. As $P \subset P' \subseteq \downarrow F'$ and $F \subset F'$, choosing $x_{n_1} \in P$, $y_{n_1} \in F$ arbitrarily close to x, then we can take $x'_{n2} \in P'$ and $y'_{n_2} \in F'$ arbitrarily close to y such that $x_{n_1} \ll_i x'_{n_2} \ll_i y'_{n_2} \ll_i y_{n_1}$.

(b) Let us prove the continuity of π first. It suffices to prove that for any sequence $\{(P_n, F_n)\}_n$ such that $(P_0, F_0) \in L((P_n, F_n))$ then $p_n := \pi((P_n, F_n)) \to p_0 := \pi((P_0, F_0))$.[13] So, by contradiction, suppose that $p_n \not\to p_0$. Consider also an auxiliary complete Riemannian metric h in M_0 with associated distance d_h. As the part a) of the proof ensures that $\hat{\pi}$ and

[13]This is a general property of sequential spaces (see for example, [19, Lemma 3.1]). In fact, assume that this property holds. Let C be any closed subset in $\overline{M}_i^*$, and let us prove that $\pi^{-1}(C)$ is chr.-closed on $\overline{M}$. Let σ be any sequence in $\pi^{-1}(C)$ such that $(P,F) \in L(\sigma)$. Then, $\pi(\sigma)$ is contained in C, and $\pi(\sigma)$ converges to $\pi(P,F)$. As C is closed in $\overline{M}_i^*$, necessarily $\pi(P,F) \in C$, i.e., $(P,F) \in \pi^{-1}(C)$, as required.

$\check{\pi}$ agree, we will prove just the continuity of $\hat{\pi}$ and assume without loss of generality $P_0 \neq \emptyset$ and $P_n \neq \emptyset$.

Denote by $\{p_n^m\}_{m=1}^\infty$ a chain generating P_n (and so converging to p_n) and by $\{p_0^m\}_m$ a chain generating P_0 (and converging to p_0). As $P_0 \subset \mathrm{LI}(P_n)$, we can assume $p_0^n \in P_n$, up to a subsequence. For each n take m_n big enough such that $p_0^n \ll_i p_n^{m_n}$ and $d_h(p_n^{m_n}, p_n) < \frac{1}{n}$. Let $\gamma_n : [0, a_n] \to M$ a sequence of timelike curves joining $p_0^n, p_n^{m_n}$. As $p_n \not\to p_0$ by assumption, necessarily $p_n^{m_n} \not\to p_0$ and Lemma 4.15 implies the existence of a limit curve $\gamma : [0, A] \to \overline{M}_i$.

Observe that p_0 is reachable by a future-directed curve which is necessarily timelike deformable. In particular, the point $q = \gamma(t)$ for small t is reachable by a future-directed (deformably timelike) curve c_q. If we denote $P_q = I^-[c_q]$ then the inclusions $P_0 \subsetneq P_q$ and $P_q \subset \mathrm{LI}(P_n)$ follow by reasoning as in the two itemized arguments in the part a). Summing up, $P_0 \subsetneq P_q \subset \mathrm{LI}(P_n)$ and, so, P_0 is not maximal in $\mathrm{LS}(P_n)$, in contradiction with $P_0 \in \hat{L}(P_n)$.

For the continuity of $\pi^{-1} : \overline{M}_i^* \to \overline{M}$, it is enough to prove that, for any sequence $\{p_n\}_n \subset \overline{M}_i^*$ which converges to a point $p \in \overline{M}_i^*$, then $(P, F) \in L((P_n, F_n))$, where $(P_n, F_n) = \pi^{-1}(p_n)$, $(P, F) = \pi^{-1}(p)$ (recall that $\overline{M}_i^*$ is metrizable — in particular, sequential — and Proposition 3.39(i)). In order to check $P \in \hat{L}(P_n)$ (the proof of $F \in \check{L}(P_n)$ would be analogous), we only need to consider the case $P \neq \emptyset$ (recall Definition 3.10) and work with the map $\hat{\pi}$. Consider a curve γ such that $P = I^-[\gamma]$. As γ is deformably timelike and $p_n \to p, \gamma(t) \ll_i p_n$ for n large enough and, thus, $\gamma(t) \in P_n$ for n large enough, so $P \subset LI(P_n)$. In order to conclude that $P \in \hat{L}(P_n)$, assume by contradiction that the maximality of P is violated, i.e., there exists $P' \supsetneq P$ maximal in $LS(P_n)$ and so $p' := \hat{\pi}(P') \neq p$ (recall Proposition 4.11). Take some chain $\{y'_m\}_m$ generating P', and choose a sequence $\{n_m\}_m$ so that $y'_m \in P_{n_m}$ for all m. Then, $y'_m \in P_{n_{m'}}$ for all $m' \geq m$ and P' is included in $\mathrm{LI}(P_{n_m})$, that is, $P' \in \hat{L}(P_{n_m})$. Thus, $\{p_{n_m}\}_m$ converges to both, p (by hypothesis) and p' (by applying the proved continuity of $\hat{\pi}$ on $\{P_{n_m}\}_m$), in contradiction with the Hausdorffness of M_0.

(c) Consider two points $(P, F)\overline{\ll}(P', F')$, that is, $F \cap P' \neq \emptyset$. Denote by $\gamma_F, \gamma_{P'}$ two timelike curves such that $F = I^+[\gamma_F], P' = I^-[\gamma_{P'}]$, and let $p := \pi((P, F)), p' := \pi((P', F'))$. Choose any $r \in P' \cap F$. There exists t_0, t_1 such that $p \ll_i \gamma_F(t_0) \ll r \ll \gamma_{P'}(t_1) \ll_i p'$. Thus, one can construct a curve connecting p, p' which is (piecewise) smooth and timelike in M, and shows $p \ll_i p'$.

Conversely, let $p \ll_i p'$ in $\overline{M}_i^*$, and consider a connecting curve $\gamma : [0, 1] \to \overline{M}_i^*$ as in the definition of $\ll_i$ (Remark 4.2). We have to check that

$P' \cap F \neq \emptyset$, for the pairs $(P, F) = \pi^{-1}(p)$ and $(P', F') = \pi^{-1}(p')$. But $P' = I^-[\gamma]$ and $F = I^+[\tilde{\gamma}]$, for $\tilde{\gamma}(t) = \gamma(1-t)$ (recall Proposition 4.11(1)) and, thus, $\gamma(t) \in P' \cap F$ for all $t \in (0, 1)$. $\square$

4.4 Envelopments with C^1 boundary

The regular accessibility imposed in the main result (Theorem 4.16) comprises two technical requirements, timelike deformability and transitivity. In this subsection, we explore some assumptions which imply regularity with a double aim: to make clearer and more easily computable the notion of regularity. A natural simplifying hypothesis for the (accessible) completion is to have a structure of *manifold with boundary*. About the order of smoothability, examples such as figure 12 show that C^0 is too weak, but a differentiability greater than 1 might be too restrictive to produce non-trivial results.

