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Bundle gerbes for orientifold sigma models

Krzysztof Gawędzki[1], **Rafał R. Suszek**[2] and **Konrad Waldorf**[3]

[1]Laboratoire de Physique, ENS-Lyon, 46 Allée d'Italie, F-69364 Lyon, France
kgawedzk@ens-lyon.fr
[2]Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK
Rafal-Roman.Suszek@fuw.edu.pl
[3]Department Mathematik, Universität Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany
Konrad.Waldorf@mathematik.uni-regensburg.de

Abstract

Bundle gerbes with connection and their modules play an important role in the theory of two-dimensional sigma models with a background Wess–Zumino flux: their holonomy determines the contribution of the flux to the Feynman amplitudes of classical fields. We discuss additional structures on bundle gerbes and gerbe modules needed in similar constructions for orientifold sigma models describing closed and open strings.

1 Introduction

Bundle gerbes [18] are geometric structures related to sheaves of line bundles; see [19, 25] for recent historical essays. They appear naturally in the

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mathematical context of lifting principal G-bundles to $\hat{G}$-bundles for central extensions $\hat{G}$ of a Lie group G by a circle. In the physical context, they arise in studies of quantum field theory anomalies [6] or, together with bundle gerbe modules, in a construction of groups of string theory charges [2]. The present paper has been mainly motivated by the role that bundle gerbes equipped with Hermitian connections play in the theory of two-dimensional (2D) sigma models with a Wess–Zumino (WZ) term [20, 28] in the action functional. Classically, the fields of such a sigma model are maps ϕ from a 2D surface Σ, called the worldsheet, to the target manifold M equipped with a metric and a closed 3-form H. The WZ term describes the background H-flux. Locally, it is given by integrals of the pullbacks ϕ^*B of local Kalb–Ramond 2-forms B on M such that $dB = H$. The ambiguities in defining such a functional $S_{\rm WZ}(\phi)$ globally in topologically non-trivial situations were originally studied with cohomology techniques in [1] and [9] for closed worldsheets, and in [15] for worldsheets with boundary. They may be sorted out systematically using bundle gerbes with connection over the manifold M and, in the case with boundary, bundle gerbe modules; see [5, 10, 11]. In particular, a choice of a bundle gerbe $\mathcal{G}$ with connection, whose curvature 3-form is H, determines unambiguously the Feynman amplitudes $e^{iS_{\rm WZ}(\phi)}$ on closed oriented worldsheets Σ. In the beginning of Section 1 of this paper, we recall the definition [18] of bundle gerbes with connection and review their 1-morphisms and the 2-morphisms between 1-morphisms, all together forming a 2-category [24, 26].

The WZ term in the action functional plays an essential role in Wess–Zumino–Witten (WZW) sigma models [27], assuring their conformal symmetry on the quantum level and rendering them soluble. The target space of a WZW sigma model is a compact Lie group G equipped with a bundle gerbe whose curvature is a bi-invariant closed 3-form H. Bundle gerbes and their modules are specially useful in treating the case [7] of WZW models with non-simply connected target groups G' that are quotients of their covering groups G by a finite subgroup Γ_0 of the center $Z(G)$ of G [12]. A gerbe $\mathcal{G}'$ over $G' = G/\Gamma_0$ may be thought of as a gerbe $\mathcal{G}$ over G equipped with a Γ_0-equivariant structure that picks up in a consistent way isomorphisms between the pullback gerbes $\gamma^*\mathcal{G}$ and $\mathcal{G}$ for each element $\gamma \in \Gamma_0$. The notion of equivariant structures on gerbes extends to the case of gerbes over general manifolds M on which a finite group Γ_0 acts preserving the curvature 3-form H, possibly with fixed points. A gerbe over M with such an equivariant structure may be thought of as a gerbe over the orbifold M/Γ_0 and it may be used to define the WZ action functional for sigma models with the orbifold target. Gerbes with equivariant structures with respect to actions of continuous groups will be discussed elsewhere; see also [16]. They find application in gauged sigma models with WZ term.

Motivated by the theory of unoriented strings [21, 22], one would like to define the WZ action functional for unoriented (in particular unorientable) worldsheets Σ. More exactly, one considers so-called orientifold sigma models. Their classical fields $\hat{\phi}$ map the oriented double $\hat{\Sigma}$, which is equipped with an orientation-changing involution σ such that $\Sigma = \hat{\Sigma}/\sigma$, to the target M equipped with an involution k so that $\hat{\phi} \circ \sigma = k \circ \hat{\phi}$. Assuming that $k^*H = -H$, one may define the WZ action functional for such fields using a gerbe $\mathcal{G}$ over M with curvature H additionally equipped with a Jandl structure [23]. Such a structure on $\mathcal{G}$ may be considered as a twisted version of a $\mathbb{Z}_2$-equivariant structure for the $\mathbb{Z}_2$ action on M defined by k. It picks up in a consistent way an isomorphism between $k^*\mathcal{G}$ and the *dual* gerbe $\mathcal{G}^*$.

One may consider more general orientifold sigma models with the WZ term, corresponding to an action on M of a finite group Γ with elements γ such that $\gamma^*H = \epsilon(\gamma)H$ for a homomorphism ϵ from Γ to $\{\pm 1\} \equiv \mathbb{Z}_2$. The notions of Γ_0-equivariant and Jandl structures on a gerbe $\mathcal{G}$ may be merged into the one of a (Γ, ϵ)-equivariant structure, which we shall also call a twisted-equivariant structure. Such a structure consistently picks up isomorphisms between $\gamma^*\mathcal{G}$ and either $\mathcal{G}$ or $\mathcal{G}^*$, according to the sign of $\epsilon(\gamma)$. The twisted-equivariant structures on gerbes are introduced in Section 1 and are the main topic of the present paper. A special case of such structures occurs when the normal subgroup $\Gamma_0 = \ker \epsilon$ of Γ acts on M without fixed points. If $\epsilon \equiv 1$ so that $\Gamma_0 = \Gamma$, we are back to the correspondence between gerbes over M with Γ_0-equivariant structure and gerbes over $M' = M/\Gamma_0$. If ϵ is non-trivial, so that $\Gamma/\Gamma_0 = \mathbb{Z}_2$, then the action of Γ on M induces a $\mathbb{Z}_2$-action on M', with the non-trivial element of $\mathbb{Z}_2$ acting as an involution k' inverting the sign of the projected 3-form H'. In this situation, gerbes over M with curvature H and (Γ, ϵ)-equivariant structure correspond to gerbes over M' with curvature H' and a Jandl structure. This descent theory for bundle gerbes is discussed in Section 2.

The present paper provides a geometric theory extending and completing the discussion of our previous paper [14] that was devoted to the study of gerbes with twisted-equivariant structures over simple simply connected compact Lie groups. Such gerbes are needed for applications to the orientifolds of WZW models. In [14], we used a local description of gerbes and cohomological tools. Section 3 of the present paper establishes the relation between the geometric and cohomological languages.

For oriented worldsheets Σ with boundary, the classical fields $\phi : \Sigma \longrightarrow M$ are often constrained to take values in special submanifolds D of M on the boundary components of Σ. Such submanifolds are called (D-)branes in string theory. The extension of the definition of the WZ action to this case requires a choice of a bundle gerbe $\mathcal{G}$ with curvature H and of gerbe modules over the submanifolds D. Gerbe modules may be viewed as vector bundles with connection twisted by the gerbe. In the context of the 2-category of bundle gerbes, they can also be viewed as particular 1-morphisms [26]. In Section 4, we adapt this notion to the case of gerbes with (Γ, ϵ)-equivariant structures needed for applications to orientifold sigma models on worldsheets with boundary. We also discuss a presentation of such gerbe modules in terms of local data and develop their descent theory.

The Feynman amplitudes $e^{iS_{WZ}(\phi)}$ of fields ϕ defined on closed worldsheets are given by the holonomy of gerbes [11]. In the case of unoriented worldsheets, the gerbes have to be equipped additionally with a Jandl structure [23]. For oriented worldsheets with boundary, the holonomy giving the Feynman amplitudes receives also contributions from the gerbe modules over the brane worldvolumes that provide the boundary conditions of the theory [5, 10, 11]. In this paper, we introduce a generalization of both notions for unoriented worldsheets with boundary using gerbe modules for gerbes with a Jandl structure. This is discussed in both the geometric and the local language in Section 5.

In Conclusions, we summarize the contents of the present paper and sketch the directions for further work that includes extending the discussion [14] of the orientifold WZW models on closed worldsheets to the ones on worldsheets with boundary.

2 Twisted-equivariant bundle gerbes

We review bundle gerbes and their algebraic structure in Section 2.1 and define twisted-equivariant structures on them in Section 2.2. Twisted-equivariant structures include two extremal versions: the untwisted one, which is just an ordinary equivariant structure, and the twisted $\mathbb{Z}_2$-equivariant one, which coincides, as we discuss in Section 2.3, with a Jandl structure.

2.1 The 2-category of bundle gerbes

In the whole paper, we work with the following conventions:

- *Vector bundles* are Hermitian vector bundles with unitary connection, and isomorphisms of vector bundles respect the Hermitian structure and the connections. These conventions in particular apply to line bundles.
- If $\pi : Y \twoheadrightarrow M$ is a surjective submersion between smooth manifolds, we denote by
$$Y^{[k]} := Y \times_M \cdots \times_M Y,$$

the k-fold fibre product of Y with itself (composed of the elements in the Cartesian product whose components have the same projection to M). The fibre products are, again, smooth manifolds in such a way that the canonical projections $\pi_{i_1\ldots i_r} : Y^{[k]} \longrightarrow Y^{[r]}$ are smooth maps.

In the following, we collect the basic definitions.

Definition 2.1 ([18])**.** A *bundle gerbe* $\mathcal{G}$ over a smooth manifold M is a surjective submersion $\pi : Y \longrightarrow M$, a line bundle L over $Y^{[2]}$, a 2-form $C \in \Omega^2(Y)$, and an isomorphism

$$\mu : \pi_{12}^*L \otimes \pi_{23}^*L \longrightarrow \pi_{13}^*L$$

of line bundles over $Y^{[3]}$, such that two axioms are satisfied:

(G1) The curvature of L is fixed by

$$\mathrm{curv}(L) = \pi_2^*C - \pi_1^*C.$$

(G2) μ is associative in the sense that the diagram

$$\begin{array}{ccc} \pi_{12}^*L \otimes \pi_{23}^*L \otimes \pi_{34}^*L & \xrightarrow{\pi_{123}^*\mu\otimes \mathrm{id}} & \pi_{13}^*L \otimes \pi_{34}^*L \\ {\scriptstyle \mathrm{id}\otimes\pi_{234}^*\mu}\downarrow & & \downarrow{\scriptstyle \pi_{134}^*\mu} \\ \pi_{12}^*L \otimes \pi_{24}^*L & \xrightarrow[\pi_{124}^*\mu]{} & \pi_{14}^*L \end{array}$$

of isomorphisms of line bundles over $Y^{[4]}$ is commutative.

Example 2.1. On any smooth manifold M, there is a family $\mathcal{I}_\omega$ of *trivial bundle gerbes* over M, labelled by 2-forms $\omega \in \Omega^2(M)$. The surjective submersion of $\mathcal{I}_\omega$ is $Y := M$ and the identity $\pi := \mathrm{id}_M$, the line bundle over $Y^{[2]} \cong M$ is the trivial line bundle (equipped with the trivial flat connection), and the isomorphism μ is the identity between trivial line bundles. Its 2-form is the given 2-form $C := \omega$.

Associated to a bundle gerbe $\mathcal{G}$ over M is a 3-form $H \in \Omega^3(M)$ called the *curvature* of $\mathcal{G}$. It is the unique 3-form which satisfies $\pi^*H = \mathrm{d}C$. The trivial bundle gerbe $\mathcal{I}_\omega$ has the curvature $\mathrm{d}\omega$.

We would like to compare two bundle gerbes using a notion of morphisms between bundle gerbes. The morphisms between such gerbes which we consider here have been introduced in [26]. For simplicity, we work with the convention that we do not label or write down pullbacks along canonical projection maps, such as in (2.1) and (1M1) below.

Definition 2.2. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be bundle gerbes over M. A 1-morphism

$$\mathcal{A} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$$

consists of a surjective submersion $\zeta : Z \longrightarrow Y_1 \times_M Y_2$, a vector bundle A over Z, and an isomorphism

$$\alpha : L_1 \otimes \zeta_2^* A \longrightarrow \zeta_1^* A \otimes L_2 \tag{2.1}$$

of vector bundles over $Z \times_M Z$, such that two axioms are satisfied:

(1M1) The curvature of A obeys

$$\frac{1}{n}\mathrm{tr}(\mathrm{curv}(A)) = C_2 - C_1,$$

where n is the rank of A.

(1M2) The isomorphism α commutes with the isomorphisms μ_1 and μ_2 of the gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ in the sense that the diagram

$$\begin{array}{ccc}
\zeta_{12}^* L_1 \otimes \zeta_{23}^* L_1 \otimes \zeta_3^* A & \xrightarrow{\mu_1 \otimes \mathrm{id}} & \zeta_{13}^* L_1 \otimes \zeta_3^* A \\
\downarrow{\scriptstyle \mathrm{id} \otimes \zeta_{23}^* \alpha} & & \\
\zeta_{12}^* L_1 \otimes \zeta_2^* A \otimes \zeta_{23}^* L_2 & & \downarrow{\scriptstyle \zeta_{13}^* \alpha} \\
\downarrow{\scriptstyle \zeta_{12}^* \alpha \otimes \mathrm{id}} & & \\
\zeta_1^* A \otimes \zeta_{12}^* L_2 \otimes \zeta_{23}^* L_2 & \xrightarrow[\mathrm{id} \otimes \mu_2]{} & \zeta_1^* A \otimes \zeta_{13}^* L_2
\end{array}$$

of isomorphisms of vector bundles over $Z \times_M Z \times_M Z$ is commutative.

These 1-morphisms are generalizations of so-called *stable isomorphisms* [17]. They are generalized in two aspects: we admit vector bundles of rank possibly higher than 1 (this makes it possible to describe gerbe modules by morphisms), and these vector bundles live over a more general space Z than just the fibre product $Y_1 \times_M Y_2$ (this makes the composition of morphisms easier).

A 1-morphism $\mathcal{A} : \mathcal{G} \longrightarrow \mathcal{G}'$ requires that the curvatures of the bundle gerbes $\mathcal{G}$ and $\mathcal{G}'$ coincide. This follows from axiom (1M1) and the fact that the trace of the curvature of a vector bundle is a closed form.

Example 2.2. Every bundle gerbe $\mathcal{G}$ has an associated 1-morphism

$$\mathrm{id}_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{G}$$

defined by the identity surjective submersion id_Z of $Z := Y^{[2]}$, the line bundle $A := L$ of the bundle gerbe $\mathcal{G}$ itself, and the isomorphism

$$\pi_{13}^*L \otimes \pi_{34}^*L \xrightarrow{\pi_{134}^*\mu} \pi_{14}^*L \xrightarrow{\pi_{124}^*\mu^{-1}} \pi_{12}^*L \otimes \pi_{24}^*L$$

of line bundles over $Z \times_M Z = Y^{[4]}$, where we have identified $\zeta_1 = \pi_{12}$ and $\zeta_2 = \pi_{34}$. The axioms for this 1-morphism follow from the axioms of the bundle gerbe $\mathcal{G}$.

The 2-categorial aspects of the theory of bundle gerbes enter when one wants to compare two 1-morphisms.

Definition 2.3. Let $\mathcal{A} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ and $\mathcal{A}' : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ be 1-morphisms between bundle gerbes over M. A 2-morphism

$$\beta : \mathcal{A} \Longrightarrow \mathcal{A}'$$

is a surjective submersion $\omega : W \longrightarrow Z_1 \times_P Z_2$, where $P := Y_1 \times_M Y_2$, together with a morphism $\beta_W : A_1 \longrightarrow A_2$ of vector bundles over W, such that the diagram

$$\begin{array}{ccc} L_1 \otimes \omega_2^*A_1 & \xrightarrow{\alpha_1} & \omega_1^*A_1 \otimes L_2 \\ {\scriptstyle \mathrm{id}\otimes\omega_2^*\beta_W}\downarrow & & \downarrow{\scriptstyle \omega_1^*\beta_W\otimes\mathrm{id}} \\ L_1 \otimes \omega_2^*A_2 & \xrightarrow[\alpha_2]{} & \omega_1^*A_2 \otimes L_2 \end{array} \tag{2.2}$$

of morphisms of vector bundles over $W \times_M W$ is commutative. We shall often omit the subscript in β_W if it is clear from the context that the notation refers to the bundle isomorphism.

Due to technical reasons, one has to define a certain equivalence relation on the space of 2-morphisms [26], whose precise form is not important for this paper. A 2-morphism $\beta : \mathcal{A} \Longrightarrow \mathcal{A}'$ is invertible if and only if the morphism β_W of vector bundles is invertible. This, in turn, is the case if and only if the ranks of the vector bundles of $\mathcal{A}$ and $\mathcal{A}'$ coincide.

Bundle gerbes over M, 1-morphisms and 2-morphisms as defined above form a strictly associative 2-category $\mathfrak{BGrb}(M)$ [26]. We describe below what that means. Most importantly for us, we can compose 1-morphisms:

Definition 2.4. The composition of two 1-morphisms $\mathcal{A} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ and $\mathcal{A}' : \mathcal{G}_2 \longrightarrow \mathcal{G}_3$ is the 1-morphism

$$\mathcal{A}' \circ \mathcal{A} : \mathcal{G}_1 \longrightarrow \mathcal{G}_3$$

defined by the following data: its surjective submersion is $\zeta : \tilde{Z} \longrightarrow Y_1 \times_M Y_3$ with $\tilde{Z} := Z \times_{Y_2} Z'$ and the canonical projections to Y_1 and Y_3, its vector bundle over $\tilde{Z}$ is $\tilde{A} := A \otimes A'$, and its isomorphism is given by

$$\begin{array}{c} L_1 \otimes \tilde{\zeta}_2^* \tilde{A} = L_1 \otimes \zeta_2^* A \otimes \zeta_2'^* A' \\ \Big\downarrow{\scriptstyle \alpha \otimes \mathrm{id}} \\ \zeta_1^* A \otimes L_2 \otimes \zeta_2'^* A' \\ \Big\downarrow{\scriptstyle \mathrm{id} \otimes \alpha'} \\ \zeta_1^* A \otimes \zeta_1'^* A' \otimes L_3 = \tilde{\zeta}_1^* \tilde{A} \otimes L_3. \end{array}$$

The axioms for this 1-morphism are easy to check. If we tacitly assume the category of vector spaces to be strictly monoidal, it turns out that the composition of 1-morphisms defined in this manner is, indeed, strictly associative,

$$(\mathcal{A}'' \circ \mathcal{A}') \circ \mathcal{A} = \mathcal{A}'' \circ (\mathcal{A}' \circ \mathcal{A}).$$

The simplicity of Definition 2.4 (compared, e.g., to the one given in [24]) and the strict associativity of the composition of 1-morphisms are consequences of our generalized definition of 1-morphisms. One can now show

Proposition 2.1 ([26]). *A 1-morphism $\mathcal{A} : \mathcal{G} \longrightarrow \mathcal{G}'$ is invertible, also called 1-isomorphism, if and only if its vector bundle is of rank one.*

In a 2-category, invertibility means that there exists a 1-isomorphism $\mathcal{A}^{-1}$ acting in the opposite direction, together with 2-isomorphisms

$$i_l : \mathcal{A}^{-1} \circ \mathcal{A} \Longrightarrow \mathrm{id}_{\mathcal{G}} \quad \text{and} \quad i_r : \mathrm{id}_{\mathcal{G}'} \Longrightarrow \mathcal{A} \circ \mathcal{A}^{-1}, \tag{2.3}$$

which satisfy certain coherence axioms [26]. The 2-category $\mathfrak{BGrb}(M)$ of bundle gerbes over M also provides the following structure:

(a) The *vertical composition* of two 2-morphisms $\beta_1 : \mathcal{A} \Longrightarrow \mathcal{A}'$ and $\beta_2 : \mathcal{A}' \Longrightarrow \mathcal{A}''$ to a new 2-morphism

$$\beta_2 \bullet \beta_1 : \mathcal{A} \Longrightarrow \mathcal{A}'',$$

which is associative and has units $\mathrm{id}_{\mathcal{A}}$ for any 1-morphism $\mathcal{A}$.