4.4.1 Notion of C^r conformal boundary

In order to make clear some notation and subtleties, we recall explicitly the following notion.

Definition 4.17. (C^r point). An envelopment $i : M \hookrightarrow M_0$ has a C^r ($r = 0, 1$) boundary at $z \in \partial_i M$ if z admits a C^r chart of M_0 adapted to the boundary, i.e., there exists an open neighborhood V_0 of z in M_0 and a C^r coordinate chart $\varphi : V_0 \to \mathbb{R}^N$ such that

$$\varphi(\overline{M}_i \cap V_0) = \varphi(V_0) \cap \mathbb{R}^N_+, \text{ where } \mathbb{R}^N_+ = \{(x^1, x^2, \dots, x^N) \in \mathbb{R}^N : x^N \geq 0\}.$$

If the envelopment i has a C^r ($r = 0, 1$) boundary for all $z \in \partial_i^* M$ then it is called envelopment with C^r ($r = 0, 1$) boundary.

In the case of an envelopment of C^1 boundary at some $z \in \partial_i M$, one says that a vector v tangent to M_0 at z *points out inwards* (resp. *points out outwards*) if $v^N > 0$ (resp. $v^N < 0$), where $(v^1, \dots, v^N) \in T_0\mathbb{R}^N$ is the expression of v in terms of a chart (V_0, φ) adapted to the boundary such that $\varphi(z) = 0$.

Notice that when $i : M \hookrightarrow M_0$ is an envelopment with C^r boundary, only the accessible part of the conformal completion $\overline{M}_i^* = M \cup \partial_i^* M$ is required to be a C^r manifold with boundary, and $\partial_i^* M$ is then a hypersurface embedded in M_0.

Remark 4.18. (1) The canonical conformal embedding of Lorentz–Minkowski in Einstein static universe (see for example [22, 40]), $i : \mathbb{L}^4 \hookrightarrow$

$\mathbb{L}^1 \times \mathbb{S}^3$, is an envelopment with C^0 boundary according to our definition. It is not C^1 because the points $i^{\pm}$ are included in $\partial_i^* M = \partial_i^* \mathbb{L}^4$. The spacelike infinity i^0 is included in $\overline{M}_i$ (which is *not* a C^0 manifold with boundary) but it is not included in $\overline{M}_i^*$.

(2) Assume that not only $\overline{M}_i^*$ but also all $\overline{M}_i$ is a C^1 manifold with boundary. Then, a simple argument based on partitions of the unity shows the existence of a smooth function Ω on all M_0 such that

$$\Omega(p) > 0 \Leftrightarrow p \in M, \quad \partial_i M = \Omega^{-1}(0), \quad d\Omega_p \neq 0, \; \forall p \in \partial_i M. \tag{4.4}$$

This viewpoint in the definition of conformal completion appears in, for example, [14, 40]. Of course, in favorable cases one can then restrict the function Ω on $\overline{M}_i^*$ and consider this intrinsically as a C^1 manifold with boundary.

4.4.2 Regular and strong accessibility of a C^1 boundary

As discussed above, the differences between the strong and weak versions of the chronological and causal relations defined in Remark 4.2 yield problems in order to identify the conformal boundary with the c-boundary. On one hand, the differences between the two causal relations $\leq_i, \leq_i^{\mathrm{S}}$ are connected with the loss of timelike transitivity of $\partial_i^* M$. On the other, the differences between the two chronological relations $\ll_i, \ll_i^{\mathrm{S}}$ may cause the loss of timelike deformability, as we analyze now. In fact, the main difficulty to ensure this last property is that some point of $\partial_i^* M$ may be the $\hat{\pi}$ or $\check{\pi}$ projection for different TIPs or TIFs. This will happen only if the timelike curve which generates the TIP or TIF cannot be extended as a smooth timelike curve to $\partial_i^* M$.

Definition 4.19 (Strong accessibility). Let $\gamma : [a, b) \to M$ be a future-directed (resp. past-directed) timelike curve with i-endpoint $z \in \overline{M}_i$. If, up to a reparameterization, γ can be smoothly extended to b with timelike velocity $\gamma'(b) \in T_z M_0$, then γ is called a future (resp. past) strongly timelike curve.

If z is the i-endpoint of a future (resp. past) strongly timelike curve, then z is said *future (resp. past) strongly accessible.*

A future (past) accessible point $z \in \overline{M}_i$ is *strongly accessible* if it is future (resp. past) strongly accessible.

The completion $\overline{M}_i$ is *strongly accessible* if all the accessible points are strongly accessible.

Recall that $z \in \partial_i M$ is called strongly accessible when it is an i-endpoint just for one strongly timelike curve — or two, one future-directed and one past-directed, in the case that z is future and past accessible. This is simpler than timelike deformability, as will be stressed in Remark 4.29 (2).

For C^1 boundaries, the conditions of accessibility and at least future or past strong accessibility are not restrictive:

Proposition 4.20. *Assume that the conformal boundary is C^1 at some $z \in \partial_i M$. Then z is either future or past strongly accessible.*

In particular, if the envelopment has a C^1 boundary then all the boundary is accessible, i.e., $\partial_i M = \partial_i^* M$.

Proof. Let $v \in T_z M_0$ be any timelike vector. If v points out inwards, the required strongly timelike curve can be chosen as the geodesic starting at z with initial velocity v. If v points out outwards, the timelike vector $-v$ lies in previous case. If $v \in T_z \partial_i M$, as the set of all timelike vectors at $T_z M$ constitutes an open subset, a small perturbation of v reduces the problem to previous cases. □

Remark 4.21. As non-accessible points cannot be C^1, either they represent a very particular situation (the case of i^0 for $\mathbb{L}^4 \hookrightarrow \mathbb{L}^1 \times \mathbb{S}^3$) or become rather irregular (see figures 10 and 11). So, essentially, the non-accessible part $\partial_i M \setminus \partial_i^* M$ not only is a non-causal ingredient, but also its definition would depend on the convenience for the particular phenomenon considered (asymptotic flatness, AdS/CFT correspondence), and would require a specific study in the corresponding theory.

The first of the following two lemmas will be useful to relate the strong and weak chronological relations under strong accessibility, and the second lemma to relate the causal ones.

Lemma 4.22. *Let $\gamma : [a,b] \rightarrow \overline{M}_i^*$ be a future-directed strongly timelike curve with $\gamma(a) = q \in M$ and $\gamma(b) = z \in \overline{M}_i^*$. If $\overline{M}_i$ is C^1 at z, then there exists a neighborhood U_0 of z in M_0 such that q can be joined with any $r \in U := U_0 \cap \overline{M}_i$ by means of a future-directed strongly timelike curve.*

In particular, γ is timelike deformable and all the points in $U \cap \partial_i M$ are future strongly accessible.

Proof. The non-trivial case appears when $z \in \partial_i M$, so, consider a chart (V_0, φ) adapted to the boundary with M on the $x^N > 0$ side and $z = \varphi^{-1}(0)$. Up to a reparameterization, write $\varphi(\gamma(t)) = (x_1(t), \ldots, x^N(t)), t \in [0,1]$. As the velocity satisfies $g(\dot\gamma,\dot\gamma)<0$ and $\dot\gamma([0,1])$ is a compact subset of $TV_0(\equiv \varphi(V_0)\times\mathbb{R}^N)$, there exists $\epsilon>0$ such that, for any $(x,y)\in\varphi(V_0)\times\mathbb{R}^N$ with $|x^i-x^i(t)|<\epsilon$ and $|y^i-\dot x^i(t)|<\epsilon$ (for some $t\in[0,1]$ and all $i=1,\dots,N$) one has $\varphi_*g((x,y),(x,y))<0$. Now, consider any $r\in U:=\varphi^{-1}([0,\epsilon)^N)$, and put $\varphi(r)=(\epsilon^1,\dots,\epsilon^N)$ (where $0\le\epsilon^i<\epsilon$). Then, the curve $\gamma_r(t)=\varphi^{-1}((x^1(t)+t\epsilon^1,\dots,x^N(t)+t\epsilon^N))$ connects q and r fulfilling the required properties. $\square$

Lemma 4.23. *Assume that $z\in\overline{M}_i^*$ is future strongly accessible and C^1. Then there exists a neighborhood $U_0\subset M_0$ of z such that, for all future-directed causal piecewise smooth curve $\sigma:[0,c]\to U=U_0\cap\overline{M}_i$, there exists a variation $\sigma_\epsilon:[0,c]\to U$ such that $\sigma(c)=\sigma_\epsilon(c)$ and σ_ϵ is future-directed strongly timelike for all ϵ.*

Proof. Consider a past-directed timelike vector field T defined on some neighborhood of σ in $\overline{M}_i$ which points out inwards M on the points of the boundary (say, T is obtained by extending an inward pointing timelike vector of T_zM, whose existence is ensured in the proof of Proposition 4.20). Let $-\sigma(s):=\sigma(c-s)$ the reversed curve of σ. Any variational field $V=\mu T$ on $-\sigma$ with $\mu=0$ at $-\sigma(0)$ and $\mu'>0$ everywhere, will yield curves contained in M except at the i-endpoint $\sigma(c)$ (which remains fixed). According to [31, Lemma 10.45], if the covariant derivative V' of V along $-\sigma$ satisfies $g_0(V',(-\sigma)')<0$, the longitudinal curves of the variation close to σ are strongly timelike. In order to ensure this property, notice that, $g(T,(-\sigma)')<-\epsilon$ and $|g(T',(-\sigma)')|<1/\epsilon$ on $[0,c]$ for some $\epsilon>0$ small enough. As $V'=\mu'T+\mu T'$,

$$g(V',(-\sigma)')=\mu' g(T,(-\sigma)')+\mu g(T',(-\sigma)')<-\epsilon\mu'+\mu/\epsilon.$$

Choosing $\mu(s)=e^{s/\epsilon^2}-1$, we have $\mu(0)=0$ and $\mu'>\mu/\epsilon^2$, i.e., the required inequality follows. $\square$

Definition 4.24. An envelopment $i:M\hookrightarrow M_0$ with C^1 boundary is *chronologically tame* (resp. *causally tame*) if $\ll_i\equiv\ll_i^{\mathrm{S}}$ (resp. $\le_i\equiv\le_i^{\mathrm{S}}$).

Remark 4.25. Even though these definitions are stated from a global viewpoint, it is enough to check them from a local one, which may be easier in practice. That is, the envelopment is *chronologically tame* (resp. *causally tame*) when for all $z\in\overline{M}_i$, there exists a neighborhood $V=\overline{M}_i\cap V_0$ such that V, regarded as an envelopment of $M\cap V_0$ in its own right, is chronologically tame (resp. causally tame). This is obvious for chronological tameness, and can be checked for the causal one as follows. Assume that $x\le_i y$. From local causal tameness one can construct a piecewise smooth future-directed causal curve σ in $\overline{M}_i$ from x to y. Let $\sigma(t_0)$ be a break of σ.

A standard argument shows that it can be smoothed if $\sigma(t_0) \in M$ or if one of the tangent vectors at the break $\sigma'(t_0^+), \sigma'(t_0^-)$ is timelike (see for example, the proof of Proposition 10.46 in [31, p. 235]). If both vectors are lightlike and $\sigma(t_0) \in \partial_i M$, the argument also works by taking into account that either $w = \sigma'(t_0^+) - \sigma'(t_0^-)$ points out inwards M or it is tangent to $\partial_i M$ (in the latter case, a close vector $\overline{w}$ points out inwards and satisfies $g(\overline{w}, \sigma'(t_0^-)) < 0, g(\overline{w}, \sigma'(t_0^+)) > 0$, which will be enough for the job).

Theorem 4.26. *A chronologically complete envelopment $i : M \hookrightarrow M_0$ with C^1 boundary, chronologically tame and causally tame is regularly accessible. So, its conformal completion and c-completion are equivalent (in the sense of Theorem* 4.16*).*

Proof. Let $z \in \partial_i^* M$. To check that z is timelike deformable, consider, say, a TIP P associated to z, and let γ be a future-directed timelike curve in M with endpoint z such that $P = I^-[\gamma]$. Notice that, for all $x = \gamma(t)$ chronological tameness implies $x \ll_i^{\mathrm{S}} z$, that is, there exists a connecting strongly timelike curve α. Then, Lemma 4.22 can be applied to this curve, showing that γ fulfills the definition of timelike deformability (Definition 4.10).