(b) The *horizontal composition* of two 2-morphisms $\beta_{12} : \mathcal{A}_{12} \Longrightarrow \mathcal{A}'_{12}$ and $\beta_{23} : \mathcal{A}_{23} \Longrightarrow \mathcal{A}'_{23}$ to a new 2-morphism

$$\beta_{23} \circ \beta_{12} : \mathcal{A}_{23} \circ \mathcal{A}_{12} \Longrightarrow \mathcal{A}'_{23} \circ \mathcal{A}'_{12},$$

which is compatible with the vertical composition.

(c) Natural 2-isomorphisms

$$\rho_{\mathcal{A}} : \mathrm{id}_{\mathcal{G}_2} \circ \mathcal{A} \Longrightarrow \mathcal{A} \quad \text{and} \quad \lambda_{\mathcal{A}} : \mathcal{A} \circ \mathrm{id}_{\mathcal{G}_1} \Longrightarrow \mathcal{A} \tag{2.4}$$

associated to any 1-morphism $\mathcal{A} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$, which satisfy the equality

$$\mathrm{id}_{\mathcal{A}'} \circ \rho_{\mathcal{A}} = \lambda_{\mathcal{A}'} \circ \mathrm{id}_{\mathcal{A}}. \tag{2.5}$$

The 2-category of bundle gerbes has pullbacks: for every smooth map $f : M \longrightarrow N$, there is a strict 2-functor

$$f^* : \mathfrak{BGrb}(N) \longrightarrow \mathfrak{BGrb}(M). \tag{2.6}$$

Thus, for any bundle gerbe $\mathcal{G}$, we have a pullback bundle gerbe $f^*\mathcal{G}$; for any 1-morphism $\mathcal{A} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$, a pullback 1-morphism $f^*\mathcal{A} : f^*\mathcal{G}_1 \longrightarrow f^*\mathcal{G}_2$; and for every 2-morphism $\beta : \mathcal{A} \Longrightarrow \mathcal{A}'$, a pullback 2-morphism $f^*\beta : f^*\mathcal{A} \Longrightarrow f^*\mathcal{A}'$. These pullbacks are essentially defined as pullbacks of the surjective submersions and the structure thereon, details can be found in [26]. If a bundle gerbe $\mathcal{G}$ has curvature H, its pullback $f^*\mathcal{G}$ has curvature f^*H. The strictness of the 2-functor (2.6) means that $f^*\mathrm{id}_{\mathcal{G}} = \mathrm{id}_{f^*\mathcal{G}}$ and $f^*(\mathcal{A}' \circ \mathcal{A}) = f^*\mathcal{A}' \circ f^*\mathcal{A}$ whenever $\mathcal{A}$ and $\mathcal{A}'$ are composable 1-morphisms. If $g : X \longrightarrow M$ is another map, we find $(f \circ g)^* = g^* \circ f^*$.

In order to concentrate on what we need in this paper, we define the dual $\mathcal{G}^*$ of a bundle gerbe $\mathcal{G}$ without emphasizing its role in the 2-categorial context. $\mathcal{G}^*$ consists of the same surjective submersion $\pi : Y \longrightarrow M$ as $\mathcal{G}$, the 2-form $-C \in \Omega^2(Y)$, the line bundle L^* over $Y^{[2]}$ and the inverse of the dual of the isomorphism μ, which is an isomorphism

$$\mu^{*-1} : \pi_{12}^* L^* \otimes \pi_{23}^* L^* \longrightarrow \pi_{13}^* L^*$$

of line bundles over $Y^{[3]}$. If H is the curvature of $\mathcal{G}$, the curvature of $\mathcal{G}^*$ is $-H$. If $\mathcal{A} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ is a 1-morphism, we define an *adjoint* 1-morphism

$$\mathcal{A}^\dagger : \mathcal{G}_1^* \longrightarrow \mathcal{G}_2^*$$

in the following way: it consists of the same surjective submersion $\zeta : Z \longrightarrow Y_1 \times_M Y_2$ as $\mathcal{A}$, it has the vector bundle A^* over Z, and the isomorphism

$$\alpha^{*-1} : L_1^* \otimes \zeta_1^* A^* \longrightarrow \zeta_2^* A^* \otimes L_2^*$$

of vector bundles over $Z \times_M Z$. The axioms for this 1-morphism follow immediately from those for $\mathcal{A}$. Finally, for a 2-*iso*morphism $\beta : \mathcal{A}_1 \Longrightarrow \mathcal{A}_2$, we define an *adjoint* 2-isomorphism

$$\beta^\dagger : \mathcal{A}_1^\dagger \Longrightarrow \mathcal{A}_2^\dagger.$$

It has the same surjective submersion $\omega : W \longrightarrow Z_1 \times_P Z_2$ as β, and the isomorphism $\beta_W^{*-1} : A_1^* \longrightarrow A_2^*$ of vector bundles over W. Notice that all these operations are strictly involutive:

$$\mathcal{G}^{**} = \mathcal{G}, \quad \mathcal{A}^{\dagger\dagger} = \mathcal{A} \quad \text{and} \quad \beta^{\dagger\dagger} = \beta. \tag{2.7}$$

Remark 2.1. In the context of some more structures in the 2-category $\mathfrak{BGrb}(M)$, as described in [26], namely a duality 2-functor $()^*$ and a functor assigning inverses $\mathcal{A}^{-1}$ to 1-isomorphisms $\mathcal{A}$, and certain 2-morphisms $\beta^\# : \mathcal{A}_2^{-1} \Longrightarrow \mathcal{A}_1^{-1}$ to 2-isomorphisms $\beta : \mathcal{A}_1 \Longrightarrow \mathcal{A}_2$, we find $\mathcal{A}^\dagger = \mathcal{A}^{*-1}$ and $\beta^\dagger = \beta^{\#-1}$.

2.2 Twisted-equivariant structures

An *orientifold group* (Γ, ϵ) for a smooth manifold M is a finite group Γ acting smoothly on the left on M, together with a group homomorphism $\epsilon : \Gamma \longrightarrow \mathbb{Z}_2 = \{-1, 1\}$. We label the diffeomorphisms implementing the action by the group elements themselves, for instance $\gamma : M \longrightarrow M$. Note that $\gamma_2 \circ \gamma_1 = \gamma_2\gamma_1$.

Next, we define an action of the orientifold group (Γ, ϵ) on bundle gerbes over M and their 1- and 2-morphisms. The value $\epsilon(\gamma)$ indicates whether a group element $\gamma \in \Gamma$ acts just by pullback along γ^{-1} or also by additionally taking adjoints. Explicitly, for a bundle gerbe $\mathcal{G}$, we set

$$\gamma\mathcal{G} := \begin{cases} (\gamma^{-1})^*\mathcal{G}, & \text{if } \epsilon(\gamma) = 1, \\ (\gamma^{-1})^*\mathcal{G}^*, & \text{if } \epsilon(\gamma) = -1. \end{cases}$$

Similarly, for a 1-morphism $\mathcal{A} : \mathcal{G} \longrightarrow \mathcal{H}$, we have a 1-morphism

$$\gamma\mathcal{A} : \gamma\mathcal{G} \longrightarrow \gamma\mathcal{H}$$

defined by

$$\gamma\mathcal{A} := \begin{cases} (\gamma^{-1})^*\mathcal{A}, & \text{if } \epsilon(\gamma) = 1, \\ (\gamma^{-1})^*\mathcal{A}^\dagger, & \text{if } \epsilon(\gamma) = -1. \end{cases}$$

Finally, for a 2-*iso*morphism $\beta : \mathcal{A} \Longrightarrow \mathcal{A}'$, we have a 2-isomorphism

$$\gamma\beta : \gamma\mathcal{A} \Longrightarrow \gamma\mathcal{A}'$$

defined by

$$\gamma\beta := \begin{cases} (\gamma^{-1})^*\beta, & \text{if } \epsilon(\gamma) = 1, \\ (\gamma^{-1})^*\beta^\dagger, & \text{if } \epsilon(\gamma) = -1. \end{cases}$$

Note that our conventions and (2.7) imply

$$(\gamma_1\gamma_2) = \gamma_1\gamma_2,$$

so that γ is a left action on gerbes and their 1- and 2-morphisms. We use the same notation for differential forms, i.e., $\gamma\omega := \epsilon(\gamma)(\gamma^{-1})^*\omega$ for any differential form ω on M. If H is the curvature of a bundle gerbe $\mathcal{G}$, the curvature of $\gamma\mathcal{G}$ is γH.

Definition 2.5. Let (Γ, ϵ) be an orientifold group for M and let $\mathcal{G}$ be a bundle gerbe over M. A (Γ, ϵ)-*equivariant structure* on $\mathcal{G}$ consists of 1-isomorphisms

$$\mathcal{A}_\gamma : \mathcal{G} \longrightarrow \gamma\mathcal{G}$$

for each $\gamma \in \Gamma$, and of 2-isomorphisms

$$\varphi_{\gamma_1,\gamma_2} : \gamma_1\mathcal{A}_{\gamma_2} \circ \mathcal{A}_{\gamma_1} \Longrightarrow \mathcal{A}_{\gamma_1\gamma_2}$$

for each pair $\gamma_1, \gamma_2 \in \Gamma$, such that the diagram

$$\begin{array}{ccc} \gamma_1\gamma_2\mathcal{A}_{\gamma_3} \circ \gamma_1\mathcal{A}_{\gamma_2} \circ \mathcal{A}_{\gamma_1} & \xRightarrow{\mathrm{id}\circ\varphi_{\gamma_1,\gamma_2}} & \gamma_1\gamma_2\mathcal{A}_{\gamma_3} \circ \mathcal{A}_{\gamma_1\gamma_2} \\ {\scriptstyle \gamma_1\varphi_{\gamma_2,\gamma_3}\circ\mathrm{id}}\Downarrow & & \Downarrow{\scriptstyle \varphi_{\gamma_1\gamma_2,\gamma_3}} \\ \gamma_1\mathcal{A}_{\gamma_2\gamma_3} \circ \mathcal{A}_{\gamma_1} & \xRightarrow[\varphi_{\gamma_1,\gamma_2\gamma_3}]{} & \mathcal{A}_{\gamma_1\gamma_2\gamma_3} \end{array} \tag{2.8}$$

of 2-isomorphisms is commutative.

We call a bundle gerbe $\mathcal{G}$ with (Γ, ϵ)-equivariant structure a (Γ, ϵ)-*equivariant bundle gerbe* or twisted-equivariant bundle gerbe. If ϵ is constant, a (Γ, ϵ)-equivariant bundle gerbe $\mathcal{G}$ is just called a Γ-*equivariant bundle gerbe.* The curvature H of a twisted-equivariant bundle gerbe satisfies $\gamma H = H$. A twisted-equivariant structure on a bundle gerbe $\mathcal{G}$ is called *normalized* if the following choices concerning the neutral group element $1 \in \Gamma$ have been made:

(a) the 1-isomorphism $\mathcal{A}_1 : \mathcal{G} \longrightarrow \mathcal{G}$ is the identity 1-isomorphism $\mathrm{id}_{\mathcal{G}}$;
(b) the 2-isomorphism $\varphi_{1,\gamma} : \mathcal{A}_\gamma \circ \mathrm{id}_{\mathcal{G}} \Longrightarrow \mathcal{A}_\gamma$ is the natural 2-isomorphism $\lambda_{\mathcal{A}_\gamma}$ from the 2-category of bundle gerbes;
(c) accordingly, the 2-isomorphism $\varphi_{\gamma,1} : \mathrm{id}_{\gamma\mathcal{G}} \circ \mathcal{A}_\gamma \Longrightarrow \mathcal{A}_\gamma$ is the natural 2-isomorphism $\rho_{\mathcal{A}_\gamma}$ from the 2-category of bundle gerbes.

Bundle gerbes with normalized twisted-equivariant structures will give rise to elements in normalized group cohomology, as we shall see in Section 3. A twisted-equivariant structure on a bundle gerbe $\mathcal{G}$ is called *descended* if all surjective submersions, i.e., the surjective submersions ζ^γ of the 1-isomorphisms $\mathcal{A}_\gamma$ and the surjective submersions $\omega^{\gamma_1,\gamma_2}$ of the 2-isomorphisms $\varphi_{\gamma_1,\gamma_2}$, are identities. This will be important in Section 2.

Example 2.3. As an example, let us equip the trivial bundle gerbe $\mathcal{I}_\omega$ from Example 2.1 with a twisted-equivariant structure, for any orientifold group (Γ, ϵ) of M. This is possible for 2-forms $\omega \in \Omega^2(M)$ with $\gamma\omega = \omega$ for all $\gamma \in \Gamma$. Since then $\gamma\mathcal{I}_\omega = \mathcal{I}_{\gamma\omega} = \mathcal{I}_\omega$, we may choose $\mathcal{A}_\gamma := \mathrm{id}_{\mathcal{I}_\omega}$ for all $\gamma \in \Gamma$. Accordingly, we can also choose

$$\varphi_{\gamma_1,\gamma_2} := \rho_{\mathrm{id}_{\mathcal{I}_\omega}} = \lambda_{\mathrm{id}_{\mathcal{I}_\omega}} : \mathrm{id}_{\mathcal{I}_\omega} \circ \mathrm{id}_{\mathcal{I}_\omega} \Longrightarrow \mathrm{id}_{\mathcal{I}_\omega}.$$

Diagram (2.8) commutes due to condition (2.5). We denote this canonical (Γ, ϵ)-equivariant structure by $\mathcal{J}_\omega$. It is normalized and descended.

We recall that there exist canonical bundle gerbes over all compact simple Lie groups [12, 16]. All normalized twisted-equivariant structures on these canonical bundle gerbes were classified (up to equivalence defined below) in [14] using cohomological considerations (see also Section 3): they arise in numbers ranging from 2 to 16. The corresponding geometrical constructions will appear in [13].

Let us formulate the definition of a (Γ, ϵ)-equivariant structure in terms of line bundles and their isomorphisms. For convenience, we assume the (Γ, ϵ)-equivariant structure to be descended (see also Lemma 2.1 below). The pullback bundle gerbe $(\gamma^{-1})^*\mathcal{G}$ has the surjective submersion $\pi_\gamma : Y_\gamma \longrightarrow M$ in the commutative pullback diagram

$$\begin{array}{ccc} Y_\gamma & \longrightarrow & Y \\ {\scriptstyle \pi_\gamma}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[\gamma^{-1}]{} & M \end{array} ,$$

with $Y_\gamma := Y$ and $\pi_\gamma := \gamma \circ \pi$, and the rest of the data is the same as for $\mathcal{G}$. The 1-isomorphism $\mathcal{A}_\gamma : \mathcal{G} \longrightarrow \gamma\mathcal{G}$ is now a line bundle A_γ over $Z^\gamma := Y \times_M Y_\gamma$ and an isomorphism

$$\alpha_\gamma : \pi_{13}^* L \otimes \pi_{34}^* A_\gamma \longrightarrow \pi_{12}^* A_\gamma \otimes \pi_{24}^* L^{\epsilon(\gamma)} \tag{2.9}$$

of line bundles over $Z^\gamma \times_M Z^\gamma$, satisfying the compatibility axiom (1M2), namely

$$\begin{array}{ccc} \pi_{13}^* L \otimes \pi_{35}^* L \otimes \pi_{56}^* A_\gamma & \xrightarrow{\pi_{135}^*\mu \otimes \mathrm{id}} & \pi_{15}^* L \otimes \pi_{56}^* A_\gamma \\ {\scriptstyle \mathrm{id}\otimes\pi_{3456}^*\alpha_\gamma}\downarrow & & \\ \pi_{13}^* L \otimes \pi_{34}^* A_\gamma \otimes \pi_{46}^* L^{\epsilon(\gamma)} & & \downarrow{\scriptstyle \pi_{1256}^*\alpha_\gamma} \\ {\scriptstyle \pi_{1234}^*\alpha_\gamma\otimes\mathrm{id}}\downarrow & & \\ \pi_{12}^* A_\gamma \otimes \pi_{24}^* L^{\epsilon(\gamma)} \otimes \pi_{46}^* L^{\epsilon(\gamma)} & \xrightarrow[\mathrm{id}\otimes\pi_{246}^*\mu^{\epsilon(\gamma)}]{} & \pi_{12}^* A_\gamma \otimes \pi_{26}^* L^{\epsilon(\gamma)}. \end{array} \tag{2.10}$$

Here, and in the following, we have regarded the fibre products of Z^γ with itself as a subset of Y^4 and Y^6 respectively, and used the projections

$\pi_{ij} : Y^k \longrightarrow Y^2$ carefully: in (2.9) we have well-defined projections $\pi_{12}, \pi_{34} : Z^\gamma \times_M Z^\gamma \longrightarrow Z^\gamma$ and $\pi_{13}, \pi_{24} : Z^\gamma \times_M Z^\gamma \longrightarrow Y^{[2]}$, and similarly in (2.10). Furthermore, $L^{\epsilon(\gamma)}$ stands for the dual line bundle L^* and $\mu^{\epsilon(\gamma)}$ for the isomorphism μ^{*-1} if $\epsilon(\gamma) = -1$.