To check that z is timelike transitive, let U be a neighborhood as in Lemma 4.23 and take any $x, y \in U$ such that $x \ll_i z \leq_i y$ (the other case is analogous). By tameness $x \ll_i^{\mathrm{S}} z \leq_i^{\mathrm{S}} y$ and, so, there exists a piecewise smooth causal curve $\sigma : [0, 2] \to \overline{M}_i$ from x to y such that $\sigma(1) = z$ and $\sigma|_{[0,1]}$ is timelike. By this last property, some timelike curve σ_ϵ close to $\sigma\,|_{[1,2]}$, and provided by Lemma 4.23, can be modified close to 0 in order to ensure $x \ll_i y$. $\square$

Figure 13 is an example of envelopment with conformal boundary different to its c-boundary (at the topological level) because of the lack of chronological tameness. In order to apply Theorem 4.26, we are going to show that chronological tameness can be replaced by strong accessibility, which is more easily computable.

Lemma 4.27. *Assume that i is causally tame. If $z \in \partial_i^* M$ is a future strongly accessible C^1 boundary point, then all future-directed timelike curves $\sigma : [a, b) \to M$ with i-endpoint z, have the same chronological past. Moreover, if $\partial_i^* M$ is C^1 and strongly accessible then i is chronologically tame.*

Proof. Fix a future-directed strongly timelike curve $\hat{\sigma} : [a, b) \to M$ with i-endpoint z, and let $P = I^-(\sigma), \hat{P} = I^-(\hat{\sigma})$. From Lemma 4.22, $I^+(\hat{\sigma}(t))$ contains a neighborhood of z for each t, and thus, it also contains some point of σ. So, $\hat{\sigma}(t) \in I^-[\sigma]$ and $\hat{P} \subset P$.

In order to prove the reversed inclusion, take an arbitrary $q \in P$. By causal tameness, q can be joined with z by means of a smooth future-directed causal curve $\gamma : [a, b) \to M$ which is smoothly extensible to b. Then, consider two cases:

(a) γ is extensible to a strongly timelike curve. In this case, the roles of γ and $\hat{\sigma}$ are interchangeable, and $q \in I^-[\gamma] = I^-[\hat{\sigma}] = \hat{P}$, as required.

(b) γ is only extensible as a smooth curve at b (necessarily causal in M_0). Lemma 4.23 allows to find arbitrarily close strongly timelike curves γ_ϵ depending on the parameter ϵ and, from the previous case, $I^-[\gamma_\epsilon] = \hat{P}$. Now, the open set $I^+(q)$ contains some point in γ_ϵ for ϵ small enough. Therefore, $q \in I^-[\gamma_\epsilon] = \hat{P}$, as required.

In order to check the last assertion, let $p, q \in \overline{M}_i$ such that $p \ll_i q$. There exists a future-directed timelike curve σ joining them. The previous proof shows $I^-[\sigma] = I^-[\gamma]$ for any future-directed strongly timelike curve γ with i-endpoint q. This property ensures the existence of a strongly timelike curve joining p and q, i.e, $p \ll_i^{\mathrm{S}} q$. $\square$

Summing up, from Theorem 4.26 and Lemma 4.27:

Corollary 4.28. *The conformal and c-completions are equivalent if the conformal boundary is C^1, strongly accessible and causally tame.*

Remark 4.29. (1) Theorem 4.26 and Corollary 4.28 yield sufficient conditions to ensure regular accessibility, which become reasonably general and easy to compute. In fact, the sufficient conditions (C^1 boundary, causal tameness and either chronological tameness or just strong accessibility) are local, recall Remark 4.25. So, one can determine first the points were these conditions are fulfilled (typically, most of them) and then concentrate on the boundary points were they are not. For example, in the canonical envelopment of $\mathbb{L}^4$ in Einstein static universe, the accessible boundary $\partial_i^*\mathbb{L}^4$ is C^1 and satisfies the hypotheses of Corollary 4.28 at all the points but $i^\pm$ (see also Theorem 4.32 below). And, easily, $i^\pm$ satisfy the definition of regular accessibility.

(2) About the hypotheses of Corollary 4.28, we emphasize first that they are very easy to check in practice. Typically, C^1 smoothability can be checked by inspection. As a difference with regular accessibility, the strong accessibility of some $z \in \partial_i M$ must be checked only when z is accessible and, then, one must check it *at most for two cases*, the future and past direction (if z is both, future and past accessible). This is an important simplification in comparison with *timelike deformability* (one of the two conditions

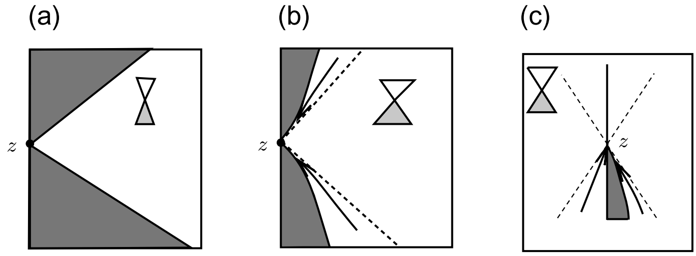

Figure 6: In figure (a), point z is not accessible, because it is not reachable by a timelike curve from M. In figure (b), any timelike curve reaching z must be null at z, i.e., z is future and past accessible, but not strongly accessible. In figure (c), point z is the endpoint of two future-directed timelike curves, one of them is strongly timelike and the other one is not.

for regular accessibility), as this condition has to be checked for all the TIPs and TIFs which project onto z. For example, figure 6 (c) shows a strongly accessible $z \in \partial_i^* M$ which is not timelike deformable, as none of its associated TIPs can be defined by means of a deformably timelike curve (this is possible because z is not C^1). Notice also that both conditions, C^1 and strong accessibility, are independent and necessary for Corollary 4.28, as figures 9 and 13 show easily.

About causal tameness, notice that it may fail only when there exists some curve $\gamma : [a, b] \to \overline{M}_i$ satisfying: (a) γ is not smooth at some point (perhaps one endpoint), (b) γ is continuous causal, regarded as a curve in M_0, and (c) no smooth causal curve contained in $\overline{M}_i$ joins its endpoints. It is easy to construct an example of this situation if the boundary is not C^1 (recall, for instance, figure 4). Under our additional hypotheses (C^1 and strong accessibility), it is conceivable the existence of such an example. Nevertheless, to construct explicitly one seems really difficult, and its existence should be regarded as a very pathological case which would not happen in physical examples. In fact, it is not necessary to check causal tameness in the cases to be studied below (globally hyperbolic spacetimes and asymptotic conformally flat ends). This also happens in the two-dimensional case and, so, when Penrose–Carter diagrams are studied (as one focuses on a Lorentzian surface, dropping other two dimensions).

4.4.3 C^1 conformal boundary with no timelike points

Next, our aim is to exploit the causal character of the tangent hyperplane to $\partial_i M (= \partial_i^* M)$ for a C^1 boundary.

Definition 4.30. Let $i : M \hookrightarrow M_0$ be an envelopment with C^1 boundary. A point $z \in \partial_i M$ is spacelike (resp. null; timelike) if $T_z(\partial_i M)$ is spacelike (resp. degenerate; timelike).

Proposition 4.31. *Any timelike* (*resp. spacelike*) *point of a C^1 conformal boundary is past and future* (*resp. either past or future*) *accessible and, then, strongly accessible.*

Proof. Let $z \in \partial_i M$. If z is timelike, there exist both, a future-directed and a past-directed timelike vector in $T_z M_0$ which points out inwards M. Therefore, reasoning as in the proof of Proposition 4.20, we find that z is not only both, future and past accessible, but also future and past strongly accessible. If z is spacelike, any causal vector at $T_z M_0$ points out inwards for one time-orientation and outwards for the reversed one. This implies that z is only future or past accessible — and then strongly accessible, notice also Proposition 4.20. $\square$

Next, our aim is to prove the following theorem.

Theorem 4.32. *Let $i : M \hookrightarrow M_0$ be a chronologically complete envelopment with C^1 boundary. If the boundary $\partial_i M (= \partial_i^* M)$ has no timelike points, then it is causally tame and strongly accessible. Therefore, $\partial_i M$ is regularly accessible and the conformal and c-completions are equivalent.*

Before starting the proof, recall that if $z \in \partial_i M$ is null or spacelike, one of the time-cones (future or past) is pointing inwards M. Moreover, due to the continuity of the cones and the fact that there are no timelike points, we can choose a neighborhood $W_0 (\subset M_0)$ of p such that $W_0 \backslash \partial_i M$ has two connected components (one of them $M \cap W_0$), and all the cones with the same time-orientation at any $q \in W_0 \cap \partial_i M$ point out the same direction (outwards or inwards).

Lemma 4.33. *Under previous assumptions, any smooth timelike curve $\gamma : [a, b] \to W_0 \subset M_0$ touches the boundary $\partial_i M$ at most once.*

Proof. Assume that γ touches $\partial_i M$ at least twice. As $\partial_i M$ has no timelike points, it is crossed transversally by γ and, without loss of generality, we can assume that γ touches the boundary exactly at $\gamma(a)$ and $\gamma(b)$. Then, $\gamma'(a), \gamma'(b)$ has the same time-orientation, but point out different directions (inwards and outwards), in contradiction with the definition of W_0. $\square$

Proof of Theorem 4.32. Choose W_0 as above and assume without lost of generality that the past cones point out inwards at any $q \in W_0 \cap \partial_i M$.

In order to prove strong accessibility, notice that all the points in $\partial_i M \cap W_0$ are future strongly accessible. So, it is enough to prove that they are not past accessible, even at any null point of $\partial_i M \cap W_0$. Assume by contradiction that σ is a past-directed timelike curve in $M \cap W_0$ with i-endpoint $p \in \partial_i M \cap W_0$. Let V_0 be a globally hyperbolic neighborhood of p in M_0 contained in W_0 (for their existence, see for example [28, Theorem 2.14]) and consider t_0 such that $\sigma(t_0)$ is contained in V_0. By the globally hyperbolic character of V_0, there exists a smooth past-directed timelike curve $\gamma : [0,1] \to V_0$ joining $\sigma(t_0)$ and p. As the previous lemma is applicable to γ, $\gamma([0,1)) \subset V_0 \cap M$ and, then, the timelike vector $\gamma'(1)$ points out outwards. This contradicts that the past time-cone in p points out inwards.

In order to prove that the envelopment is causally tame, consider again $z \in \partial_i M$ and a neighborhood $V_0 \subset M_0$ as above. We can also assume that all the causal geodesics in $\overline{V_0}$ maximize the (Lorentzian) distance for g_0. Consider a point $q \in V_0 \cap \overline{M}_i$ such that $q \leq_i p$ (the case $p \leq_i q$ is analogous). This implies the existence of a causal past-directed maximizing geodesic $\gamma : [0,1] \to V_0$. If γ is lightlike, then the only possible continuous causal curve in M_0 joining p and q is (up to a reparameterization) γ and, from the fact that $p \leq_i q$, necessarily $\gamma \subset \overline{M}_i$, i.e., $p \leq_i^{\mathrm{S}} q$. If γ is timelike, previous lemma ensures that $\gamma \subset \overline{M}_i$ (notice that the past cone of p points out inwards) and again $p \leq_i^{\mathrm{S}} q$ (moreover, $p \ll_i^{\mathrm{S}} q$). $\square$

As straightforward consequences of these results, we can obtain the following two assertions, which are more or less implicitly assumed by researchers. About the first one, the implication to the right is well known, but we are not aware of a rigorous proof for the implication to the left.

Corollary 4.34. *A spacetime which admits a chronologically complete envelopment with C^1 boundary $\partial_i M (= \partial_i^* M)$ is globally hyperbolic if and only if $\partial_i M$ does not have timelike points.*

Proof. ($\Rightarrow$). Take a timelike point z of the boundary. From Proposition 4.31, it is future and past (strongly) accessible. Take a future and a past-directed timelike curve in M with i-endpoint z, and fix points q_1, q_2 at each curve. By construction, $J^+(q_1) \cap J^-(q_2)$ is not compact.