The surjective submersion of the 1-isomorphism ${}_{\gamma_1}\mathcal{A}_{\gamma_2}$ is the identity on $Z^{\gamma_2}_{\gamma_1} := Y_{\gamma_1} \times_M Y_{\gamma_1\gamma_2}$, whose projection to the base space M makes the diagram

$$\begin{array}{ccc} Z^{\gamma_2}_{\gamma_1} & = & Z^{\gamma_2} \\ \downarrow & & \downarrow \\ M & \xrightarrow[\gamma_1^{-1}]{} & M \end{array}$$

commutative. Further, ${}_{\gamma_1}\mathcal{A}_{\gamma_2}$ consists of the line bundle $A^{\epsilon(\gamma_1)}_{\gamma_2}$ over $Z^{\gamma_2}_{\gamma_1}$, and of the isomorphism $\alpha^{\epsilon(\gamma_1)}_{\gamma_2}$, which stands for $\alpha^{*-1}_{\gamma_2}$ if $\epsilon(\gamma_1) = -1$. Next, applying Definition 2.4 for the composition of two 1-morphisms to ${}_{\gamma_1}\mathcal{A}_{\gamma_2} \circ \mathcal{A}_{\gamma_1}$, we have to form the fibre product

$$Z^{\gamma_1,\gamma_2} := Z^{\gamma_1} \times_{Y_{\gamma_1}} Z^{\gamma_2}_{\gamma_1} \cong Y \times_M Y_{\gamma_1} \times_M Y_{\gamma_1\gamma_2}$$

with the surjective submersion $\pi_{13} : Z^{\gamma_1,\gamma_2} \longrightarrow Y \times_M Y_{\gamma_1\gamma_2}$. The line bundle of the composition ${}_{\gamma_1}\mathcal{A}_{\gamma_2} \circ \mathcal{A}_{\gamma_1}$ is the line bundle $\pi_{12}^* A_{\gamma_1} \otimes \pi_{23}^* A^{\epsilon(\gamma_1)}_{\gamma_2}$ over Z^{γ_1,γ_2}, and its isomorphism is

$$\begin{aligned}(\mathrm{id} \otimes \pi_{2356}^* \alpha^{\epsilon(\gamma_1)}_{\gamma_2}) \circ (\pi_{1245}^* \alpha_{\gamma_1} \otimes \mathrm{id}) &: \pi_{14}^* L \otimes \pi_{45}^* A_{\gamma_1} \otimes \pi_{56}^* A^{\epsilon(\gamma_1)}_{\gamma_2} \\ &\longrightarrow \pi_{12}^* A_{\gamma_1} \otimes \pi_{23}^* A^{\epsilon(\gamma_1)}_{\gamma_2} \otimes \pi_{36}^* L^{\epsilon(\gamma_1\gamma_2)}\end{aligned}$$

Finally, we come to the 2-isomorphisms $\varphi_{\gamma_1,\gamma_2}$, whose surjective submersion ω is by assumption the identity on Z^{γ_1,γ_2}, so that they induce the isomorphisms

$$\varphi_{\gamma_1,\gamma_2} : \pi_{12}^* A_{\gamma_1} \otimes \pi_{23}^* A^{\epsilon(\gamma_1)}_{\gamma_2} \longrightarrow \pi_{13}^* A_{\gamma_1\gamma_2}$$

of line bundles over Z^{γ_1,γ_2} satisfying the compatibility condition (2.2) for 2-morphisms, which, here, amounts to the commutativity of the diagram (turned by 90 degrees compared to (2.2) for presentational reasons)

$$\begin{array}{ccc}
\pi_{14}^*L \otimes \pi_{45}^*A_{\gamma_1} \otimes \pi_{56}^*A_{\gamma_2}^{\epsilon(\gamma_1)} & \xrightarrow{\mathrm{id}\otimes\pi_{456}^*\varphi_{\gamma_1,\gamma_2}} & \pi_{14}^*L \otimes \pi_{46}^*A_{\gamma_1\gamma_2} \\
{\scriptstyle \pi_{1245}^*\alpha_{\gamma_1}\otimes\mathrm{id}}\big\downarrow & & \\
\pi_{12}^*A_{\gamma_1} \otimes \pi_{25}^*L^{\epsilon(\gamma_1)} \otimes \pi_{56}^*A_{\gamma_2}^{\epsilon(\gamma_1)} & & \big\downarrow{\scriptstyle \pi_{1346}^*\alpha_{\gamma_1\gamma_2}} \\
{\scriptstyle \mathrm{id}\otimes\pi_{2356}^*\alpha_{\gamma_2}^{\epsilon(\gamma_1)}}\big\downarrow & & \\
\pi_{12}^*A_{\gamma_1} \otimes \pi_{23}^*A_{\gamma_2}^{\epsilon(\gamma_1)} \otimes \pi_{36}^*L^{\epsilon(\gamma_1\gamma_2)} & \xrightarrow[\pi_{123}^*\varphi_{\gamma_1,\gamma_2}\otimes\mathrm{id}]{} & \pi_{13}^*A_{\gamma_1\gamma_2} \otimes \pi_{36}^*L^{\epsilon(\gamma_1\gamma_2)}
\end{array} \tag{2.11}$$

of isomorphisms of line bundles over $Z^{\gamma_1,\gamma_2} \times_M Z^{\gamma_1,\gamma_2}$. The commutativity of diagram (2.8) from Definition 2.5 is equivalent to that of the diagram

$$\begin{array}{ccc}
\pi_{12}^*A_{\gamma_1} \otimes \pi_{23}^*A_{\gamma_2}^{\epsilon(\gamma_1)} \otimes \pi_{34}^*A_{\gamma_3}^{\epsilon(\gamma_1\gamma_2)} & \xrightarrow{\mathrm{id}\otimes\pi_{234}^*\varphi_{\gamma_2,\gamma_3}^{\epsilon(\gamma_1)}} & \pi_{12}^*A_{\gamma_1} \otimes \pi_{24}^*A_{\gamma_2\gamma_3}^{\epsilon(\gamma_1)} \\
{\scriptstyle \pi_{123}^*\varphi_{\gamma_1,\gamma_2}\otimes\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{124}^*\varphi_{\gamma_1,\gamma_2\gamma_3}} \\
\pi_{13}^*A_{\gamma_1\gamma_2} \otimes \pi_{34}^*A_{\gamma_3}^{\epsilon(\gamma_1\gamma_2)} & \xrightarrow[\pi_{134}^*\varphi_{\gamma_1\gamma_2,\gamma_3}]{} & \pi_{14}^*A_{\gamma_1\gamma_2\gamma_3}
\end{array} \tag{2.12}$$

of isomorphisms of line bundles over $Z^{\gamma_1,\gamma_2,\gamma_3} \cong Y \times_M Y_{\gamma_1} \times_M Y_{\gamma_1\gamma_2} \times_M Y_{\gamma_1\gamma_2\gamma_3}$.

Summarizing, a descended (Γ, ϵ)-equivariant structure on the bundle gerbe $\mathcal{G}$ is

1. A line bundle A_γ over Z^γ of curvature $\mathrm{curv}(A_\gamma) = \epsilon(\gamma)\pi_2^*C - \pi_1^*C$ for each $\gamma \in \Gamma$.
2. For each $\gamma \in \Gamma$, an isomorphism

$$\alpha_\gamma : \pi_{13}^*L \otimes \pi_{34}^*A_\gamma \longrightarrow \pi_{12}^*A_\gamma \otimes \pi_{24}^*L^{\epsilon(\gamma)}$$

   of line bundles over $Z^\gamma \times_M Z^\gamma$ such that the diagram (2.10) is commutative.
3. For each pair $(\gamma_1, \gamma_2) \in \Gamma \times \Gamma$, an isomorphism

$$\varphi_{\gamma_1,\gamma_2} : \pi_{12}^*A_{\gamma_1} \otimes \pi_{23}^*A_{\gamma_2}^{\epsilon(\gamma_1)} \longrightarrow \pi_{13}^*A_{\gamma_1\gamma_2}$$

of line bundles over Z^{γ_1,γ_2} such that the diagrams (2.11) and (2.12) are commutative.

If the (Γ,ϵ)-equivariant structure is normalized, we have $A_1 := L$ and the isomorphism $\alpha_1 := \pi_{124}^*\mu^{-1} \circ \pi_{134}^*\mu$. The normalization constraints $\varphi_{\gamma,1} = \rho_{\mathcal{A}_\gamma}$ and $\varphi_{1,\gamma} = \lambda_{\mathcal{A}_\gamma}$ imply $\varphi_{1,1} = \mu$.

Next, we would like to compare two (Γ,ϵ)-equivariant bundle gerbes.

Definition 2.6. Let (Γ,ϵ) be an orientifold group for M and let $\mathcal{G}^a$ and $\mathcal{G}^b$ be bundle gerbes over M equipped with (Γ,ϵ)-equivariant structures $\mathcal{J}^a = (\mathcal{A}^a_\gamma, \varphi^a_{\gamma_1,\gamma_2})$ and $\mathcal{J}^b = (\mathcal{A}^b_\gamma, \varphi^b_{\gamma_1,\gamma_2})$, respectively. An equivariant 1-morphism

$$(\mathcal{B},\eta_\gamma) : (\mathcal{G}^a,\mathcal{J}^a) \longrightarrow (\mathcal{G}^b,\mathcal{J}^b)$$

is a 1-morphism $\mathcal{B} : \mathcal{G}^a \longrightarrow \mathcal{G}^b$ of the underlying bundle gerbes together with a family of 2-isomorphisms

$$\eta_\gamma : \gamma\mathcal{B} \circ \mathcal{A}^a_\gamma \implies \mathcal{A}^b_\gamma \circ \mathcal{B},$$

one for each $\gamma \in \Gamma$, such that the diagram

$$\begin{array}{ccc}
\gamma_1\gamma_2\mathcal{B} \circ \gamma_1\mathcal{A}^a_{\gamma_2} \circ \mathcal{A}^a_{\gamma_1} & \xRightarrow{\mathrm{id}_{\gamma_1\gamma_2\mathcal{B}} \circ \varphi^a_{\gamma_1,\gamma_2}} & \gamma_1\gamma_2\mathcal{B} \circ \mathcal{A}^a_{\gamma_1\gamma_2} \\
\Big\Downarrow {\scriptstyle \gamma_1\eta_{\gamma_2} \circ \mathrm{id}_{\mathcal{A}^a_{\gamma_1}}} & & \\
\gamma_1\mathcal{A}^b_{\gamma_2} \circ \gamma_1\mathcal{B} \circ \mathcal{A}^a_{\gamma_1} & & \Big\Downarrow {\scriptstyle \eta_{\gamma_1\gamma_2}} \\
\Big\Downarrow {\scriptstyle \mathrm{id}_{\gamma_1\mathcal{A}^b_{\gamma_2}} \circ \eta_{\gamma_1}} & & \\
\gamma_1\mathcal{A}^b_{\gamma_2} \circ \mathcal{A}^b_{\gamma_1} \circ \mathcal{B} & \xRightarrow[\varphi^b_{\gamma_1,\gamma_2} \circ \mathrm{id}_{\mathcal{B}}]{} & \mathcal{A}^b_{\gamma_1\gamma_2} \circ \mathcal{B}
\end{array} \tag{2.13}$$

of 2-isomorphisms is commutative.

Equivariant 1-morphisms can be composed in a natural way: if

$$(\mathcal{G}^a,\mathcal{J}^a) \xrightarrow{(\mathcal{B},\eta_\gamma)} (\mathcal{G}^b,\mathcal{J}^b) \xrightarrow{(\mathcal{B}',\eta'_\gamma)} (\mathcal{G}^c,\mathcal{J}^c)$$

are two composable equivariant 1-morphisms, their composition consists of the 1-morphism $\mathcal{B}' \circ \mathcal{B} : \mathcal{G}^a \longrightarrow \mathcal{G}^c$ and the 2-morphisms

$$\gamma(\mathcal{B}' \circ \mathcal{B}) \circ \mathcal{A}^a_\gamma = \gamma\mathcal{B}' \circ \gamma\mathcal{B} \circ \mathcal{A}^a_\gamma \xRightarrow{\mathrm{id}\circ\eta_\gamma} \gamma\mathcal{B}' \circ \mathcal{A}^b_\gamma \circ \mathcal{B} \xRightarrow{\eta'_\gamma \circ \mathrm{id}} \mathcal{A}^c_\gamma \circ (\mathcal{B}' \circ \mathcal{B}) \cdot$$

This composition is associative. We also have an identity equivariant 1-morphism associated to a (Γ, ϵ)-equivariant bundle gerbe $(\mathcal{G}, \mathcal{J})$ given by $(\mathrm{id}_{\mathcal{G}}, \lambda^{-1}_{\mathcal{A}_\gamma} \bullet \rho_{\mathcal{A}_\gamma})$. An equivariant 1-morphism $(\mathcal{B}, \eta_\gamma)$ is called *invertible* or equivariant 1-*iso*morphism if the 1-morphism $\mathcal{B}$ is invertible. In this case, an inverse is given by $(\mathcal{B}^{-1}, \eta_\gamma^{-1})$. Hence, equivariant 1-isomorphisms furnish an equivalence relation on the set of (Γ, ϵ)-equivariant bundle gerbes over M.

Definition 2.7. Two twisted-equivariant bundle gerbes over M are called *equivalent* if there exists an equivariant 1-isomorphism between them.

The set of equivalence classes of twisted-equivariant bundle gerbes over M will be further investigated in Sections 2 and 3. Let us anticipate here the following fact.

Lemma 2.1. *Every twisted-equivariant bundle gerbe is equivalent to one with descended twisted-equivariant structure.*

Proof. We recall Theorem 1 of [26]: for every 1-morphism $\mathcal{A} : \mathcal{G} \longrightarrow \mathcal{H}$, there exists a "descended" 1-morphism $\mathrm{Des}(\mathcal{A}) : \mathcal{G} \longrightarrow \mathcal{H}$ whose surjective submersion $\zeta : Z \longrightarrow Y \times_M Y'$ is the identity, together with a 2-isomorphism $\sigma_{\mathcal{A}} : \mathcal{A} \Longrightarrow \mathrm{Des}(\mathcal{A})$. For every 2-morphism $\varphi : \mathcal{A}^a \Longrightarrow \mathcal{A}^b$, there exists a 2-morphism

$$\mathrm{Des}(\varphi) : \mathrm{Des}(\mathcal{A}^a) \Longrightarrow \mathrm{Des}(\mathcal{A}^b)$$

such that the diagram

$$\begin{array}{ccc} \mathcal{A}^a & \xRightarrow{\sigma_{\mathcal{A}^a}} & \mathrm{Des}(\mathcal{A}^a) \\ {\scriptstyle \varphi}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \mathrm{Des}(\varphi)} \\ \mathcal{A}^b & \xRightarrow[\sigma_{\mathcal{A}^b}]{} & \mathrm{Des}(\mathcal{A}^b) \end{array} \tag{2.14}$$

is commutative. For a given (Γ, ϵ)-equivariant structure $\mathcal{J} = (\mathcal{A}_\gamma, \varphi_{\gamma_1,\gamma_2})$ on a bundle gerbe $\mathcal{G}$, we define $\mathcal{A}'_\gamma := \mathrm{Des}(\mathcal{A}_\gamma)$ and $\varphi'_{\gamma_1,\gamma_2} := \mathrm{Des}(\varphi_{\gamma_1,\gamma_2})$. Due to the commutativity of (2.14), the new $\varphi'_{\gamma_1,\gamma_2}$ still satisfy condition (2.8) for (Γ, ϵ)-equivariant structures. Then, the choices $\mathcal{B} = \mathrm{id}_{\mathcal{G}}$ and $\eta_\gamma = \sigma_{\mathcal{A}_\gamma}$ define an equivariant 1-isomorphism which establishes the claimed equivalence. □

We call an equivariant 1-morphism between bundle gerbes with normalized (Γ, ϵ)-equivariant structures *normalized* if $\eta_1 : \mathcal{B} \circ \mathrm{id}_{\mathcal{G}^a} \Longrightarrow \mathrm{id}_{\mathcal{G}^b} \circ \mathcal{B}$ is given by the natural 2-morphisms of the 2-category, $\eta_1 = \rho_{\mathcal{B}}^{-1} \bullet \lambda_{\mathcal{B}}$. We call an equivariant 1-morphism *descended* if the surjective submersion of $\mathcal{B}$ is the identity.

Let us, again, describe what an equivariant 1-morphism $(\mathcal{B}, \eta_\gamma)$ is in terms of line bundles and isomorphisms thereof. We assume it to be descended for simplicity. The 1-isomorphism $\mathcal{B} : \mathcal{G}^a \longrightarrow \mathcal{G}^b$ consists of a vector bundle B over $Z := Y^a \times_M Y^b$ and of an isomorphism $\beta : \pi_{13}^* L^a \otimes \pi_{34}^* B \longrightarrow \pi_{12}^* B \otimes \pi_{24}^* L^b$ over $Z^{[2]}$ satisfying axiom (1M1). The composition $\gamma\mathcal{B} \circ \mathcal{A}_\gamma^a$ we have to consider is the 1-morphism with the vector bundle $\pi_{12}^* A_\gamma^a \otimes \pi_{23}^* B^{\epsilon(\gamma)}$ over

$$Z_1^\gamma := (Z^a)^\gamma \times_{Y_\gamma^a} Z_\gamma \cong Y^a \times_M Y_\gamma^a \times_M Y_\gamma^b,$$

and with the isomorphism

$$\begin{aligned}(\mathrm{id} \otimes \pi_{2356}^* \beta^{\epsilon(\gamma)}) \circ (\pi_{1245}^* \alpha_\gamma^a \otimes \mathrm{id}) : \pi_{14}^* L^a \otimes \pi_{45}^* A_\gamma^a \otimes \pi_{56}^* B^{\epsilon(\gamma)} \\ \longrightarrow \pi_{12}^* A_\gamma^a \otimes \pi_{23}^* B^{\epsilon(\gamma)} \otimes \pi_{36}^* (L^b)^{\epsilon(\gamma)}\end{aligned} \tag{2.15}$$

of vector bundles over $(Z_1^\gamma)^{[2]}$. The other composition, $\mathcal{A}_\gamma^b \circ \mathcal{B}$, is the 1-morphism with the vector bundle $\pi_{12}^* B \otimes \pi_{23}^* A_\gamma^b$ over

$$Z_2^\gamma := Z \times_{Y^b} (Z^b)^\gamma \cong Y^a \times_M Y^b \times_M Y_\gamma^b,$$

and with the isomorphism

$$\begin{aligned}(\mathrm{id} \otimes \pi_{2356}^* \alpha_\gamma^b) \circ (\pi_{1245}^* \beta \otimes \mathrm{id}) : \pi_{14}^* L^a \otimes \pi_{45}^* B \otimes \pi_{56}^* A_\gamma^b \\ \longrightarrow \pi_{12}^* B \otimes \pi_{23}^* A_\gamma^b \otimes \pi_{36}^* (L^b)^{\epsilon(\gamma)}\end{aligned} \tag{2.16}$$

of vector bundles over $(Z_2^\gamma)^{[2]}$. The 2-isomorphisms η_γ correspond now to isomorphisms

$$\eta_\gamma : \pi_{12}^* A_\gamma^a \otimes \pi_{24}^* B^{\epsilon(\gamma)} \longrightarrow \pi_{13}^* B \otimes \pi_{34}^* A_\gamma^b \tag{2.17}$$

of vector bundles over $Z_1^\gamma \times_P Z_2^\gamma \cong Y^a \times_M Y_\gamma^a \times_M Y^b \times_M Y_\gamma^b$, where $P := Y^a \times_M Y_\gamma^b$, and these isomorphisms satisfy the compatibility condition

$$\begin{array}{ccc} \pi_{15}^* L^a \otimes \pi_{56}^* A_\gamma^a \otimes \pi_{68}^* B^{\epsilon(\gamma)} & \longrightarrow & \pi_{12}^* A_\gamma^a \otimes \pi_{24}^* B^{\epsilon(\gamma)} \otimes \pi_{48}^* (L^b)^{\epsilon(\gamma)} \\ {\scriptstyle \mathrm{id} \otimes \pi_{5678}^* \eta_\gamma} \downarrow & & \downarrow {\scriptstyle \pi_{1234}^* \eta_\gamma \otimes \mathrm{id}} \\ \pi_{15}^* L^a \otimes \pi_{57}^* B \otimes \pi_{78}^* A_\gamma^b & \longrightarrow & \pi_{13}^* B \otimes \pi_{34}^* A_\gamma^b \otimes \pi_{48}^* (L^b)^{\epsilon(\gamma)}, \end{array} \tag{2.18}$$

where the horizontal arrows are given by (2.15) and (2.16), respectively. Finally, the commutativity of diagram (2.13) implies the commutativity of the diagram

$$\begin{array}{ccc}
\pi_{12}^*A^a_{\gamma_1}\otimes\pi_{23}^*(A^a_{\gamma_2})^{\epsilon(\gamma_1)}\otimes\pi_{36}^*B^{\epsilon(\gamma_1\gamma_2)} & \xrightarrow{\pi_{123}^*\varphi^a_{\gamma_1,\gamma_2}\otimes\mathrm{id}} & \pi_{13}^*A^a_{\gamma_1\gamma_2}\otimes\pi_{36}^*B^{\epsilon(\gamma_1\gamma_2)} \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes\pi_{2356}^*\eta_{\gamma_2}^{\epsilon(\gamma_1)}} & & \\
\pi_{12}^*A^a_{\gamma_1}\otimes\pi_{25}^*B^{\epsilon(\gamma_1)}\otimes\pi_{56}^*(A^b_{\gamma_2})^{\epsilon(\gamma_1)} & & \Big\downarrow{\scriptstyle \pi_{1346}^*\eta_{\gamma_1\gamma_2}} \\
\Big\downarrow{\scriptstyle \pi_{1245}^*\eta_{\gamma_1}\otimes\mathrm{id}} & & \\
\pi_{14}^*B\otimes\pi_{45}^*A^b_{\gamma_1}\otimes\pi_{56}^*(A^b_{\gamma_2})^{\epsilon(\gamma_1)} & \xrightarrow[\mathrm{id}\otimes\pi_{456}^*\varphi^b_{\gamma_1,\gamma_2}]{} & \pi_{14}^*B\otimes\pi_{46}^*A^b_{\gamma_1\gamma_2}.
\end{array} \tag{2.19}$$

For completeness, and as a preparation for Section 5, we would also like to introduce equivariant 2-morphisms. Suppose that we have (Γ,ϵ)-equivariant bundle gerbes $(\mathcal{G}^a,\mathcal{J}^a)$ and $(\mathcal{G}^b,\mathcal{J}^b)$, and that we have two equivariant 1-morphisms $(\mathcal{B},\eta_\gamma)$ and $(\mathcal{B}',\eta'_\gamma)$ between these. An *equivariant 2-morphism*

$$\phi:(\mathcal{B},\eta_\gamma)\implies(\mathcal{B}',\eta'_\gamma)$$

is a 2-morphism $\phi:\mathcal{B}\implies\mathcal{B}'$, which is compatible with the 2-morphisms η_γ and η'_γ in the sense that the diagram

$$\begin{array}{ccc}
\gamma\mathcal{B}\circ\mathcal{A}^a_\gamma & \xRightarrow{\eta_\gamma} & \mathcal{A}^b_\gamma\circ\mathcal{B} \\
\Big\Downarrow{\scriptstyle \gamma\phi\circ\mathrm{id}_{\mathcal{A}^a_\gamma}} & & \Big\Downarrow{\scriptstyle \mathrm{id}_{\mathcal{A}^b_\gamma}\circ\phi} \\
\gamma\mathcal{B}'\circ\mathcal{A}^a_\gamma & \xRightarrow[\eta'_\gamma]{} & \mathcal{A}^b_\gamma\circ\mathcal{B}'
\end{array} \tag{2.20}$$

of 2-morphisms is commutative.