($\Leftarrow$). Assume by contradiction that M is not globally hyperbolic and, so, the c-boundary admits a pair $(P, F) \in \partial M$ with $P \neq \emptyset \neq F$ (Theorem 3.29). By Theorem 4.32, $z = \pi((P,F))$ is well defined and it is the i-endpoint of both, a past and a future-directed strongly timelike curve. These curves yield both, a future and a past-directed timelike vector at z pointing outwards, in contradiction with the inexistence of timelike points. $\square$

The second consequence of Theorem 4.32 concerns the boundary of conformally flat and asymptotically conformally flat ends (see Section 3.4). Obviously, there are no timelike points in the null part of the canonical conformal boundary $\partial_i\mathbb{L}^4$. So, this part agrees with a part of its c-boundary (recall Remark 4.29), and can be regarded also as intrinsic for any conformally flat end. This intrinsic character also holds for any asymptotically conformally flat end E. More precisely, the existence of some conformal envelopment without timelike points is assumed in the axioms for asymptotic flatness at null infinity (and spacelike infinity), see [2, 40]. The corresponding part $\partial_i^{\mathrm{asym}}E$ of the accessible conformal boundary $\partial_i^* M$ will agree with a part of the c-boundary ∂M. From a formal viewpoint, $\partial_i^{\mathrm{asym}}E$ can be more precisely defined. Let E be an asymptotically conformally flat end, i.e., E is a suitable connected component of $M\backslash K$, for some compact $K \subset \overline{M}$ (Section 3.4). In principle, the part $\partial_i^{\mathrm{asym}}E$ of the boundary of E can be distinguished from the remainder of $\partial_i^* E$ (which involves points in the boundary of K, and must not be regarded as asymptotic) because the axioms for asymptotic flatness distinguish a null boundary. At any case, $\partial_i^{\mathrm{asym}}E$ is equal to $\lim_{n\to\infty}\partial_i(M\backslash K_n)$, where $\{K_n\}_n$ is a sequence of compact subsets such that each K_{n+1} is included in the interior of K_n, and $K = \cap_n K_n$. Here, $\partial_i^*(M\backslash K_n)$ denotes the conformal boundary of $M\backslash K_n$ for the fixed conformal envelopment i which defines asymptotic conformal flatness, and a point $z \in M_0$ belongs to the limit of the boundaries if $z \in \partial_i^*(M\backslash K_n)$ for all n greater than some n_z. Summing up:

Corollary 4.35. *Let E be an asymptotically conformally flat end. The points in $\partial_i^{\mathrm{asym}}E$ are regularly accessible and, then, they correspond with a subset of the c-boundary ∂M.*

4.5 Examples and counterexamples: the conformal boundary fauna

Previous results on the conformal boundary simplify the computation of the c-boundary in some particular cases (conformally flat, those which admit a natural Penrose–Carter diagram). Nevertheless, more specific techniques are needed for general classes of spacetimes, where a conformal embedding is not expected. This is the case of plane wave type [13], static [1, 10] or stationary [12] spacetimes (as well as spacetimes conformal to them, which include other well-known families as, for example, Generalized Robertson–Walker spacetimes).

However, when the conditions which ensure the equivalence between the conformal and the c-boundary are not fulfilled, the conformal boundary may present a very erratic behavior. Here, we collect some examples, which

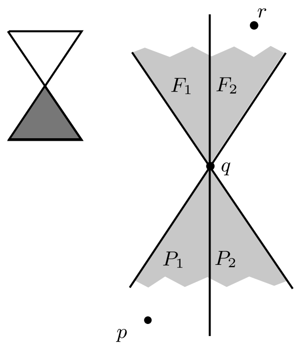

Figure 7: Trivial artificial identifications in the conformal boundary of c-boundary points, and non-transitivity of $\ll_i$.

are instructive in order to understand our previous results. They are quite simple, as we consider always an envelopment $i : M \hookrightarrow M_0$ where the aphysical space M_0 is always $\mathbb{L}^2$, $\mathbb{L}^3$ or some trivial quotient, with the timecones oriented as depicted in the figures. (Notice that for the figures, the time variable is written here as the last one.)

Figure 7: Consider $M = \mathbb{L}^2 \setminus \{(x,t) : x = 0, t \in [-1,1]\}$ and $M_0 = \mathbb{L}^2$. The depicted point q corresponds with two points of the c-boundary, $(P_1, F_1), (P_2, F_2)$. Trivially, $p \ll_i q \ll_i r$, but $p \not\ll_i r$.

Figure 8: The artificial character of the identifications of the two points (P_j, F_j), $j = 1, 2$, in previous example is stressed in the following one. Consider the spacetime (M, g) given by $M = (0, 2\pi) \times \mathbb{R}$, $g = dx^2 - dt^2$. If we take the natural inclusion $i : M \hookrightarrow \mathbb{L}^2$, the conformal boundary $\partial_i M$ is obviously $M = \{0, 2\pi\} \times \mathbb{R}$, and it can be identified with the c-boundary. Nevertheless, regarding x as an angular coordinate, one finds a conformal embedding $\tilde{\imath}$ of M into the cylinder $C = (\mathbb{S}^1 \times \mathbb{R}, d\theta^2 - dt^2)$ such that $\tilde{\imath}(M) = (\mathbb{S}^1 \setminus \{\theta = 0\}) \times \mathbb{R}$. So, the conformal boundary is now a quotient of previous one, $\partial_{\tilde{\imath}} M \equiv \partial_i M / \sim$, where $(0, t) \sim (2\pi, t)$ for all $t \in \mathbb{R}$, which shows the non-intrinsic character of this boundary. Notice that $\partial_i M$ is a smooth hypersurface of M_0, and strongly accessible. Nevertheless, $\tilde{\imath} : \overline{M}_{\tilde{\imath}} \to M_0$ is not an envelopment with C^0 boundary $\partial_{\tilde{\imath}} M$, according to Definition 4.17. As in previous figure, $\ll_{\tilde{\imath}}$ is not transitive.

Figure 9: In the two previous figures, some points of the c-boundary were artificially identified to a single point in the conformal boundary. In the following one, there is also an identification, but it seems much more natural, as it yields a Hausdorff boundary. Let $M_0 = \mathbb{L}^2$ and define M just removing the subset $C = \{(x,t) : t \geq |2x|\}$. Now, consider the following two

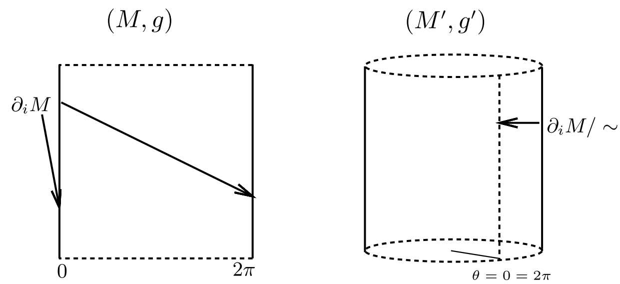

Figure 8: More refined artificial identifications of c-boundary points in open conformal embeddings.

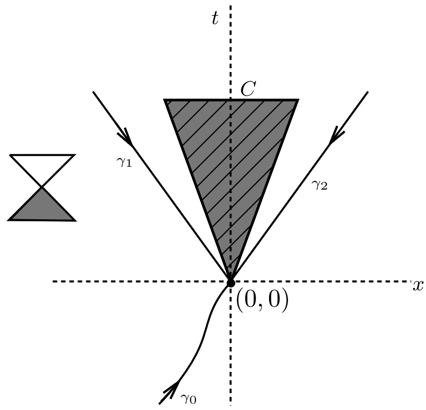

Figure 9: Non-Hausdorff c-boundary but Hausdorff conformal one.

past-directed strongly timelike curves:

$$\gamma_1 : [-1,0) \to M, \gamma_1(x) = (x, -\tfrac{3}{2}x)$$
$$\gamma_2 : [0,1) \to M, \gamma_2(x) = (1-x, \tfrac{3}{2}(1-x)).$$

Obviously, these two curves converge in $\overline{M}_i$ to the single point $(0,0)$. However, γ_i determine two different c-boundary points, namely $(P_0, F_1), (P_0, F_2)$ where $F_j = I^+[\gamma_j], j = 1,2$ and $P_0 = I^-[\gamma_0]$. These two points are non-Hausdorff related in the c-completion $\overline{M}$. According to Harris [21], P_0 is not a *regular* point of the future chronological completion $\hat{M}$ of M, because the future of P_0 is not an indecomposable future set. In the original construction by Geroch *et al.* [17], the Hausdorffness of the c-boundary was imposed a priori. However, this imposition implied very undesirable consequences (see, for example, [35]).

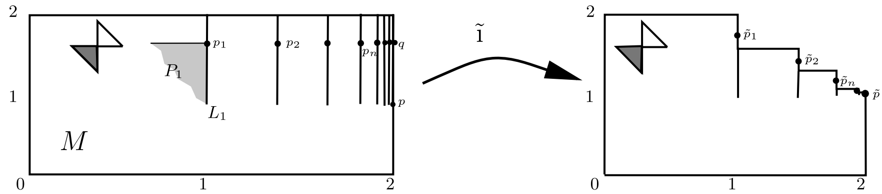

Figure 10: Non-uniqueness of conformal boundary solved by the c-boundary.

Figure 10: The lack of uniqueness and the ambiguities of a general conformal boundary are stressed here. In fact, the open subset M of $M_0 = \mathbb{L}^2$ inherits a conformal boundary $\partial_i M$ through its natural inclusion i. The behavior of $\partial_i M$ is very different to the behavior of the c-boundary ∂M. This boundary seems to be very unappropriate, especially from the topological viewpoint, because the sequence of points $\{p_n\}_n$ converges to q in $\mathbb{L}^2$, but the corresponding sequence $\{P_n = (I^-(p_n), \emptyset)\}_n$ in the c-boundary converges to $P = (I^-(p), \emptyset)$. Nevertheless, one can find a second envelopment $\tilde{\imath} : M \to M_0 = \mathbb{L}^2$. Under this envelopment, the behavior of ∂M agrees with the expected one from the conformal boundary $\partial_{\tilde{\imath}} M$, namely, $\tilde{p}_n := \tilde{\imath}(p_n) \to \tilde{p} := \tilde{\imath}(p)$. In fact, $\partial_{\tilde{\imath}} M$ and ∂M are essentially equivalent (the differences between them appears in elements which are not relevant for the example: some identifications in the removed vertical lines, similar to those in figure 7, and the non-accessibility of the vertexes $(2,0), (0,2)$).

More precisely, M is constructed from $\mathbb{L}^2$ in null coordinates as:

$$M = \{(u,v) \in (0,2) \times (0,2)\} \setminus (\cup_n L_n),$$
$$L_n := \left\{ \left(2 - \frac{1}{n}, v\right), \, v \in (1,2) \right\} \quad \forall n \in \mathbb{N},$$

so that $(P_n, \emptyset), (P, \emptyset) \in \partial M$ project on $p_n, p \in \overline{M}_i$. The convergence with the chr. topology of $\{(P_n, \emptyset)\}_n$ to $(P, \emptyset)$ is straightforward from Definition 3.10. As $\{p_n = \pi((P_n, \emptyset))\}_n \to q \in \partial M \backslash \partial^* M$, the projection $\pi : \overline{M} \to \overline{M}_i^*$ is well defined but non-continuous.

Now, consider a second envelopment $\tilde{\imath} : M \to \mathbb{L}^2$ given by $\tilde{\imath}(u,v) = (u, h_{n(u)}(v))$. Here, the non-negative integer $n(u)$ is determined by $2 - \frac{1}{n(u)} < u \leq 2 - \frac{1}{n(u)+1}$ and $h_n : [0,2] \to [0, 1 + \frac{2}{n+2}]$ is a smooth function with $h_n' > 0$ and $h_n(v) = v$ for all $v \in [0, 1 + \frac{1}{n+2}]$. Clearly, $\tilde{\imath}$ is a conformal map, because it maps lightlike lines in lightlike lines. Putting $\tilde{p} = \tilde{\imath}(p)$, $\tilde{p_n} = \tilde{\imath}(p_n)$ $\tilde{P} = \tilde{\imath}(P)$, $\tilde{P}_n = \tilde{\imath}(P)$ one has $\{(\tilde{P}_n, \emptyset)\}_n \to (\tilde{P}, \emptyset)$ and the projection onto de new conformal boundary $\tilde{\pi} : \partial M \to \partial_{\tilde{\imath}}^* M$ becomes continuous: $\{\tilde{\pi}((\tilde{P}_n, \emptyset))\}_n = \{\tilde{p}_n\}_n \to \tilde{p} = \{\tilde{\pi}((\tilde{P}, \emptyset))\}$.

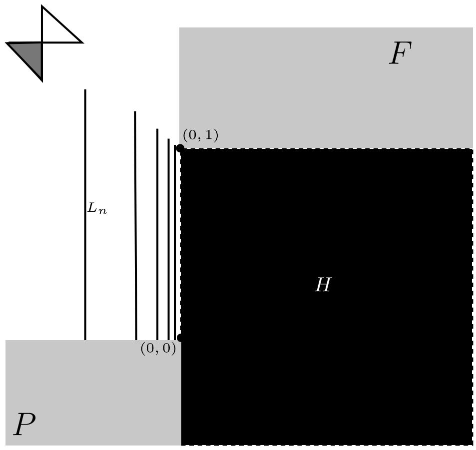

Figure 11: Natural projection $\pi : \partial M \to \partial_i M$ not well-defined.