In terms of morphisms between vector bundles, ϕ is just a morphism $\phi:B\longrightarrow B'$ of vector bundles over $Z=Y^a\times_M Y^b$ which is compatible with the isomorphisms β and β' in the sense that the diagram

$$\begin{array}{ccc}
\pi_{13}^*L^a\otimes\pi_{34}^*B & \xrightarrow{\beta} & \pi_{12}^*B\otimes\pi_{24}^*L^b \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes\pi_{34}^*\phi} & & \Big\downarrow{\scriptstyle \pi_{12}^*\phi\otimes\mathrm{id}} \\
\pi_{13}^*L^a\otimes\pi_{34}^*B' & \xrightarrow[\beta']{} & \pi_{12}^*B'\otimes\pi_{24}^*L^b
\end{array} \tag{2.21}$$

is commutative, and diagram (2.20) imposes the commutativity of

$$\begin{array}{ccc} \pi_{12}^*A_\gamma^a \otimes \pi_{24}^*B^{\epsilon(\gamma)} & \xrightarrow{\eta_\gamma} & \pi_{13}^*B \otimes \pi_{34}^*A_\gamma^b \\ {\scriptstyle \mathrm{id}\otimes\pi_{24}^*\phi^{\epsilon(\gamma)}}\downarrow & & \downarrow{\scriptstyle \pi_{13}^*\phi\otimes\mathrm{id}} \\ \pi_{12}^*A_\gamma^a \otimes \pi_{24}^*B'^{\epsilon(\gamma)} & \xrightarrow[\eta'_\gamma]{} & \pi_{13}^*B' \otimes \pi_{34}^*A_\gamma^b. \end{array} \tag{2.22}$$

Naturally, twisted-equivariant bundle gerbes, equivariant 1-morphisms and equivariant 2-morphisms form, again, a 2-category, but we will not stress this point.

2.3 Jandl gerbes

In this section, we consider the particular orientifold group $(\mathbb{Z}_2, \mathrm{id})$. The non-trivial group element of $\mathbb{Z}_2$ is denoted by k, and its action $k : M \longrightarrow M$ is an involution. According to Definition 2.5, a (normalized) $\mathbb{Z}_2^{\mathrm{id}}$-equivariant structure is a single 1-isomorphism

$$\mathcal{A}_k : \mathcal{G} \longrightarrow k^*\mathcal{G}^*$$

and single 2-isomorphism

$$\varphi_{k,k} : k^*\mathcal{A}_k^\dagger \circ \mathcal{A}_k \Longrightarrow \mathrm{id}_\mathcal{G}$$

such that

$$\lambda_{\mathcal{A}_k} \bullet (\mathrm{id} \circ \varphi_{k,k}) = \rho_{\mathcal{A}_k} \bullet (k^*\varphi_{k,k}^\dagger \circ \mathrm{id}). \tag{2.23}$$

It is easy to see that this is exactly the same as a Jandl structure [23]: the 1-isomorphism $\mathcal{A} := k^*\mathcal{A}_k : k^*\mathcal{G} \longrightarrow \mathcal{G}^*$ and the 2-isomorphism φ defined by

$$k^*\mathcal{A} \xRightarrow{\rho_{k^*\mathcal{A}}^{-1}} \mathrm{id}_\mathcal{G} \circ k^*\mathcal{A} \xRightarrow{i_r \circ \mathrm{id}_{k^*\mathcal{A}}} \mathcal{A}^* \circ \mathcal{A}^\dagger \circ k^*\mathcal{A} \xRightarrow{\mathrm{id}_{\mathcal{A}^*} \circ \varphi_{k,k}} \mathcal{A}^* \circ \mathrm{id}_\mathcal{G} \xRightarrow{\lambda_{\mathcal{A}^*}} \mathcal{A}^*$$

yield a Jandl structure as described in [26]. Hence, we will call a $(\mathbb{Z}_2, \mathrm{id})$-equivariant structure just a *Jandl structure* and a $(\mathbb{Z}_2, \mathrm{id})$-equivariant bundle gerbe *Jandl gerbe.*

Now, we elaborate the details of a Jandl structure, which we may assume to be descended according to Lemma 2.1. The 1-isomorphism $\mathcal{A}_k$ consists of a line bundle A_k over $Z^k := Y \times_M Y_k$ of curvature

$$\mathrm{curv}(A_k) = -(\pi_2^* C + \pi_1^* C) \tag{2.24}$$

in the notation of Section 2.2, together with an isomorphism

$$\alpha_k : \pi_{13}^* L \otimes \pi_{34}^* A_k \longrightarrow \pi_{12}^* A_k \otimes \pi_{24}^* L^* \tag{2.25}$$

of line bundles over $Z^k \times_M Z^k$ satisfying axiom (1M2). The composition $k^* \mathcal{A}_k^\dagger \circ \mathcal{A}_k$ is the 1-isomorphism with the surjective submersion id on

$$Z^{k,k} = Z^k \times_{Y_k} Z_k^k \cong Y \times_M Y_k \times_M Y,$$

the line bundle $\pi_{12}^* A_k \otimes \pi_{23}^* A_k^*$ over $Z^{k,k}$ and the isomorphism

$$\begin{aligned}(\mathrm{id} \otimes \pi_{2356}^* \alpha_k^{*-1}) \circ (\pi_{1245}^* \alpha_k \otimes \mathrm{id}) &: \pi_{14}^* L \otimes \pi_{45}^* A_k \otimes \pi_{56}^* A_k^* \\ &\longrightarrow \pi_{12}^* A_k \otimes \pi_{23}^* A_k^* \otimes \pi_{36}^* L\end{aligned}$$

of line bundles over $Z^{k,k} \times_M Z^{k,k}$. The 2-isomorphism $\varphi_{k,k}$ corresponds to a bundle isomorphism

$$\varphi_{k,k} : \pi_{12}^* A_k \otimes \pi_{23}^* A_k^* \longrightarrow \pi_{13}^* L, \tag{2.26}$$

compatible with α_k by virtue of the commutativity of the diagram

$$\begin{array}{ccc} \pi_{14}^* L \otimes \pi_{45}^* A_k \otimes \pi_{56}^* A_k^* & \longrightarrow & \pi_{12}^* A_k \otimes \pi_{23}^* A_k^* \otimes \pi_{36}^* L \\ \Big\downarrow {\scriptstyle \mathrm{id} \otimes \pi_{456}^* \varphi_{k,k}} & & \Big\downarrow {\scriptstyle \pi_{123}^* \varphi_{k,k} \otimes \mathrm{id}} \\ \pi_{14}^* L \otimes \pi_{46}^* L & \longrightarrow & \pi_{13}^* L \otimes \pi_{36}^* L. \end{array}$$

Finally, (2.23) gives the commutativity of

$$\begin{array}{ccc} \pi_{12}^*A_k \otimes \pi_{23}^*A_k^* \otimes \pi_{34}^*A_k & \xrightarrow{\mathrm{id}\otimes\pi_{234}^*\varphi_{k,k}^{*-1}} & \pi_{12}^*A_k \otimes \pi_{24}^*L^* \\ {\scriptstyle \pi_{123}^*\varphi_{k,k}\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \pi_{124}^*\rho} \\ \pi_{13}^*L \otimes \pi_{34}^*A_k & \xrightarrow[\pi_{134}^*\lambda]{} & \pi_{14}^*A_k. \end{array} \tag{2.27}$$

It is worthwhile to discuss a *trivialized Jandl gerbe*. This is a Jandl gerbe $(\mathcal{G}, \mathcal{J})$ equipped with a trivialization, i.e., a 1-isomorphism $\mathcal{T} : \mathcal{G} \longrightarrow \mathcal{I}_\rho$. Here, the 2-categorial formalism can be used fruitfully. In [26], a functor

$$\mathcal{B}un : \mathfrak{Hom}(\mathcal{I}_{\rho_1}, \mathcal{I}_{\rho_2}) \longrightarrow \mathfrak{Bun}(M) \tag{2.28}$$

is defined: for every 1-morphism $\mathcal{A} : \mathcal{I}_{\rho_1} \longrightarrow \mathcal{I}_{\rho_2}$ between trivial bundle gerbes over M, it provides a vector bundle $\mathcal{B}un(\mathcal{A})$ over M; and for every 2-morphism $\beta : \mathcal{A} \Longrightarrow \mathcal{A}'$, it provides a morphism

$$\mathcal{B}un(\beta) : \mathcal{B}un(\mathcal{A}) \longrightarrow \mathcal{B}un(\mathcal{A}')$$

of vector bundles over M. If the 1-morphism $\mathcal{A} : \mathcal{I}_{\rho_1} \longrightarrow \mathcal{I}_{\rho_2}$ has a vector bundle A over $\zeta : Z \longrightarrow M \times_M M \cong M$, the vector bundle $\mathcal{B}un(\mathcal{A})$ is uniquely characterized by the property that $\zeta^*\mathcal{B}un(\mathcal{A}) \cong A$. Accordingly, the rank n of $\mathcal{B}un(\mathcal{A})$ is equal to the rank of A, and its curvature satisfies, by axiom (1M1),

$$\frac{1}{n}\mathrm{tr}(\mathrm{curv}(\mathcal{B}un(\mathcal{A}))) = \rho_2 - \rho_1.$$

The functor $\mathcal{B}un$ has the following compatibility properties:

- $\mathcal{B}un(\mathcal{A}_2 \circ \mathcal{A}_1) = \mathcal{B}un(\mathcal{A}_1) \otimes \mathcal{B}un(\mathcal{A}_2)$;
- $\mathcal{B}un(\mathrm{id}_{\mathcal{I}_\rho}) = \mathbb{1}$;
- $\mathcal{B}un(f^*\mathcal{A}) = f^*\mathcal{B}un(\mathcal{A})$;
- $\mathcal{B}un(\mathcal{A}^\dagger) = \mathcal{B}un(\mathcal{A})^*$;

in which $\mathbb{1}$ denotes the trivial line bundle with the trivial flat connection.

Let us return to the Jandl gerbe $(\mathcal{G}, \mathcal{J})$ and the trivialization $\mathcal{T} : \mathcal{G} \longrightarrow \mathcal{I}_\rho$. First, we form a 1-isomorphism $\mathcal{R} : \mathcal{I}_\rho \longrightarrow \mathcal{I}_{-k^*\rho}$ by composing

$$\mathcal{I}_\rho \xrightarrow{\mathcal{T}^{-1}} \mathcal{G} \xrightarrow{\mathcal{A}_k} k^*\mathcal{G}^* \xrightarrow{k^*\mathcal{T}^\dagger} \mathcal{I}_{-k^*\rho}, \tag{2.29}$$

and a 2-isomorphism $\psi : k^*\mathcal{R}^\dagger \circ \mathcal{R} \implies \mathrm{id}_{\mathcal{I}_\rho}$ by composing

$$\begin{array}{c} k^*\mathcal{R}^\dagger \circ \mathcal{R} = \mathcal{T} \circ k^*\mathcal{A}_k^\dagger \circ k^*\mathcal{T}^{\dagger -1} \circ k^*\mathcal{T}^\dagger \circ \mathcal{A}_k \circ \mathcal{T}^{-1} \\ \Big\Downarrow{\scriptstyle \mathrm{id}\circ i_l \circ \mathrm{id}} \\ \mathcal{T} \circ k^*\mathcal{A}_k^\dagger \circ \mathrm{id}_{k^*\mathcal{G}^*} \circ \mathcal{A}_k \circ \mathcal{T}^{-1} \\ \Big\Downarrow{\scriptstyle \mathrm{id}\circ \rho_{\mathcal{A}_k} \circ \mathrm{id}} \\ \mathcal{T} \circ k^*\mathcal{A}_k^\dagger \circ \mathcal{A}_k \circ \mathcal{T}^{-1} \\ \Big\Downarrow{\scriptstyle \mathrm{id}\circ \varphi_{k,k} \circ \mathrm{id}} \\ \mathcal{T} \circ \mathrm{id}_{\mathcal{G}} \circ \mathcal{T}^{-1} \\ \Big\Downarrow{\scriptstyle \lambda_{\mathcal{T}} \circ \mathrm{id}} \\ \mathcal{T} \circ \mathcal{T}^{-1} \xRightarrow[i_r^{-1}]{} \mathrm{id}_{\mathcal{I}_\rho}. \end{array} \tag{2.30}$$

In this definition, we have used the 2-isomorphisms i_l and i_r from (2.3) associated to the inverse 1-isomorphism $k^*\mathcal{T}^{\dagger -1}$, and the 2-isomorphisms ρ and λ from (2.4). Equation (2.5) assures that it is not important whether one uses λ or ρ.

Now we apply the functor $\mathcal{B}un$ to the 1-isomorphism $\mathcal{R}$ and the 2-isomorphism ψ. The first yields a line bundle $R := \mathcal{B}un(\mathcal{R})$ over M of curvature $-(k^*\rho + \rho)$, and the second (using the above rules) an isomorphism

$$\phi := \mathcal{B}un(\psi) : R \otimes k^*R^* \longrightarrow 1\!\!1$$

of line bundles over M. Finally, condition (2.23) implies that

$$\phi \otimes \mathrm{id}_R = \mathrm{id}_R \otimes k^*\phi^\dagger$$

as isomorphisms from $R \otimes k^*R^* \otimes R$ to R. In other words, the pair (R, ϕ) is a k-equivariant line bundle over M. Summarizing, every trivialized Jandl gerbe gives rise to an equivariant line bundle.

Remark 2.2. One could also use the functor $\mathcal{B}un$ to express a trivialized twisted-equivariant structure in terms of bundles over M in the case of a general orientifold group (Γ, ϵ). The result is not (as probably expected) a

Γ-equivariant line bundle over M but a *family* R_γ of line bundles over M of curvature $\gamma\rho - \rho$, together with isomorphisms

$$\phi_{\gamma_1,\gamma_2} : R_{\gamma_1} \otimes \gamma_1 R_{\gamma_2} \longrightarrow R_{\gamma_1\gamma_2}$$

of line bundles, which satisfy a coherence condition on triples $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$.

It will be useful to investigate the relation between the equivariant line bundles associated to two equivariantly isomorphic Jandl gerbes. Suppose that $(\mathcal{G}^a, \mathcal{J}^a)$ and $(\mathcal{G}^b, \mathcal{J}^b)$ are Jandl gerbes over M with respect to the same involution $k : M \longrightarrow M$, and suppose further that $(\mathcal{B}, \eta_k) : (\mathcal{G}^a, \mathcal{J}^a) \longrightarrow (\mathcal{G}^b, \mathcal{J}^b)$ is an equivariant 1-isomorphism. Let $\mathcal{T}^b : \mathcal{G}^b \longrightarrow \mathcal{I}_\rho$ be a trivialization of $\mathcal{G}^a$ and let $\mathcal{T}^a := \mathcal{T}^b \circ \mathcal{B}$ be the induced trivialization of $\mathcal{G}^a$. We obtain the k-equivariant line bundles (R^a, ϕ^a) and (R^b, ϕ^b) over M in the manner described above.

Lemma 2.2. *The 2-isomorphism η_k induces an isomorphism $\kappa : R^a \longrightarrow R^b$ of line bundles over M that respects the equivariant structures in the sense that the diagram*

$$\begin{array}{ccc} R^a \otimes k^*(R^a)^* & \xrightarrow{\phi^a} & 1\!\!1 \\ {\scriptstyle \kappa\otimes k^*\kappa^{*-1}}\big\downarrow & & \big\| \\ R^b \otimes k^*(R^b)^* & \xrightarrow[\phi^b]{} & 1\!\!1 \end{array}$$

of isomorphisms of line bundles over M is commutative.

Proof. The 2-isomorphism $\eta_k : k^*\mathcal{B}^\dagger \circ \mathcal{A}_k^a \Longrightarrow \mathcal{A}_k^b \circ \mathcal{B}$ induces an isomorphism

$$k^*\mathcal{B}^\dagger \circ \mathcal{A}_k^a \circ \mathcal{B}^{-1} \xRightarrow{\eta_k \circ \mathrm{id}} \mathcal{A}_k^b \circ \mathcal{B} \circ \mathcal{B}^{-1} \xRightarrow{\mathrm{id}\circ i_r^{-1}} \mathcal{A}_k^b \circ \mathrm{id}_{\mathcal{G}^b} \xRightarrow{\lambda_{\mathcal{A}_k^b}} \mathcal{A}_k^b.$$

The composition of the above 1-morphisms with $(\mathcal{T}^b)^{-1}$ from the right and with $k^*(\mathcal{T}^b)^\dagger$ from the left yields a 2-isomorphism $\eta'_k : \mathcal{R}^a \Longrightarrow \mathcal{R}^b$ according to (2.29). Then, we have $\kappa := \mathcal{B}un(\eta'_k)$. It is straightforward to check that the 2-isomorphism η'_k and the two 2-isomorphisms ψ^a and ψ^b from (2.30) fit into a commutative diagram, such that applying the functor $\mathcal{B}un$ yields the assertion we had to show. □

3 Descent theory for Jandl gerbes

In this section, we consider an orientifold group (Γ, ϵ) whose normal subgroup $\Gamma_0 := \ker(\epsilon)$ of Γ acts on M without fixed points, so that the quotient $M' := M/\Gamma_0$ is equipped with a canonical smooth-manifold structure such that the projection $p : M \longrightarrow M'$ is a smooth map. We remark that there is a remaining smooth group action of $\Gamma' := \Gamma/\Gamma_0$ on M'. We also still have a group homomorphism $\epsilon' : \Gamma' \longrightarrow \mathbb{Z}_2$, so that (Γ', ϵ') is an orientifold group for M'.

Theorem 3.1. *Let (Γ, ϵ) be an orientifold group for M with Γ_0 acting without fixed points, and let (Γ', ϵ') be the quotient orientifold group for the quotient $M' := M/\Gamma_0$. Then, there is a canonical bijection*

$$\left\{ \begin{array}{c} \textit{Equivalence classes} \\ \textit{of } (\Gamma, \epsilon)\textit{-equivariant} \\ \textit{bundle gerbes over } M \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \textit{Equivalence classes} \\ \textit{of } (\Gamma', \epsilon')\textit{-equivariant} \\ \textit{bundle gerbes over } M' \end{array} \right\}.$$

Note that Theorem 3.1 unites two interesting cases:

1. The original group homomorphism $\epsilon : \Gamma \longrightarrow \mathbb{Z}_2$ is constant $\epsilon(\gamma) = 1$. In this case, $\Gamma_0 = \Gamma$ and (Γ', ϵ') is the trivial (orientifold) group. Here, Theorem 3.1 reduces to the well-known bijection

$$\left\{ \begin{array}{c} \text{Equivalence classes} \\ \text{of } \Gamma\text{-equivariant} \\ \text{bundle gerbes over } M \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{bundle gerbes over } M' \end{array} \right\}.$$

   This bijection was used in [12] to construct bundle gerbes on non-simply connected Lie groups G/Γ_0 from bundle gerbes over the universal covering group G.
2. The original group homomorphism $\epsilon : \Gamma \longrightarrow \mathbb{Z}_2$ is non-trivial. In this case, $\Gamma' = \mathbb{Z}_2$ and $\epsilon' = \mathrm{id}$. Here Theorem 3.1 reduces to a bijection

$$\left\{ \begin{array}{c} \text{Equivalence classes} \\ \text{of } (\Gamma, \epsilon)\text{-equivariant} \\ \text{bundle gerbes over } M \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{Equivalence classes of} \\ \text{Jandl gerbes over } M' \end{array} \right\}.$$

   We will use this bijection in [13] to construct Jandl gerbes over non-simply connected Lie groups.