Figure 11: In this example, one can see easily that the projection $\pi : \partial M \to \partial_i M$ is not always well defined, even for a chronologically complete envelopment. Notice that, when $(P, F) \in \partial M$ satisfy $P \neq \emptyset \neq F$, π is defined as $\pi((P, F)) = \hat{\pi}(P) = \check{\pi}(F)$. But this last equality does not hold for the depicted TIP P and TIF F, as $\hat{\pi}(P) = (0, 0), \check{\pi}(F) = (0, 1)$.

Formally $M_0 = \mathbb{L}^2$ in null coordinates (u, v) and $M = \mathbb{R}^2 \setminus (\cup_n L_n \cup H)$, where $L_n = \{(-\frac{1}{n}, v) : 0 \leq v \leq 1 + \frac{1}{n}\}$ and $H = \{(u, v) : u \geq 0, v \leq 1\}$. The points $(0, 0), (0, 1) \in M_0$ are associated to the TIP $P = I^-((0, 0))$ and the TIF $F = I^+((0, 1))$, resp. Notice that $P \sim_S F$ in M (in fact, $P = \downarrow F, F = \uparrow P$), i.e., $(P, F) \in \partial M$ because of the removed elements of M_0.

Figure 12: As in previous figure, this example shows that π is not always well defined. Nevertheless, this case is more sophisticated because the envelopment has now a C^0 boundary.

Let $M_0 = \mathbb{L}^3$ in natural coordinates, and consider only the region $x > 0$. In this region, M is obtained by removing the following two subsets:

$$\{(x, y, t) : (x, y, t) \in J^+((0, 0, 0)) \cap \{(x, y, t) : y \leq 1\}\},$$
$$\{(x, y, t) : (x, y, t) \in J^-((0, 1, 1)) \cap \{(x, y, t) : y \geq 0\}.$$

Here, the points $(0, 0, 0), (0, 1, 1) \in \partial_i M$ are reachable by a future-directed and past-directed timelike curve resp. They determine, resp. a TIP P and a TIF F (not drawn). Taking into account that the coordinate $u = t + y$ is null, one sees that $P \subset \downarrow F$ and $F \subset \uparrow P$. Moreover, the suppressed subsets remove the TIPs and TIFs, which violate the maximality of P in

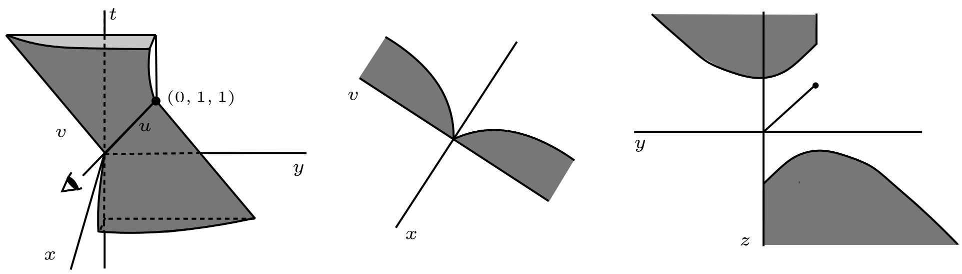

Figure 12: Refinement of the example in Fig. 11 with a (topologically embedded) C^0 conformal boundary.

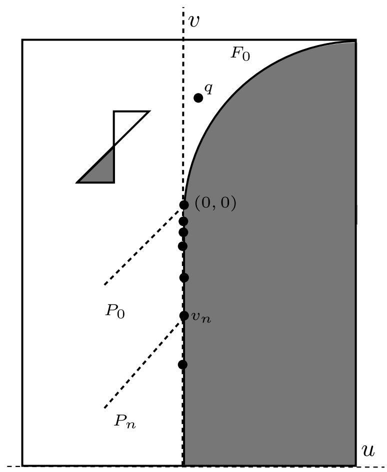

Figure 13: Topological differences between the c-boundary and a C^1 conformal boundary.

$\downarrow F$, and vice versa, that is, $P \sim_S F$. Summing up, $(P, F) \in \partial M$. But $\check{\pi}(F) = (0, 1, 1) \neq (0, 0, 0) = \hat{\pi}(P)$.

Figure 13: This very simple example stresses the differences between the topologies of the conformal and c-boundaries, even for a C^1 envelopment (when strong accessibility is not ensured).

Consider the following open subset of $\mathbb{L}^2$,

$$M = \{(u, v) \in \mathbb{R}^2 : u < 0\} \cup \{(u, v) \in \mathbb{R}^2 : u < v^2, u \geq 0\},$$

with constant null cones depicted in the figure. The natural inclusion in $M_0 = \mathbb{L}^2$ is an envelopment with C^1 boundary. The points $(0, v)$ with $v < 0$ correspond to pairs of the form $(P, \emptyset)$. The point $(0, 0)$ corresponds to a pair

(P_0, F_0) (observe that $(0,0)$ is reachable by a past-directed timelike curve, but it is not strongly accessible).

Consider the sequence $\{(0, v_n)\}_n$ where $v_n \nearrow 0$. This sequence in $\overline{M}_i$ is the projection of a sequence $\{(P_n, \emptyset)\}_n$ in $\overline{M}$. Obviously, $\{(0, v_n)\}_n \to (0,0)$ with the topology induced from $\mathbb{L}^2$, but the sequence $\{(P_n, \emptyset)\}_n$ does not converge to (P_0, F_0) with the chronological topology. Moreover, choose $q \in F_0$ and recall that $(P_0, F_0) \in I^-(q, \overline{M}_i)$ but $(P_n, \emptyset) \notin I^-(q, \overline{M}_i)$.

In fact, this last behavior is a consequence of the following properties of ∂M: (i) $(P_0, F_0) \overline{\ll} q$ and (ii) $I^-(q, \overline{M})$ is open. So, one must drop one of the following elements in the conformal completion: (a) the relation $\ll_i$ as the chronology of $\overline{M}_i$, (b) the induced topology from M_0 as the topology of $\overline{M}_i$, or (c) the open character of chronological futures and pasts in $\overline{M}_i$. Elements (a) and (c) are intrinsic to M, but (b) is extrinsic.

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