In the sequel of this section, we shall prove Theorem 3.1 assuming, for simplicity, that all equivariant structures are normalized. First, we start with a given (Γ, ϵ)-equivariant bundle gerbe over M and construct an associated quotient bundle gerbe $\mathcal{G}'$ over M' along the lines of [11].

By Lemma 2.1, we may assume that the (Γ, ϵ)-equivariant structure is descended. If $\pi : Y \longrightarrow M$ is the surjective submersion of $\mathcal{G}$, the fibre products of the surjective submersion $\omega : Y \longrightarrow M'$ with itself, defined by $\omega := p \circ \pi$, are disjoint unions

$$Y \times_{M'} Y \cong \bigsqcup_{\gamma \in \Gamma_0} Z^{\gamma} \quad \text{and} \quad Y \times_{M'} Y \times_{M'} Y \cong \bigsqcup_{(\gamma_1, \gamma_2) \in \Gamma_0^2} Z^{\gamma_1, \gamma_2}.$$

We recall that the (Γ, ϵ)-equivariant structure on $\mathcal{G}$ in particular has a line bundle A_γ over Z^γ of curvature

$$\mathrm{curv}(A_\gamma) = \epsilon(\gamma)\pi_2^* C - \pi_1^* C \tag{3.1}$$

for each $\gamma \in \Gamma$, and for each pair $(\gamma_1, \gamma_2) \in \Gamma^2$ an isomorphism

$$\varphi_{\gamma_1,\gamma_2} : \pi_{12}^* A_{\gamma_1} \otimes \pi_{23}^* A_{\gamma_2}^{\epsilon(\gamma_1)} \longrightarrow \pi_{13}^* A_{\gamma_1 \gamma_2}$$

of line bundles over Z^{γ_1,γ_2}, such that diagram (2.12) is commutative.

Definition 3.1. The quotient bundle gerbe $\mathcal{G}'$ over M' is defined as follows:

(i) its surjective submersion $\omega : Y \longrightarrow M'$ is the composition of the surjective submersion $\pi : Y \longrightarrow M$ of $\mathcal{G}$ with the quotient map $p : M \longrightarrow M'$;
(ii) its 2-form is the 2-form $C \in \Omega^2(Y)$ of $\mathcal{G}$;
(iii) its line bundle A over $Y \times_{M'} Y$ is given by the line bundle $A|_{Z^\gamma} := A_\gamma$ over each component Z^γ of $Y \times_{M'} Y$;
(iv) its isomorphism is given by the isomorphism $\varphi_{\gamma_1,\gamma_2}$ over each component Z^{γ_1,γ_2} of $Y \times_{M'} Y \times_{M'} Y$.

The axioms (G1) and (G2) for the quotient bundle gerbe follow from (3.1) and (2.12), respectively.

In the case of non-trivial ϵ, we enhance the quotient bundle gerbe $\mathcal{G}'$ to a Jandl gerbe. Let us, for simplicity, denote by $\Gamma_- \subset \Gamma$ the subset of elements $\gamma \in \Gamma$ with $\epsilon(\gamma) = -1$. To define a Jandl structure $\mathcal{J}'$ on the quotient bundle gerbe $\mathcal{G}'$, we use the line bundles A_γ over Z^γ for $\gamma \in \Gamma_-$ (which have not been used in Definition 3.1), and the isomorphisms $\varphi_{\gamma_1,\gamma_2}$ for elements $\gamma_1, \gamma_2 \in \Gamma$

with either $\gamma_1 \in \Gamma_-$ or $\gamma_2 \in \Gamma_-$ (which have not been used yet either). The 1-isomorphism

$$\mathcal{A}'_k : \mathcal{G}' \longrightarrow k^*\mathcal{G}'^* \tag{3.2}$$

is defined as follows: the fibre product $P := Y \times_{M'} Y_k$ of the surjective submersions of the two bundle gerbes can be written as

$$P \cong \bigsqcup_{\gamma \in \Gamma_-} Z^{\gamma},$$

and the line bundle A'_k over P is defined as $A'_k|_{Z^\gamma} := A_\gamma$. It has the correct curvature $\mathrm{curv}(A) = -\pi_2^*C - \pi_1^*C$ in the sense of axiom (1M1). The two-fold fibre product has the components

$$P \times_{M'} P \cong \bigsqcup_{\gamma_1,\gamma_2,\gamma_3 \in \Gamma_-} Z^{\gamma_1,\gamma_2,\gamma_3}.$$

Now, we have to define an isomorphism α of line bundles over $P \times_{M'} P$, which is an isomorphism

$$\alpha|_{Z^{\gamma_1,\gamma_2,\gamma_3}} : \pi_{13}^* A_{\gamma_1\gamma_2} \otimes \pi_{34}^* A_{\gamma_3} \longrightarrow \pi_{12}^* A_{\gamma_1} \otimes \pi_{24}^* A^*_{\gamma_2\gamma_3},$$

on the component $Z^{\gamma_1,\gamma_2,\gamma_3}$, where we have a dual line bundle because the target of the isomorphism $\mathcal{A}$ is the dual bundle gerbe. We define this isomorphism as the composition of

$$\pi_{134}^*\varphi_{\gamma_1\gamma_2,\gamma_3} : \pi_{13}^* A_{\gamma_1\gamma_2} \otimes \pi_{34}^* A_{\gamma_3} \longrightarrow \pi_{14}^* A_{\gamma_1\gamma_2\gamma_3},$$

with no dual line bundle on the left since $\epsilon(\gamma_1\gamma_2) = 1$, with the inverse of

$$\pi_{124}^*\varphi_{\gamma_1,\gamma_2\gamma_3} : \pi_{12}^* A_{\gamma_1} \otimes \pi_{24}^* A^*_{\gamma_1\gamma_3} \longrightarrow \pi_{14}^* A_{\gamma_1\gamma_2\gamma_3}.$$

The isomorphism α defined in this manner satisfies axiom (1M2) for 1-morphisms due to the commutativity condition (2.12) for the isomorphisms $\varphi_{\gamma_1,\gamma_2}$. This completes the definition of the 1-isomorphism $\mathcal{A}'_k$.

We are left with the definition of the 2-isomorphism $\varphi'_{k,k} : k^*\mathcal{A}'^{\dagger}_k \circ \mathcal{A}'_k \Longrightarrow \mathrm{id}_{\mathcal{G}'}$ for which we use the remaining structure, namely the 1-isomorphisms $\varphi_{\gamma_1,\gamma_2}$ with $\gamma_1, \gamma_2 \in \Gamma_-$. The 1-morphism $k^*\mathcal{A}'^{\dagger}_k \circ \mathcal{A}'_k$ has a surjective submersion $\omega : W \to Y \times_{M'} Y$ for $W = Y \times_{M'} Y_k \times_{M'} Y$. Upon the identification

$$W \cong \bigsqcup_{\gamma_1,\gamma_2 \in \Gamma_-} Z^{\gamma_1,\gamma_2},$$

ω is induced by the natural maps $Z^{\gamma_1,\gamma_2} \to Z^{\gamma_1\gamma_2}$. Over the component Z^{γ_1,γ_2}, the line bundle of $k^*\mathcal{A}_k'^{\dagger} \circ \mathcal{A}_k'$ is equal to $\pi_{12}^*A_{\gamma_1} \otimes \pi_{23}^*A_{\gamma_2}^*$ and we define the bundle isomorphism $\varphi'_{k,k}|_{Z^{\gamma_1,\gamma_2}}$ as

$$\varphi_{\gamma_1,\gamma_2} : \pi_{12}^*A_{\gamma_1} \otimes \pi_{23}^*A_{\gamma_2}^* \longrightarrow \pi_{13}^*A_{\gamma_1\gamma_2}.$$

Indeed, the 1-isomorphism $\mathrm{id}_{\mathcal{G}}$ has the line bundle of the bundle gerbe $\mathcal{G}'$ which is $A_{\gamma_1\gamma_2}$ over $Z^{\gamma_1\gamma_2}$. The axiom for $\varphi'_{k,k}$ can be deduced from the commutativity condition for the 2-isomorphisms $\varphi_{\gamma_1,\gamma_2}$.

Finally, we have to assure that the 2-isomorphism $\varphi'_{k,k}$ satisfies (2.23) for Jandl structures. To see this, we have to express the natural 2-isomorphisms $\rho_{\mathcal{A}}$ and $\lambda_{\mathcal{A}}$ by the given 2-isomorphisms $\varphi_{\gamma_1,\gamma_2}$. According to their definition, we find $\rho_{\mathcal{A}}|_{Z^{\gamma_1,\gamma_2}} = \varphi_{\gamma_1,\gamma_2}$ and $\lambda_{\mathcal{A}}|_{Z^{\gamma_1,\gamma_2}} = \varphi_{\gamma_2,\gamma_1}$ for $\gamma_1 \in \Gamma_-$ and $\gamma_2 \in \Gamma_0$. Then, equation (2.23) reduces to the commutativity condition for the 2-isomorphisms $\varphi_{\gamma_1,\gamma_2}$. This completes the definition of the Jandl structure $\mathcal{J}'$ on the quotient bundle gerbe $\mathcal{G}'$.

The second step in the proof of Theorem 3.1 is to demonstrate that the procedure described above is well-defined on equivalence classes. For this purpose, we show that an equivariant 1-morphism

$$(\mathcal{B}, \eta_\gamma) : (\mathcal{G}^a, \mathcal{J}^a) \longrightarrow (\mathcal{G}^b, \mathcal{J}^b)$$

between (Γ, ϵ)-equivariant bundle gerbes over M induces a 1-morphism $\mathcal{B}'$ between the quotient bundle gerbes $\mathcal{G}^{a\prime}$ and $\mathcal{G}^{b\prime}$. We may assume again that the 1-morphism $\mathcal{B}$ is descended. Then, it consists of a line bundle B over $Z := Y^a \times_M Y^b$, and of an isomorphism β of line bundles over $Z \times_M Z$. The additional 2-isomorphisms correspond to bundle isomorphisms

$$\eta_\gamma : \pi_{12}^*A_\gamma^a \otimes \pi_{24}^*B^{\epsilon(\gamma)} \longrightarrow \pi_{13}^*B \otimes \pi_{34}^*A_\gamma^b.$$

of line bundles over $Y^a \times_M Y_\gamma^a \times_M Y^b \times_M Y_\gamma^b$; see (2.17).

The quotient 1-morphism $\mathcal{B}'$ is defined as follows. Its surjective submersion is the disjoint union $\tilde{Z}$ of $\tilde{Z}^\gamma := Z \times_{Y^b} (Z^b)^\gamma \cong Y^a \times_M Y^b \times_M Y_\gamma^b$ over all $\gamma \in \Gamma_0$, together with the projection

$$\pi_{13} : \tilde{Z}^\gamma \longrightarrow Y^a \times_M Y_\gamma^b$$

whose codomain is the γ-component of the fibre product of the surjective submersions of the two quotient bundle gerbes. Its line bundle B' is defined as $B'|_{\tilde{Z}^\gamma} \equiv B'_\gamma := \pi_{12}^* B \otimes \pi_{23}^* A_\gamma^b$, which has the correct curvature:

$$\begin{aligned}\mathrm{curv}(B'_\gamma) &= \pi_{12}^*\mathrm{curv}(B) + \pi_{23}^*\mathrm{curv}(A_\gamma^b)\\ &= \pi_2^* C^b - \pi_1^* C^a + \pi_3^* C^b - \pi_2^* C^b = \pi_3^* C^b - \pi_1^* C^a.\end{aligned}$$

In order to define the isomorphism of $\mathcal{B}'$, we have to consider the fibre product

$$\tilde{Z} \times_{M'} \tilde{Z} \cong \bigsqcup_{\gamma,\gamma',\gamma''\in\Gamma_0} Y^a \times_M Y^b \times_M Y_\gamma^b \times_M Y_{\gamma\gamma'}^a \times_M Y_{\gamma\gamma'}^b \times_M Y_{\gamma\gamma'\gamma''}^b$$

and set

$$\begin{array}{ccc}\pi_{14}^* A_{\gamma\gamma'}^a \otimes \pi_{456}^* B_{\gamma''} & = & \pi_{14}^* A_{\gamma\gamma'}^a \otimes \pi_{45}^* B \otimes \pi_{56}^* A_{\gamma''}^b \\ & & \Big\downarrow {\scriptstyle \pi_{1245}^*\eta_{\gamma\gamma'}\otimes\mathrm{id}} \\ & & \pi_{12}^* B \otimes \pi_{25}^* A_{\gamma\gamma'}^b \otimes \pi_{56}^* A_{\gamma''}^b \\ & & \Big\downarrow {\scriptstyle \mathrm{id}\otimes\pi_{256}^*\varphi^b_{\gamma\gamma',\gamma''}} \\ & & \pi_{12}^* B \otimes \pi_{26}^* A_{\gamma\gamma'\gamma''}^b \\ & & \Big\downarrow {\scriptstyle \mathrm{id}\otimes\pi_{236}^*\varphi^{b-1}_{\gamma,\gamma'\gamma''}} \\ & & \pi_{12}^* B \otimes \pi_{23}^* A_\gamma^b \otimes \pi_{36}^* A_{\gamma'\gamma''}^b = \pi_{123}^* B'_\gamma \otimes \pi_{36}^* A_{\gamma'\gamma''}^b.\end{array}$$

This isomorphism satisfies axiom (1M2) due to the commutativity of the diagram for the isomorphisms η_γ from Definition 2.6 and the one for the $\varphi^b_{\gamma_1,\gamma_2}$ from Definition 2.5.

In the case of ϵ non-trivial, we enhance the quotient 1-morphism $\mathcal{B}'$ to an equivariant 1-morphism

$$(\mathcal{B}', \eta'_k) : (\mathcal{G}^{a\prime}, \mathcal{J}^{a\prime}) \longrightarrow (\mathcal{G}^{b\prime}, \mathcal{J}^{b\prime})$$

between Jandl gerbes over M'. To this end, we have to define the 2-isomorphism

$$\eta'_k : k^*\mathcal{B}'^\dagger \circ \mathcal{A}'^a_k \implies \mathcal{A}'^b_k \circ \mathcal{B}' \tag{3.3}$$

for k the non-trivial group element of $\Gamma' \cong \mathbb{Z}_2$, and $\mathcal{A}'^a_k$ and $\mathcal{A}'^b_k$ the 1-isomorphisms (3.2) of the quotient Jandl structures on $\mathcal{G}^{a\prime}$ and $\mathcal{G}^{b\prime}$, respectively. Collecting all definitions, we establish that the 1-morphism $k^*\mathcal{B}'^\dagger \circ$

$\mathcal{A}'^a_k$ on the left has the submersion $Z^l := P^a \times_{Y^a_k} \tilde{Z}_k$ which, for $\tilde{Z} \cong \bigsqcup_{\gamma\in\Gamma_0} Y^a \times_M Y^b \times_M Y^b_\gamma$, becomes a disjoint union over the fibre products $Y^a \times_M Y^a_{\gamma'} \times_M Y^b_{\gamma'} \times_M Y^b_{\gamma''}$ for $\gamma', \gamma'' \in \Gamma_-$. Over this space, it has a line bundle A^l defined componentwise as $\pi^*_{12}A^a_{\gamma'} \otimes \pi^*_{23}B^* \otimes \pi^*_{34}(A^b_{\gamma'^{-1}\gamma''})^*$. The 1-morphism $\mathcal{A}'^b_k \circ \mathcal{B}'$ on the right has the submersion $Z^r := \tilde{Z} \times_{Y^b} P^b$, which is the disjoint union of the fibre products $Y^a \times_M Y^b \times_M Y^b_\gamma \times Y^b_{\gamma''}$, with $\gamma \in \Gamma_0$ and $\gamma'' \in \Gamma_-$, and over that, a line bundle A^r defined componentwise as $\pi^*_{12}B \otimes \pi^*_{23}A^b_\gamma \otimes \pi^*_{34}A^b_{\gamma^{-1}\gamma''}$. The 2-morphism (3.3) is now defined on the surjective submersion

$$W := \bigsqcup_{\gamma\in\Gamma_0,\gamma',\gamma''\in\Gamma_-} Y^a \times_M Y^a_{\gamma'} \times_M Y^b \times_M Y^b_{\gamma'} \times_M Y^b_\gamma \times_M Y^b_{\gamma''}$$

with the projections π_{1246} to Z^l and π_{1356} to Z^r. We then declare the following isomorphism between the pullbacks of A^l and A^r to W:

$$\begin{array}{c}
\pi^*_{1246}A^l = \pi^*_{12}A^a_{\gamma'} \otimes \pi^*_{24}B^* \otimes \pi^*_{46}(A^b_{\gamma'^{-1}\gamma''})^* \\
\Big\downarrow {\scriptstyle \pi^*_{1234}\eta_{\gamma'}\otimes \mathrm{id}} \\
\pi^*_{13}B \otimes \pi^*_{34}A^b_{\gamma'} \otimes \pi^*_{46}(A^b_{\gamma'^{-1}\gamma''})^* \\
\Big\downarrow {\scriptstyle \mathrm{id}\otimes\pi^*_{346}\varphi^b_{\gamma',\gamma'^{-1}\gamma''}} \\
\pi^*_{13}B \otimes \pi^*_{36}A^b_{\gamma''} \\
\Big\downarrow {\scriptstyle \mathrm{id}\otimes\pi^*_{356}(\varphi^b_{\gamma,\gamma^{-1}\gamma''})^{-1}} \\
\pi^*_{13}B \otimes \pi^*_{35}A^b_\gamma \otimes \pi^*_{56}A^b_{\gamma^{-1}\gamma''} = \pi^*_{1356}A^r.
\end{array}$$

It involves the isomorphism $\eta_{\gamma'}$ belonging to the equivariant 1-morphism; see (2.17). Since $\gamma' \in \Gamma_-$ here, we have, by now, used all the structure of $(\mathcal{B}, \eta_\gamma)$. It is straightforward to check that these isomorphisms satisfy the compatibility axiom and make the diagram (2.13) commutative.

We have, so far, obtained a well-defined map

$$q \;:\; \left\{\begin{array}{c}\text{Equivalence classes}\\ \text{of } (\Gamma,\epsilon)\text{-equivariant}\\ \text{bundle gerbes over } M\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Equivalence classes}\\ \text{of } (\Gamma',\epsilon')\text{-equivariant}\\ \text{bundle gerbes over } M'\end{array}\right\}.$$

In order to finish the proof of Theorem 3.1, we now show that q is surjective and injective. Let us start with the observation that the pullback $\mathcal{G} := p^*\mathcal{G}'$ of any bundle gerbe $\mathcal{G}'$ over M' along the projection $p : M \longrightarrow M'$ has a canonical Γ_0-equivariant structure. This comes from the fact that $\mathcal{G} = \gamma\mathcal{G}$ for all $\gamma \in \Gamma_0$, so that $\mathcal{A}_\gamma := \mathrm{id}_{\mathcal{G}}$ and

$$\varphi_{\gamma_1,\gamma_2} := \lambda_{\mathrm{id}_{\mathcal{G}}} = \rho_{\mathrm{id}_{\mathcal{G}}} : \mathrm{id}_{\mathcal{G}} \circ \mathrm{id}_{\mathcal{G}} \Longrightarrow \mathrm{id}_{\mathcal{G}}$$

define a Γ_0-equivariant structure.

Let us have a closer look at this Γ_0-equivariant structure. To this end, notice that $\mathcal{G}$ has the surjective submersion $Y := M \times_{M'} Y' \longrightarrow M$ whose fibre products have, each, an obvious projection $Y^{[k]} \longrightarrow Y'^{[k]}$. The line bundle L and the isomorphism μ of $\mathcal{G}$ are pullbacks of L' and μ' along this projection. The 1-isomorphisms A_γ have the surjective submersion $Z^\gamma \cong Y^{[2]}$ and the line bundles $A_\gamma := L$, see Example 2.2. The 2-isomorphisms $\varphi_{\gamma_1,\gamma_2}$ have the surjective submersions $Z^{\gamma_1,\gamma_2} \cong Y^{[3]}$ and the isomorphisms $\varphi_{\gamma_1,\gamma_2} := \mu$.

We can, next, pass to the quotient bundle gerbe $\mathcal{G}''$. Its surjective submersion is $Y'' := Y$ whose fibre products $Y''^{[k]} = \Gamma_0^{k-1} \times Y^{[k]}$ come, each, with an obvious projection to $Y^{[k]}$. In fact, the line bundle L'' and the multiplication μ'' for $\mathcal{G}''$ are pullbacks of L and μ, respectively, along these projections. Summarizing, the quotient bundle gerbe $\mathcal{G}''$ has the structure of $\mathcal{G}'$ pulled back along the composed projections $Y''^{[k]} \longrightarrow Y'^{[k]}$, which are fibre-preserving. It is well-known that such bundle gerbes are isomorphic.

Let, now, $\mathcal{J}'$ be a Jandl structure on $\mathcal{G}'$ consisting of a line bundle A'_k over $Z' := Y' \times_{M'} Y'_k$, and of an isomorphism α of line bundles over $Z'^{[2]}$. To the above canonical Γ_0-equivariant structure, we add 1-isomorphisms $\mathcal{A}_\gamma : \mathcal{G} \longrightarrow \gamma\mathcal{G}$ for $\gamma \in \Gamma_-$. The relevant fibre product $Z^\gamma = Y \times_M Y_\gamma$ projects to Z', so that the line bundle A_γ is defined as the pullback of A'_k along this projection. Similarly, the isomorphism α' pulls back to the isomorphism of $\mathcal{A}_\gamma$. The Jandl structure $\mathcal{J}'$ also contains a 2-isomorphism $\varphi'_{k,k}$, which naturally pulls back to the 2-isomorphisms $\varphi_{\gamma_1,\gamma_2}$ that we need to complete the definition of a canonical (Γ,ϵ)-equivariant structure $\mathcal{J}$ on the pullback bundle gerbe $\mathcal{G}$. We conclude, along the lines of the above discussion, that $(\mathcal{G},\mathcal{J})$ descends to a Jandl gerbe over M' which is equivariantly isomorphic to $(\mathcal{G}',\mathcal{J}')$.

It remains to prove that the map q is injective. Thus, we assume that two (Γ,ϵ)-equivariant bundle gerbes $(\mathcal{G}^a,\mathcal{J}^a)$ and $(\mathcal{G}^b,\mathcal{J}^b)$ descend to a pair of equivariantly isomorphic Jandl gerbes $(\mathcal{G}^{a\prime},\mathcal{J}^{a\prime})$ and $(\mathcal{G}^{b\prime},\mathcal{J}^{b\prime})$ over M'. Let $(\mathcal{B}',\eta'_k)$ be an equivariant 1-isomorphism between the latter Jandl gerbes. If we assume it to be descended, $\mathcal{B}'$ is based on the fibre product

$$Z' = Y^{a\prime} \times_{M'} Y^{b\prime} \cong \bigsqcup_{\gamma\in\Gamma_0} Y^a \times_M Y^b_\gamma$$

and the line bundle B' over Z' has components B'_γ. Its isomorphism is

$$\beta'_{\gamma_1,\gamma_2,\gamma_3} : \pi_{13}^* A^a_{\gamma_1\gamma_2} \otimes \pi_{34}^* B'_{\gamma_3} \longrightarrow \pi_{12}^* B'_{\gamma_1} \otimes \pi_{24}^* A^b_{\gamma_2\gamma_3}. \tag{3.4}$$

The additional 2-isomorphism η'_k is, in the notation of Section 2.2, an isomorphism of line bundles over

$$Z_1'^k \times_{P'} Z_2'^k \cong \bigsqcup_{\gamma_1,\gamma_2,\gamma_3\in\Gamma_-} Y^a \times_M Y^a_{\gamma_1} \times_M Y^b_{\gamma_1\gamma_2} \times_M Y^b_{\gamma_1\gamma_2\gamma_3},$$

with components

$$(\eta'_k)_{\gamma_1,\gamma_2,\gamma_3} : \pi_{12}^* A^a_{\gamma_1} \otimes \pi_{24}^* B'^*_{\gamma_2\gamma_3} \longrightarrow \pi_{13}^* B'_{\gamma_1\gamma_2} \otimes \pi_{34}^* A^b_{\gamma_3}\,, \tag{3.5}$$

according to (2.17). Let us, now, construct an equivariant 1-isomorphism

$$(\mathcal{B}, \eta_\gamma) : (\mathcal{G}^a, \mathcal{J}^a) \longrightarrow (\mathcal{G}^b, \mathcal{J}^b)$$

out of this structure. Its existence will prove that the map q is injective.

The 1-isomorphism $\mathcal{B} : \mathcal{G}^a \longrightarrow \mathcal{G}^b$ has the surjective submersion $Y^a \times_M Y^b$ and we take the line bundle B'_1 over that space as its line bundle B (all the other line bundles B'_γ with $\gamma \neq 1$ are to be ignored). Similarly, the isomorphism $\beta'_{1,1,1}$ serves as the isomorphism β of $\mathcal{B}$. It remains to construct the 2-isomorphisms η_γ. For $\gamma \in \Gamma_-$, these we define them as the isomorphisms

$$(\eta'_k)_{\gamma,\gamma^{-1},\gamma} : \pi_{12}^* A^a_\gamma \otimes \pi_{24}^* B'^*_1 \longrightarrow \pi_{13}^* B'_1 \otimes \pi_{34}^* A^b_\gamma$$

from (3.5), while for $\gamma \in \Gamma_0$, as the isomorphisms

$$\beta'_{1,\gamma,1} : \pi_{13}^* A^a_\gamma \otimes \pi_{34}^* B'_1 \longrightarrow \pi_{12}^* B'_1 \otimes \pi_{24}^* A^b_\gamma$$

from (3.4) (pulled back along the map that exchanges the second and the third factor). Finally, all the relations that these isomorphisms should obey are readily seen to be satisfied.

4 Twisted-equivariant Deligne cohomology

In this section, we relate the geometric theory developed in Sections 1 and 2 to its cohomological counterpart introduced in [14].

4.1 Local data

Let (Γ, ϵ) be an orientifold group for M, and let $\mathcal{G}$ be a bundle gerbe over M with (Γ, ϵ)-equivariant structure $\mathcal{J}$, consisting of 1-isomorphisms $\mathcal{A}_\gamma$ and of 2-isomorphisms $\varphi_{\gamma_1,\gamma_2}$. We assume that there is a covering $\mathfrak{O} = \{O_i\}_{i\in I}$ and a left action of Γ on the index set I such that $\gamma(O_i) = O_{\gamma i}$, and such that there exist sections $s_i : O_i \longrightarrow Y$. We define

$$M_{\mathfrak{O}} := \bigsqcup_{i\in I} O_i$$

and construct the smooth map

$$s : M_{\mathfrak{O}} \longrightarrow Y : (x, i) \longmapsto s_i(x).$$

There are induced maps $s : M_{\mathfrak{O}}^{[k]} \longrightarrow Y^{[k]}$ on all fibre products, where $M_{\mathfrak{O}}^{[k]}$ is just the disjoint union of all non-empty k-fold intersections $O_{i_1\ldots i_k} := O_{i_1} \cap \ldots \cap O_{i_k}$ of open sets in $\mathfrak{O}$. They may be used to pull back the line bundle L, the 2-form C and the isomorphism μ of $\mathcal{G}$. Choose, now, sections $\sigma_{ij} : O_{ij} \longrightarrow s^*L$ (of unit length) and define smooth functions $g_{ijk} : O_{ijk} \longrightarrow U(1)$ by

$$s^*\mu(\sigma_{ij} \otimes \sigma_{jk}) = g_{ijk} \cdot \sigma_{ik},$$

extract local connection 1-forms $A_{ij} \in \Omega^1(O_{ij})$ such that

$$s^*\nabla\sigma_{ij} = \tfrac{1}{\mathrm{i}} A_{ij}\, \sigma_{ij}\,,$$

where ∇ stands for the covariant derivative, and define 2-forms $B_i := s_i^*C \in \Omega^2(O_i)$. Axiom (G1) gives the equation

$$\mathrm{d}A_{ij} = B_j - B_i \quad \text{on } O_{ij}. \tag{4.1}$$

Since μ preserves connections, one obtains

$$A_{ij} - A_{ik} + A_{jk} = \mathrm{i} g_{ijk}^{-1}\mathrm{d}g_{ijk} \quad \text{on } O_{ijk}, \tag{4.2}$$

and axiom (G2) infers the cocycle condition

$$g_{ijl} \cdot g_{jkl} = g_{ikl} \cdot g_{ijk} \quad \text{on } O_{ijkl}. \tag{4.3}$$

One can always choose the sections σ_{ij} such that A_{ij} and g_{ijk} have the antisymmetry property $A_{ij} = -A_{ji}$ and $g_{ijk} = g_{jik}^{-1} = g_{ikj}^{-1} = g_{kji}^{-1}$.

We continue by extracting local data of the 1-isomorphisms

$$\mathcal{A}_\gamma : \mathcal{G} \longrightarrow \gamma\mathcal{G}$$

consisting, each, of a line bundle A_γ over $Z^\gamma = Y \times_M Y_\gamma$ and of an isomorphism

$$\alpha_\gamma : \pi_{13}^* L \otimes \pi_{34}^* A_\gamma \longrightarrow \pi_{12}^* A_\gamma \otimes \pi_{24}^* L^{\epsilon(\gamma)}$$

of line bundles over $Z^\gamma \times_M Z^\gamma$, see the discussion in Section 2.2. Note that $s_{\gamma^{-1}i} \circ \gamma^{-1} : O_i \longrightarrow Y_\gamma$ is a section into Y_γ, i.e., $\pi_\gamma \circ (s_{\gamma^{-1}i} \circ \gamma^{-1}) = \mathrm{id}_{O_i}$, and so we get the map $s_\gamma : M_{\mathfrak{O}} \longrightarrow Y_\gamma : (x, i) \longmapsto s_{\gamma^{-1}i}(\gamma^{-1}(x))$ compatible with the projections to M. Note that also $\sigma_{\gamma^{-1}i\gamma^{-1}j} \circ \gamma^{-1} : O_{ij} \longrightarrow s_\gamma^* L$ is a section. Furthermore, we have a mixed map

$$z_\gamma : M_{\mathfrak{O}} \longrightarrow Z^\gamma : (x, i) \longmapsto (s_i(x), s_{\gamma^{-1}i}(\gamma^{-1}(x))) \tag{4.4}$$

into the space Z^γ. Note that $\pi_1 \circ z_\gamma = s$ and $\pi_2 \circ z_\gamma = s_\gamma$, so that the pull-back of α_γ along z_γ is an isomorphism

$$z_\gamma^* \alpha_\gamma : s^* L \otimes \pi_2^* z_\gamma^* A_\gamma \longrightarrow \pi_1^* z_\gamma^* A_\gamma \otimes s_\gamma^* L^{\epsilon(\gamma)}$$

of line bundles over $M_{\mathfrak{O}} \times_M M_{\mathfrak{O}}$ (the two maps π_1, π_2 above are the canonical projections from $M_{\mathfrak{O}} \times_M M_{\mathfrak{O}}$ to $M_{\mathfrak{O}}$). Choose new unit-length sections $\sigma_i^\gamma : O_i \longrightarrow z_\gamma^* A_\gamma$ and obtain local connection 1-forms $\Pi_i^\gamma \in \Omega^1(O_i)$, as well as smooth functions $\chi_{ij}^\gamma : O_{ij} \longrightarrow U(1)$ by the relation

$$z_\gamma^* \alpha_\gamma(\sigma_{ij} \otimes \sigma_j^\gamma) = \chi_{ij}^\gamma \cdot (\sigma_i^\gamma \otimes (\sigma_{\gamma^{-1}i\gamma^{-1}j}^{\epsilon(\gamma)} \circ \gamma^{-1})).$$

In order to simplify the notation in the following discussion, we write

$$\gamma f_i := ((\gamma^{-1})^* f_{\gamma^{-1}i})^{\epsilon(\gamma)} \quad \text{and} \quad \gamma\Pi_i := \epsilon(\gamma)(\gamma^{-1})^* \Pi_{\gamma^{-1}i} \tag{4.5}$$

for $U(1)$-valued functions f_i and 1-forms Π_i, respectively, and likewise for components of generic p-form-valued Čech cochains encountered below. In this notation, axiom (1M1) gives

$$\gamma B_i - B_i = \mathrm{d}\Pi_i^\gamma. \tag{4.6}$$

Since α_γ preserves connections, one obtains

$$\gamma A_{ij} - A_{ij} = \Pi_j^\gamma - \Pi_i^\gamma - \mathrm{i}\chi_{ij}^{\gamma\,-1}\mathrm{d}\chi_{ij}^\gamma, \tag{4.7}$$

and axiom (1M2) gives

$$\gamma g_{ijk} \cdot g_{ijk}^{-1} = \chi_{ij}^{\gamma\,-1} \cdot \chi_{ik}^\gamma \cdot \chi_{jk}^{\gamma\,-1}. \tag{4.8}$$

One can, again, choose the sections such that $\chi_{ij}^\gamma = (\chi_{ji}^\gamma)^{-1}$.

Finally, we extract local data from the 2-isomorphisms $\varphi_{\gamma_1,\gamma_2}$. Consider the space $Z^{\gamma_1,\gamma_2} = Y \times_M Y_{\gamma_1} \times_M Y_{\gamma_1\gamma_2}$ and the isomorphism

$$\varphi_{\gamma_1,\gamma_2} : \pi_{12}^* A_{\gamma_1} \otimes \pi_{23}^* A_{\gamma_2}^{\epsilon(\gamma_1)} \longrightarrow \pi_{13}^* A_{\gamma_1\gamma_2}$$

of line bundles over Z^{γ_1,γ_2}. We use the map

$$\begin{aligned} z_{\gamma_1,\gamma_2} : M_{\mathfrak{O}} &\longrightarrow Z^{\gamma_1,\gamma_2}, \\ (x,i) &\longmapsto (s_i(x), s_{\gamma_1^{-1}i}(\gamma_1^{-1}(x)), s_{\gamma_2^{-1}\gamma_1^{-1}i}(\gamma_2^{-1}(\gamma_1^{-1}(x)))) \end{aligned}$$

to pull back the isomorphism $\varphi_{\gamma_1,\gamma_2}$ to $M_{\mathfrak{O}}$. Note that $\pi_{12} \circ z_{\gamma_1,\gamma_2} = z_{\gamma_1}$, $\pi_{23} \circ z_{\gamma_1,\gamma_2} = z_{\gamma_2} \circ \gamma_1^{-1}$ and $\pi_{13} \circ z_{\gamma_1,\gamma_2} = z_{\gamma_1\gamma_2}$. Hence, we may use the sections $\sigma_i^\gamma : O_i \longrightarrow z_\gamma^* A_\gamma$ to extract smooth functions $f_i^{\gamma_1,\gamma_2} : O_i \longrightarrow U(1)$ by the relation

$$z_{\gamma_1,\gamma_2}^* \varphi_{\gamma_1,\gamma_2}(\sigma_i^{\gamma_1} \otimes (\sigma_{\gamma_1^{-1}i}^{\gamma_2} \circ \gamma_1^{-1})^{\epsilon(\gamma_1)}) = f_i^{\gamma_1,\gamma_2} \cdot \sigma_i^{\gamma_1\gamma_2}.$$

From the requirement that $\varphi_{\gamma_1,\gamma_2}$ respect connections, it follows that

$$\gamma_1 \Pi_i^{\gamma_2} - \Pi_i^{\gamma_1\gamma_2} + \Pi_i^{\gamma_1} = \mathrm{i}(f_i^{\gamma_1,\gamma_2})^{-1}\mathrm{d}f_i^{\gamma_1,\gamma_2}. \tag{4.9}$$

The compatibility condition (2.11) for the 2-morphism $\varphi_{\gamma_1,\gamma_2}$ becomes

$$\gamma_1 \chi_{ij}^{\gamma_2} \cdot (\chi_{ij}^{\gamma_1\gamma_2})^{-1} \cdot \chi_{ij}^{\gamma_1} = (f_i^{\gamma_1,\gamma_2})^{-1} \cdot f_j^{\gamma_1,\gamma_2}, \tag{4.10}$$

and the condition imposed on the morphisms $\varphi_{\gamma_1,\gamma_2}$ in Definition 2.5, equivalent to the commutativity of diagram (2.12), reads

$$\gamma_1 f_i^{\gamma_2,\gamma_3} \cdot (f_i^{\gamma_1\gamma_2,\gamma_3})^{-1} \cdot f_i^{\gamma_1,\gamma_2\gamma_3} \cdot (f_i^{\gamma_1,\gamma_2})^{-1} = 1. \tag{4.11}$$

Summarizing, the bundle gerbe has local data $c := (B_i, A_{ij}, g_{ijk})$, and the (Γ, ϵ)-equivariant structure has local data $b_\gamma := (\Pi_i^\gamma, \chi_{ij}^\gamma)$ and $a_{\gamma_1,\gamma_2} := (f_i^{\gamma_1,\gamma_2})$. If the (Γ, ϵ)-equivariant structure is normalized, the mixed map $z_1 : M_{\mathfrak{O}} \longrightarrow Z^1 = Y^{[2]}$ defined in (4.4) is $z_1 = \Delta \circ s$, so we may choose $\sigma_i^1 := \Delta^*\sigma_{ii}$ and obtain $\Pi_i^1 = A_{ii} = 0$. With this choice, we find, for $\alpha_1 = (1 \otimes \pi_{124}^*\mu^{-1}) \circ (\pi_{134}^*\mu \otimes 1)$, the local datum $\chi_{ij}^1 = g_{iij}^{-1} \cdot g_{ijj} = 1$. For $\varphi_{1,\gamma}$, we get $f_i^{1,\gamma} = (\chi_{ii}^\gamma)^{-1} g_{iii} = 1$, and, analogously, $f_i^{\gamma,1} = \chi_{ii}^\gamma g_{iii} = 1$. Finally, $f_i^{1,1} = g_{iii} = 1$.

Let us now extract local data of an equivariant 1-morphism

$$(\mathcal{B}, \eta_\gamma) : (\mathcal{G}, \mathcal{J}) \longrightarrow (\mathcal{G}', \mathcal{J}')$$

between two (Γ, ϵ)-equivariant bundle gerbes over M. We may choose an open covering $\mathfrak{O}$ of M with an action of Γ on its index set as above, such that it admits sections $s_i : O_i \longrightarrow Y$ and $s_i' : O_i \longrightarrow Y'$ for both bundle gerbes. The 1-morphism $\mathcal{B}$ provides a vector bundle B over $Z := Y \times_M Y'$ of some rank n. Generalizing the mixed map (4.4), we have a map

$$z : M_{\mathfrak{O}} \longrightarrow Z : (x, i) \longmapsto (s_i(x), s_i'(x)). \tag{4.12}$$

Just as above, we choose an orthonormal frame of sections $\sigma_i^a : O_i \longrightarrow z^*B$, $a = 1, \ldots, n$, and obtain local connection 1-forms $\Lambda_i \in \Omega^1(O_i, \mathfrak{u}(n))$ with values in the set of Hermitian $(n \times n)$-matrices, alongside smooth functions $G_{ij} : O_{ij} \longrightarrow U(n)$ defined by the relation

$$(z^*\nabla)\sigma_i^a = \tfrac{1}{\mathrm{i}}(\Lambda_i)_b{}^a\, \sigma_i^b$$
$$z^*\beta(\sigma_{ij} \otimes \sigma_j^a) = (G_{ij})_b{}^a\, (\sigma_i^b \otimes \sigma_{ij}'),$$

where β is the isomorphism of $\mathcal{B}$ over $Z \times_M Z$ and σ_{ij} and σ_{ij}' are the sections chosen to extract local data of the two bundle gerbes $\mathcal{G}$ and $\mathcal{G}'$. Analogously to (4.6), (4.7) and (4.8), we have

$$\begin{aligned} B_i' - B_i &= \tfrac{1}{n}\mathrm{tr}(\mathrm{d}\Lambda_i), \\ A_{ij}' - A_{ij} &= \Lambda_j - G_{ij}^{-1} \cdot \Lambda_i \cdot G_{ij} - \mathrm{i}G_{ij}^{-1}\mathrm{d}G_{ij}, \\ g_{ijk}' \cdot g_{ijk}^{-1} &= G_{ik} \cdot G_{jk}^{-1} \cdot G_{ij}^{-1}. \end{aligned} \tag{4.13}$$

Again, the sections σ_i^a can be chosen such that $G_{ij} = G_{ji}^{-1}$. The 2-isomorphisms $\eta_\gamma : \gamma\mathcal{B} \circ \mathcal{A}_\gamma \Longrightarrow \mathcal{A}_\gamma' \circ \mathcal{B}$ are, following the discussion in Section 2.2, isomorphisms

$$\eta_\gamma : \pi_{12}^* A_\gamma \otimes \pi_{24}^* B^{\epsilon(\gamma)} \longrightarrow \pi_{13}^* B \otimes \pi_{34}^* A'_\gamma$$

of vector bundles over $Z_1^\gamma \times_P Z_2^\gamma \cong Y \times_M Y_\gamma \times_M Y' \times_M Y'_\gamma$; see (2.17). We compose a map

$$\begin{aligned} z_\gamma : M_{\mathfrak{D}} \longrightarrow\ & Z_1^\gamma \times_P Z_2^\gamma : (x,i) \\ \longmapsto\ & (s_i(x), s_{\gamma^{-1}i}(\gamma^{-1}(x)), s'_i(x), s'_{\gamma^{-1}i}(\gamma^{-1}(x))), \end{aligned}$$

into that space and define smooth functions $H_i^\gamma : O_i \longrightarrow U(n)$ by

$$z_\gamma^* \eta_\gamma(\sigma_i^\gamma \otimes (\sigma_{\gamma^{-1}i}^a \circ \gamma^{-1})^{\epsilon(\gamma)}) = (H_i^\gamma)_b{}^a\, (\sigma_i^b \otimes \sigma_i'^\gamma).$$

Since η_γ preserves the connections, we obtain

$$\gamma\Lambda_i = (H_i^\gamma)^{-1} \cdot \Lambda_i \cdot H_i^\gamma + \mathrm{i}(H_i^\gamma)^{-1}\mathrm{d}H_i^\gamma + \Pi_i'^\gamma - \Pi_i^\gamma. \tag{4.14}$$

The commutativity of diagram (2.18) implies that

$$\gamma G_{ij} = \chi_{ij}'^\gamma \cdot (\chi_{ij}^\gamma)^{-1} \cdot (H_i^\gamma)^{-1} \cdot G_{ij} \cdot H_j^\gamma. \tag{4.15}$$

Here, we have extended the notation of (4.5) to $U(n)$-valued functions and $\mathfrak{u}(n)$-valued 1-forms in the following way:

$$\begin{aligned} \gamma\Lambda_i &:= \begin{cases} (\gamma^{-1})^*\Lambda_{\gamma^{-1}i}, & \text{if } \epsilon(\gamma)=1, \\ -(\gamma^{-1})^*\overline{\Lambda_{\gamma^{-1}i}}, & \text{if } \epsilon(\gamma)=-1, \end{cases} \\ \gamma G_i &:= \begin{cases} (\gamma^{-1})^* G_{\gamma^{-1}i}, & \text{if } \epsilon(\gamma)=1, \\ (\gamma^{-1})^*\overline{G_{\gamma^{-1}i}}, & \text{if } \epsilon(\gamma)=-1, \end{cases} \end{aligned} \tag{4.16}$$

with the overbar denoting the complex conjugation. These definitions coincide with (4.5) for $n=1$. The commutativity of diagram (2.19) leads to

$$H_i^{\gamma_1\gamma_2} \cdot f_i^{\gamma_1,\gamma_2} = f_i'^{\gamma_1,\gamma_2} \cdot H_i^{\gamma_1} \cdot \gamma_1 H_i^{\gamma_2}. \tag{4.17}$$

Thus, an equivariant 1-morphism has local data $\beta := (\Lambda_i, G_{ij})$ and $\eta_\gamma := (H_i^\gamma)$ satisfying (4.14), (4.15) and (4.17).

Finally, for completeness, assume that $\phi : (\mathcal{B}, \eta_\gamma) \Longrightarrow (\mathcal{B}', \eta'_\gamma)$ is an equivariant 2-morphism, i.e., $\phi : B \longrightarrow B'$ is a morphism of vector bundles over Z subject to the two conditions of Section 2.2. We use the map $z : M_{\mathfrak{D}} \longrightarrow Z$

from (4.12) to pull back ϕ and to extract smooth functions $(U_i)_{a'}^{\ a}$ defined on O_i by

$$z^*\phi(\sigma_i^a) = (U_i)_{a'}^{\ a}\, \sigma_i'^{a'}.$$

Here, $a = 1, \dots, n$ and $a' = 1, \dots, n'$, where n and n' are ranks of the vector bundles B and B', respectively. The condition that ϕ preserves Hermitian metrics and the connections yields

$$U_i^\dagger \cdot U_i = 1 \quad \text{and} \quad \Lambda_i = U_i^\dagger \cdot \Lambda_i' \cdot U_i + \mathrm{i} U_i^\dagger \mathrm{d} U_i. \tag{4.18}$$

The two remaining conditions, namely the commutativity of diagrams (2.21) and (2.22) impose the relations

$$G_{ij} = U_i^\dagger \cdot G_{ij}' \cdot U_j \quad \text{and} \quad H_i^\gamma = U_i^\dagger \cdot H_i'^\gamma \cdot \gamma U_i. \tag{4.19}$$

4.2 Deligne cohomology-valued group cohomology

It is convenient to put the local data extracted above into the context of Deligne hypercohomology. We denote by $\mathcal{U}$ the sheaf of smooth $U(1)$-valued functions on M, and by Λ^q the sheaf of q-forms on M. The Deligne complex in degree 2, denoted by $\mathcal{D}(2)$, is the complex

$$0 \longrightarrow \mathcal{U} \xrightarrow{\frac{1}{\mathrm{i}}\mathrm{dlog}} \Lambda^1 \xrightarrow{\mathrm{d}} \Lambda^2 \tag{4.20}$$

of sheaves over M. Together with the Čech complex of the open cover $\mathfrak{O}$, it gives a double complex whose total complex $K(\mathcal{D}(2))$

$$0 \longrightarrow A^0 \xrightarrow{D_0} A^1 \xrightarrow{D_1} A^2 \xrightarrow{D_2} A^3 \tag{4.21}$$

has the cochain groups

$$\begin{aligned} A^0 &= C^0(\mathcal{U}), \\ A^1 &= C^0(\Lambda^1) \oplus C^1(\mathcal{U}), \\ A^2 &= C^0(\Lambda^2) \oplus C^1(\Lambda^1) \oplus C^2(\mathcal{U}), \\ A^3 &= C^1(\Lambda^2) \oplus C^2(\Lambda^1) \oplus C^3(\mathcal{U}), \end{aligned}$$

and the differentials

$$\begin{aligned} D_0(f_i) &= (-\mathrm{i}f_i^{-1}\mathrm{d}f_i\ ,\ f_j^{-1}\cdot f_i),\\ D_1(\Pi_i,\chi_{ij}) &= (\mathrm{d}\Pi_i\ ,\ -\mathrm{i}\chi_{ij}^{-1}\mathrm{d}\chi_{ij}+\Pi_j-\Pi_i\ ,\ \chi_{jk}^{-1}\cdot\chi_{ik}\cdot\chi_{ij}^{-1}),\\ D_2(B_i,A_{ij},g_{ijk}) &= (\mathrm{d}A_{ij}-B_j+B_i, -\mathrm{i}g_{ijk}^{-1}\mathrm{d}g_{ijk}+A_{jk}-A_{ik}\\ &\qquad + A_{ij}, g_{jkl}^{-1}\cdot g_{ikl}\cdot g_{ijl}^{-1}\cdot g_{ijk}). \end{aligned}$$

The cohomology of this complex is the hypercohomology of the double complex induced from (4.20). Its groups are denoted by $\mathbb{H}^k(M,\mathcal{D}(2))$. The local data $c=(B_i,A_{ij},g_{ijk})$ extracted above from the bundle gerbe $\mathcal{G}$ are an element of A^2, and the properties (4.1), (4.2) and (4.3) show that $D_2(c)=0$. This is the Deligne cocycle of a bundle gerbe [9].

In [14], we turned the complex A^n into a complex of left Γ-modules, where the action of Γ is given by (4.5) and its extension to higher order forms. Thus, γc are the local data of $\gamma\mathcal{G}$ for the same choice of sections. The local data $b_\gamma=(\Pi_i^\gamma,\chi_{ij}^\gamma)$ extracted above from 1-isomorphisms $\mathcal{A}_\gamma$ give an element $b_\gamma\in A^1$, and the properties (4.6), (4.7) and (4.8) amount to the relations

$$\gamma c - c = D_1 b_\gamma. \tag{4.22}$$

Furthermore, the local data $a_{\gamma_1,\gamma_2}=(f_i^{\gamma_1,\gamma_2})$ extracted from 2-isomorphisms $\varphi_{\gamma_1,\gamma_2}$ give an element $a_{\gamma_1,\gamma_2}\in A^0$, and its properties (4.9) and (4.10) are equivalent to the identities

$$\gamma_1 b_{\gamma_2} - b_{\gamma_1\gamma_2} + b_{\gamma_2} = -D_0 a_{\gamma_1\gamma_2}. \tag{4.23}$$

The additional constraint (4.11) reads:

$$\gamma_1 a_{\gamma_2,\gamma_3} - a_{\gamma_1\gamma_2,\gamma_3} + a_{\gamma_1,\gamma_2\gamma_3} - a_{\gamma_1,\gamma_2} = 0 \tag{4.24}$$

(where the right-hand side represents the trivial Čech cochain (1) in the additive notation). As recognized in [14], the equations above show that group cohomology is relevant in the present context. For each Γ-module A^n, we form the group of Γ-cochains $C^k(A^n)=\mathrm{Map}(\Gamma^k,A^n)$, together with the Γ-coboundary operator

$$\delta : C^k(A^n) \longrightarrow C^{k+1}(A^n)$$

defined as

$$\begin{aligned} (\delta n)_{\gamma_1,\dots,\gamma_{k+1}} &= \gamma_1 n_{\gamma_2,\dots,\gamma_{k+1}} - n_{\gamma_1\gamma_2,\dots,\gamma_{k+1}}\\ &\qquad + \dots + (-1)^k n_{\gamma_1,\dots,\gamma_k\gamma_{k+1}} + (-1)^{k+1} n_{\gamma_1,\dots,\gamma_k}. \end{aligned}$$

When expressed in terms of this coboundary operator, (4.22), (4.23) and (4.24) become

$$(\delta c)_\gamma = D_1 b_\gamma, \quad (\delta b)_{\gamma_1,\gamma_2} = -D_0 a_{\gamma_1,\gamma_2} \quad \text{and} \quad (\delta a)_{\gamma_1,\gamma_2,\gamma_3} = 0. \tag{4.25}$$

These are (2.13)–(2.14) of [14]. In group cohomology, a cochain $n_{\gamma_1,\ldots,\gamma_n}$ is called *normalized* if $n_{\gamma_1,\ldots,\gamma_n} = 0$ whenever some $\gamma_i = 1$. Notice that the cochains η_γ and a_{γ_1,γ_2} are normalized if the (Γ,ϵ)-equivariant structure on $\mathcal{G}$ is normalized.

The coboundary operator δ commutes with the differential D_n, and so the groups $C^k(A^n)$ form again a double complex. Its hypercohomology is denoted by $\mathbb{H}^n(\Gamma, K(\mathcal{D}(2))_\epsilon)$, where the subscript ϵ on $K(\mathcal{D}(2))$ indicates that the action of Γ on the latter module is the one inherited from (4.5). The collection $(c, b_\gamma, a_{\gamma_1,\gamma_2})$, representing the bundle gerbe $\mathcal{G}$ with (Γ,ϵ)-equivariant structure, defines an element in the degree 2 cochain group of this total complex. Equations (4.25) together with $D_2 c = 0$ show that it is even a cocycle, defining a class in $\mathbb{H}^2(\Gamma, K(\mathcal{D}(2))_\epsilon)$.

In the same way as above, local data $\beta = (\Lambda_i, G_{ij})$ and $\eta_\gamma = (H_i^\gamma)$ of an equivariant 1-isomorphism fit into this framework. The restriction to 1-*iso*morphisms means that the functions Λ_i and H_i^γ take values in $U(1)$, and that the 1-forms Λ_i are real-valued. Note that the conventions (4.16) coincide with the definitions (4.5) in the abelian case. Then, $\beta \in A^1$ and $\eta_\gamma \in A^0$. Equations (4.13), (4.14) and (4.15), and (4.17) mean

$$\begin{aligned} c' = c + D_1\beta, \quad b'_\gamma = b_\gamma + (\delta\beta)_\gamma + D_0\eta_\gamma \quad \text{and} \\ a'_{\gamma_1,\gamma_2} = a_{\gamma_1,\gamma_2} - (\delta\eta)_{\gamma_1,\gamma_2}. \end{aligned} \tag{4.26}$$

These are (2.15) to (2.17) of [14]. Thus, (Γ,ϵ)-equivariant bundle gerbes, which are related by an equivariant 1-isomorphism define the same class in $\mathbb{H}^2(\Gamma, K(\mathcal{D}(2))_\epsilon)$.

Proposition 4.1. *The map*

$$\left\{ \begin{matrix} \textit{Equivalence classes} \\ \textit{of } (\Gamma,\epsilon)\textit{-equivariant} \\ \textit{bundle gerbes over } M \end{matrix} \right\} \longrightarrow \mathbb{H}^2(\Gamma, K(\mathcal{D}(2))_\epsilon)$$

defined by extracting local data as described above is a bijection.

Proof. This follows from the usual reconstruction of bundle gerbes, 1-morphisms and 2-morphisms from given local data, see, e.g., [24]. The reconstructed objects have the property that they admit local data from which they were reconstructed. Thus, in order to see the surjectivity, one reconstructs a bundle gerbe $\mathcal{G}$, the 1-morphisms $\mathcal{A}_\gamma$ and the 2-morphisms $\varphi_{\gamma_1,\gamma_2}$ from given local data. The cocycle condition assures that all necessary diagrams are commutative, so that one obtains a twisted-equivariant bundle gerbe whose local data are the given one. To see the injectivity, assume that the class of the cocycle c of a given twisted-equivariant bundle gerbe $(\mathcal{G}, \mathcal{J})$ is trivial, $c = D_1(d)$. Then, one can reconstruct an equivariant 1-isomorphism $(\mathcal{G}, \mathcal{J}) \longrightarrow (\mathcal{I}_0, \mathcal{J}_0)$ from the cochain d. □

The geometric descent theory from Section 2 implies, via Proposition 4.1, results for the cohomology theories, namely a bijection

$$\mathbb{H}^2(\Gamma, K(\mathcal{D}(2))_\epsilon) \;\cong\; \mathbb{H}^2(\Gamma', K(\mathcal{D}(2))_{\epsilon'}),$$

whenever the normal subgroup $\Gamma_0 := \ker(\epsilon)$ acts without fixed points so that (Γ', ϵ') is an orientifold group for the quotient manifold M'. In the next section, we will use Proposition 4.1 in the opposite direction.

4.3 Classification results

In this section, we present a short summary of the classification results from [14]. In general, there are obstructions to the existence of a (Γ, ϵ)-equivariant structure on a bundle gerbe $\mathcal{G}$, and if these vanish, there may be inequivalent choices thereof. We use Proposition 4.1 to study these issues in a purely cohomological way. To this end, we are looking for the image and the kernel of the homomorphism

$$\mathrm{pr} : \mathbb{H}^2(\Gamma, K(\mathcal{D}(2))_\epsilon) \longrightarrow \mathbb{H}^2(M, \mathcal{D}(2)), \tag{4.27}$$

which sends a twisted-equivariant Deligne class to the underlying ordinary Deligne class.

As shown in [14], the image can be characterized by hierarchical obstructions to the existence of twisted-equivariant structures on a given bundle gerbe $\mathcal{G}$. If we assume that the curvature H of $\mathcal{G}$ is (Γ, ϵ)-equivariant in the sense that $\gamma H = H$ for all $\gamma \in \Gamma$, these obstructions are classes

$$\begin{aligned} &o_1 \in H^2(M, U(1)), \quad o_2 \in H^2(\Gamma, H^1(M, U(1))_\epsilon) \quad \text{and} \\ &o_3 \in H^3(\Gamma, H^0(M, U(1))_\epsilon). \end{aligned} \tag{4.28}$$

The latter two are Γ-cohomology groups, with the action of Γ on the coefficients induced from (4.5). The class o_2 is well-defined if o_1 vanishes, and o_3 is well-defined if o_1 and o_2 vanish.

Let us now discuss the kernel of the homomorphism (4.27), i.e., the question what the set of equivalence classes of twisted-equivariant bundle gerbes with isomorphic underlying bundle gerbe looks like. We infer that pr is induced from the projection

$$p^n : \bigoplus_{p+q=n} C^p(A^q) \longrightarrow A^n \tag{4.29}$$

of chain complexes, whose cohomologies are $\mathbb{H}^n(\Gamma, K(\mathcal{D}(2))_\epsilon)$ and $\mathbb{H}^n(M, \mathcal{D}(2))$, respectively. The kernel of p^n forms, again, a complex whose cohomology will be denoted by $\mathcal{H}^n$. Explicitly, a class in $\mathcal{H}^2$ is represented by a pair $(b_\gamma, a_{\gamma_1,\gamma_2})$ with $b_\gamma \in A^1$ and $a_{\gamma_1,\gamma_2} \in A^0$ such that

$$\mathrm{D}_1 b_\gamma = 0, \quad (\delta b)_{\gamma_1,\gamma_2} = -\mathrm{D}_0 a_{\gamma_1,\gamma_2} \quad \text{and} \quad (\delta a)_{\gamma_1,\gamma_2\gamma_3} = 0.$$

Equivalent representatives satisfy $b'_\gamma = b_\gamma + \mathrm{D}_0\eta_\gamma$ and $a'_{\gamma_1,\gamma_2} = a_{\gamma_1,\gamma_2} - (\delta\eta)_{\gamma_1,\gamma_2}$ for a collection $\eta_\gamma \in A^0$. Comparing this with (20) in [14], we conclude that the group H_Γ, which we considered there is obtained from $\mathcal{H}^2$ by additionally identifying cocycles $(b_\gamma, a_{\gamma_1,\gamma_2})$ and $(b'_\gamma, a'_{\gamma_1,\gamma_2})$ if there exists a $\beta \in A^1$ such that $\mathrm{D}_1\beta = 0$ and $b'_\gamma = b_\gamma + (\delta\beta)_\gamma$.

Lemma 4.1. *The group $\mathcal{H}^2$ fits into the exact sequences*

$$0 \longrightarrow \mathcal{H}^2 \,/\, H^1(M, U(1)) \longrightarrow \mathbb{H}^2(\Gamma, K(\mathcal{D}(2))_\epsilon) \xrightarrow{\mathrm{pr}} \mathbb{H}^2(M, \mathcal{D}(2))$$

and

$$0 \longrightarrow H^2(\Gamma, H^0(M, U(1))_\epsilon) \longrightarrow \mathcal{H}^2 \longrightarrow C^1(H^1(M, U(1))).$$

Proof. The first sequence is just a piece of the long exact sequence obtained from the short exact sequence, which is (4.29) extended by its kernel to the left, together with the well-known identification $\mathbb{H}^k(M, \mathcal{D}(2)) \cong H^k(M, U(1))$ for $k = 0, 1$ [4]. The second sequence can be obtained by the same trick: we project out another factor $q^n : \ker(p^n) \longrightarrow C^1(A^{n-1})$ from the exact sequence of complexes, yielding a short exact sequence

$$0 \longrightarrow \ker(q)^\bullet \longrightarrow \ker(p)^\bullet \xrightarrow{q} C^1(A^{\bullet-1}) \longrightarrow 0$$

of chain complexes. The interesting part of its long exact sequence is

$$C^1(H^0(M, U(1))) \xrightarrow{\delta} H^2(\ker(q)) \longrightarrow \mathcal{H}^2 \longrightarrow C^1(H^1(M, U(1))),$$

for which an easy computation shows that $H^2(\ker(q))$ coincides with $\ker(\delta|_{C^2(H^0(M,U(1)))})$. □

We have now derived results on the image and the kernel of the projection (4.27). When the underlying manifold is 2-connected, $H^2(M,U(1)) = H^1(M,U(1)) = 0$ and $H^0(M,U(1))_\epsilon = U(1)_\epsilon$ as Γ-modules, so that the obstructions (4.28) and Lemma 4.1 boil down to

Proposition 4.2. *Let M be a 2-connected smooth manifold and let $\mathcal{G}$ be a bundle gerbe over M with (Γ,ϵ)-equivariant curvature.*

(a) *$\mathcal{G}$ admits (Γ,ϵ)-equivariant structures if and only if the third obstruction class $o_3 \in H^3(\Gamma, U(1)_\epsilon)$ vanishes.*
(b) *In the latter case, equivalence classes of (Γ,ϵ)-equivariant bundle gerbes whose underlying bundle gerbe is isomorphic to $\mathcal{G}$ are parameterized by the group $H^2(\Gamma, U(1)_\epsilon)$.*

This was the starting point for the calculations in finite-group cohomology of [14]. Namely, on a compact connected simple and simply connected Lie group, there is a canonical family $\mathcal{G}_k$ of bundle gerbes with (Γ,ϵ)-equivariant curvature for Γ a semidirect product of $\mathbb{Z}_2$ (generated by the ζ-twisted inversion $g \longrightarrow \zeta \cdot g^{-1}$, with ζ from the centre of G) and a subgroup of the centre of G. Since these Lie groups are 2-connected, the obstruction classes and the classifying groups for (Γ,ϵ)-equivariant structures on $\mathcal{G}_k$ may be computed by calculations in finite-group cohomology.

5 Equivariant gerbe modules

Gerbe modules can be described conveniently as 1-morphisms [26]:

Definition 5.1. Let $\mathcal{G}$ be a bundle gerbe over M. A $\mathcal{G}$-module is a 1-morphism

$$\mathcal{E} : \mathcal{G} \longrightarrow \mathcal{I}_\omega .$$

The rank of the vector bundle of $\mathcal{E}$ is called the rank of the bundle-gerbe module, and the 2-form ω is called its central curvature.

Let us extract the details of this definition. We assume that the 1-morphism $\mathcal{E}$ is descended in the sense that it consists of a vector bundle E over $Y \cong Y \times_M M$. Similarly as in Lemma 2.1, this can be assumed up to natural 2-isomorphisms; see Theorem 1 in [26]. By axiom (1M1), the curvature of the vector bundle E satisfies

$$\frac{1}{n}\mathrm{tr}(\mathrm{curv}(E)) = \pi^*\omega - C.$$

The $\mathcal{G}$-module consists also of an isomorphism

$$\rho : L \otimes \pi_2^* E \longrightarrow \pi_1^* E$$

of vector bundles over $Y^{[2]}$ which satisfies, by axiom (1M2), the condition

$$\pi_{13}^*\rho \circ (\mu \otimes \mathrm{id}) = \pi_{12}^*\rho \circ (\mathrm{id} \otimes \pi_{23}^*\rho). \tag{5.1}$$

The latter resembles the axiom for an action ρ of a monoid L on a module E, hence the terminology. The above definition of a bundle-gerbe module coincides with the usual one; see, e.g., [2, 10].

Definition 5.2. Let (Γ, ϵ) be an orientifold group for M and let $(\mathcal{G}, \mathcal{J})$ be a (Γ, ϵ)-equivariant bundle gerbe over M. A $(\mathcal{G}, \mathcal{J})$-module is a 2-form ω on M that satisfies the condition $\gamma\omega = \omega$ for all $\gamma \in \Gamma$, together with an equivariant 1-morphism

$$(\mathcal{E}, \rho_\gamma) : (\mathcal{G}, \mathcal{J}) \longrightarrow (\mathcal{I}_\omega, \mathcal{J}_\omega),$$

where $\mathcal{J}_\omega$ is the canonical (Γ, ϵ)-equivariant structure on the trivial bundle gerbe $\mathcal{I}_\omega$ from Example 2.3.

Thus, a $(\mathcal{G}, \mathcal{J})$-module is a $\mathcal{G}$-module $\mathcal{E} : \mathcal{G} \longrightarrow \mathcal{I}_\omega$ together with a 2-morphism

$$\rho_\gamma : \gamma\mathcal{E} \circ \mathcal{A}_\gamma \Longrightarrow \mathcal{E}$$

for every $\gamma \in \Gamma$, such that the diagram

$$\begin{array}{ccc} \gamma_1\gamma_2\mathcal{E} \circ \gamma_1\mathcal{A}_{\gamma_2} \circ \mathcal{A}_{\gamma_1} & \xRightarrow{\mathrm{id}_{\gamma_1\gamma_2\mathcal{E}} \circ \varphi_{\gamma_1,\gamma_2}} & \gamma_1\gamma_2\mathcal{E} \circ \mathcal{A}_{\gamma_1\gamma_2} \\ {\scriptstyle \gamma_1\rho_{\gamma_2} \circ \mathrm{id}_{\mathcal{A}_{\gamma_1}}} \Big\Downarrow & & \Big\Downarrow {\scriptstyle \rho_{\gamma_1\gamma_2}} \\ \gamma_1\mathcal{E} \circ \mathcal{A}_{\gamma_1} & \xRightarrow[\rho_{\gamma_1}]{} & \mathcal{E} \end{array} \tag{5.2}$$

is commutative for all $\gamma_1, \gamma_2 \in \Gamma$. We say that a $(\mathcal{G}, \mathcal{J})$-module is *normalized* if the equivariant 1-morphism $(\mathcal{E}, \rho_\gamma)$ is normalized.

We already discussed equivariant 1-morphisms in terms of vector bundles and isomorphisms of vector bundles in Section 2.2, so that we only have to apply these results to the particular case at hand. We recall that the (Γ, ϵ)-equivariant structure on $\mathcal{G}$ consists of a line bundle A_γ over Z^γ of curvature $\mathrm{curv}(A_\gamma) = \epsilon(\gamma)\pi_2^*C - \pi_1^*C$ for each $\gamma \in \Gamma$, and of isomorphisms

$$\alpha_\gamma : \pi_{13}^*L \otimes \pi_{34}^*A_\gamma \longrightarrow \pi_{12}^*A_\gamma \otimes \pi_{24}^*L^{\epsilon(\gamma)}$$

of line bundles over $Z^\gamma \times_M Z^\gamma$ subject to various conditions. The $\mathcal{G}$-module $\mathcal{E} : \mathcal{G} \longrightarrow \mathcal{I}_\omega$ consists of a vector bundle E over Y and of an isomorphism $\rho : L \otimes \pi_2^*E \longrightarrow \pi_1^*E$ over $Y^{[2]}$ satisfying (5.1). The 2-morphisms ρ_γ are isomorphisms

$$\rho_\gamma : \pi_{12}^*A_\gamma \otimes \pi_2^*E^{\epsilon(\gamma)} \longrightarrow \pi_1^*E \tag{5.3}$$

of vector bundles over $Z_1^\gamma \times_P Z_2^\gamma$, see (2.17), which is just Z^γ here. The compatibility condition (2.18) now reads

$$\begin{array}{ccc} \pi_{13}^*L \otimes \pi_{34}^*A_\gamma \otimes \pi_4^*E^{\epsilon(\gamma)} & \longrightarrow & \pi_{12}^*A_\gamma \otimes \pi_2^*E^{\epsilon(\gamma)} \\ {\scriptstyle \mathrm{id}\otimes\pi_{34}^*\rho_\gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_{12}^*\rho_\gamma} \\ \pi_{13}^*L \otimes \pi_3^*E & \underset{\pi_{13}^*\rho}{\longrightarrow} & \pi_1^*E, \end{array} \tag{5.4}$$

and the commutative diagram (5.2), which is a specialization of (2.19), becomes

$$\begin{array}{ccc} \pi_{12}^*A_{\gamma_1} \otimes \pi_{23}^*A_{\gamma_2}^{\epsilon(\gamma_1)} \otimes \pi_3^*E^{\epsilon(\gamma_1\gamma_2)} & \xrightarrow{\pi_{12}^*\varphi_{\gamma_1,\gamma_2}\otimes\mathrm{id}} & \pi_{13}^*A_{\gamma_1\gamma_2} \otimes \pi_3^*E^{\epsilon(\gamma_1\gamma_2)} \\ {\scriptstyle \mathrm{id}\otimes\pi_{23}^*\rho_{\gamma_2}^{\epsilon(\gamma_1)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_{13}^*\rho_{\gamma_1\gamma_2}} \\ \pi_{12}^*A_{\gamma_1} \otimes \pi_2^*E^{\epsilon(\gamma_1)} & \underset{\pi_{12}^*\rho_{\gamma_1}}{\longrightarrow} & \pi_1^*E. \end{array} \tag{5.5}$$

Definition 5.3. A $(\mathcal{G}^a, \mathcal{J}^a)$-module $(\mathcal{E}^a, \rho_\gamma^a)$ and a $(\mathcal{G}^b, \mathcal{J}^b)$-module $(\mathcal{E}^b, \rho_\gamma^b)$ are called *equivalent* if there exists an equivariant 1-isomorphism $(\mathcal{B}, \eta_\gamma) : (\mathcal{G}^a, \mathcal{J}^a) \longrightarrow (\mathcal{G}^b, \mathcal{J}^b)$ and an equivariant 2-isomorphism

$$\nu : (\mathcal{E}^b, \rho_\gamma^b) \circ (\mathcal{B}, \eta_\gamma) \Longrightarrow (\mathcal{E}^a, \rho_\gamma^a).$$

In particular, the bundle gerbes $\mathcal{G}^a$ and $\mathcal{G}^b$ are isomorphic, the 2-forms ω^a and ω^b of the two gerbe modules coincide, and the two modules have the same rank. If the 1-isomorphism $\mathcal{B}$ has a line bundle B over Z and an isomorphism β, this equivariant 2-morphism is just an isomorphism

$$\nu : B \otimes \pi_2^* E^b \longrightarrow \pi_1^* E^a \tag{5.6}$$

of line bundles over Z that satisfies the usual axiom for 2-isomorphisms and the additional equivariance condition (2.22), which now becomes the commutative diagram

$$\begin{array}{ccc} \pi_{12}^* A_\gamma^a \otimes (\pi_{24}^* B \otimes \pi_4^* E^b)^{\epsilon(\gamma)} & \xrightarrow{\pi_{34}^*\rho_\gamma^b \circ \eta_\gamma \otimes \mathrm{id}} & \pi_{13}^* B \otimes \pi_3^* E^b \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\pi_{24}^*\nu^{\epsilon(\gamma)}} & & \Big\downarrow{\scriptstyle \pi_{13}^*\nu\otimes\mathrm{id}} \\ \pi_{12}^* A_\gamma^a \otimes \pi_2^*(E^a)^{\epsilon(\gamma)} & \xrightarrow[\rho_\gamma^a]{} & \pi_1^* E^a. \end{array}$$

Concerning the local data of a $(\mathcal{G},\mathcal{J})$-module, we only have to specialize the local data of an equivariant 1-morphism to the case in which the second bundle gerbe is a trivial one equipped with its canonical (Γ,ϵ)-equivariant structure, see Section 4.1. Thus, let $c = (B_i, A_{ij}, g_{ijk})$ be local data of the bundle gerbe $\mathcal{G}$ with respect to some invariant open cover $\mathfrak{O}$, and let $b_\gamma = (\Pi_i^\gamma, \chi_{ij}^\gamma)$ and $a_{\gamma_1,\gamma_2} = (f_i^{\gamma_1,\gamma_2})$ be local data of the (Γ,ϵ)-equivariant structure. Evidently, the trivial bundle gerbe $\mathcal{I}_\omega$ has local data $c' = (\omega, 0, 1)$, and its canonical equivariant structure $\mathcal{J}_\omega$ has local data $b'_\gamma = (0,1)$ and $a'_{\gamma_1,\gamma_2} = (1)$. A $(\mathcal{G},\mathcal{J})$-module of rank n, i.e., an equivariant 1-morphism

$$(\mathcal{E}, \rho_\gamma) : (\mathcal{G},\mathcal{J}) \longrightarrow (\mathcal{I}_\omega, \mathcal{J}_\omega),$$

has local data $\beta = (\Lambda_i, G_{ij})$ and $\eta_\gamma = (H_i^\gamma)$ satisfying (4.13), (4.14), (4.15) and (4.17). Explicitly, we have 1-forms $\Lambda_i \in \Omega^1(O_i, \mathfrak{u}(n))$, and smooth functions $G_{ij} : O_{ij} \longrightarrow U(n)$ and $H_i^\gamma : O_i \longrightarrow U(n)$. The equations are

$$\omega = B_i + \tfrac{1}{n}\mathrm{tr}(\mathrm{d}\Lambda_i), \quad \Lambda_j = G_{ij}^{-1}\Lambda_i G_{ij} - A_{ij} + \mathrm{i}G_{ij}^{-1}\mathrm{d}G_{ij} \quad \text{and}$$
$$G_{ij} \cdot G_{jk} = g_{ijk} \cdot G_{ik}.$$

These are just the relations for an ordinary $\mathcal{G}$-module, see (2.3) in [10]. Equivariance is expressed by the relations

$$\begin{aligned} \gamma\Lambda_i &= (H_i^\gamma)^{-1}\Lambda_i H_i^\gamma + \mathrm{i}(H_i^\gamma)^{-1}\mathrm{d}H_i^\gamma - \Pi_i^\gamma, \\ \gamma G_{ij} &= (H_i^\gamma)^{-1} \cdot G_{ij} \cdot H_j^\gamma \cdot (\chi_{ij}^\gamma)^{-1}, \end{aligned} \tag{5.7}$$

$$H_i^{\gamma_1\gamma_2} = H_i^{\gamma_1} \cdot \gamma_1 H_i^{\gamma_2} \cdot (f_i^{\gamma_1,\gamma_2})^{-1},$$

where we have used the conventions (4.5) and (4.16).

If a $(\mathcal{G},\mathcal{J})$-module $(\mathcal{E},\rho_\gamma)$ and a $(\mathcal{G}',\mathcal{J}')$-module $(\mathcal{E}',\rho'_\gamma)$ are equivalent in the sense of Definition 5.3, and $(c, b_\gamma, a_{\gamma_1,\gamma_2})$ and $(c', b'_\gamma, a'_{\gamma_1,\gamma_2})$ are local data of the two gerbes, there exist local data (R_i, u_{ij}) and (h_i^γ) of the equivariant 1-isomorphism $(\mathcal{B},\eta_\gamma)$ satisfying (4.26). There are also functions $U_i : O_i \longrightarrow U(n)$ coming from the equivariant 2-morphism ν. If $\beta = (\Lambda_i, G_{ij})$ and $\rho_\gamma = (H_i^\gamma)$ are local data of $(\mathcal{E},\rho_\gamma)$, and, similarly, β' and ρ'_γ are those of $(\mathcal{E}',\rho'_\gamma)$, (4.18) and (4.19) take the form

$$\Lambda'_i = U_i^{-1}\Lambda_i U_i + \mathrm{i}U_i^{-1}\mathrm{d}U_i - R_i, \quad G'_{ij} = U_i^{-1}\cdot G_{ij}\cdot U_j \cdot u_{ij}^{-1} \quad \text{and}$$
$$H_i'^{\gamma} = U_i^{-1}\cdot H_i^\gamma \cdot \gamma U_i \cdot (h_i^\gamma)^{-1}.$$

A particular situation that we shall discuss explicitly is the orientifold group $(\mathbb{Z}_2, \mathrm{id})$, and a bundle gerbe $\mathcal{G}$ with (normalized) Jandl structure $\mathcal{J} = (\mathcal{A}_k, \varphi_{k,k})$. In this situation, we call a $(\mathcal{G},\mathcal{J})$-module a *Jandl module.* Given such a (normalized) Jandl module $(\mathcal{E},\rho_k)$, $\mathcal{E} : \mathcal{G} \longrightarrow \mathcal{I}_\omega$ is a bundle-gerbe module whose curvature satisfies $k^*\omega = -\omega$, and there is a single 2-isomorphism

$$\rho_k : k^*\mathcal{E}^\dagger \circ \mathcal{A}_k \implies \mathcal{E}$$

such that the diagram

$$\begin{array}{ccc} \mathcal{E}\circ k^*\mathcal{A}_k^\dagger \circ \mathcal{A}_k & \xRightarrow{\mathrm{id}\circ\varphi_{k,k}} & \mathcal{E}\circ \mathrm{id}_{\mathcal{G}} \\ {\scriptstyle k^*\rho_k^\dagger\circ\mathrm{id}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle\lambda_{\mathcal{E}}} \\ k^*\mathcal{E}^\dagger\circ\mathcal{A}_k & \xRightarrow[\rho_k]{} & \mathcal{E} \end{array} \tag{5.8}$$

of 2-isomorphisms is commutative. Still more specifically, we assume that there is a trivialization $\mathcal{T} : \mathcal{G} \longrightarrow \mathcal{I}_\rho$. As discussed in Section 2.3, the trivialization and the Jandl structure induce a k-equivariant line bundle (R,ϕ) over M of curvature $-(k^*\rho+\rho)$. This was obtained by applying the functor $\mathcal{B}un$ of (2.28) to the 1-isomorphism

$$\mathcal{R} = k^*\mathcal{T}^\dagger \circ \mathcal{A}_k \circ \mathcal{T}^{-1} : \mathcal{I}_\rho \longrightarrow \mathcal{I}_{-k^*\rho}.$$

In the same way, we form the 1-morphism $\mathcal{E} \circ \mathcal{T}^{-1} : \mathcal{I}_\rho \longrightarrow \mathcal{I}_\omega$, and get a vector bundle $E := \mathcal{B}un(\mathcal{E} \circ \mathcal{T}^{-1})$. We have, further, a 2-isomorphism

$$\begin{array}{ccc} k^*(\mathcal{E}\circ\mathcal{T}^{-1})^\dagger \circ \mathcal{R} = k^*\mathcal{E}^\dagger \circ k^*\mathcal{T}^{\dagger -1} \circ k^*\mathcal{T}^\dagger \circ \mathcal{A}_k \circ \mathcal{T}^{-1} & & \\ \Big\Downarrow {\scriptstyle \mathrm{id}\circ i_l \circ \mathrm{id}} & & \\ k^*\mathcal{E}^\dagger \circ \mathrm{id}_{k^*\mathcal{G}^*} \circ \mathcal{A}_k \circ \mathcal{T}^{-1} & & \\ \Big\Downarrow {\scriptstyle \mathrm{id}\circ \rho_{\mathcal{A}_k} \circ \mathrm{id}} & & \\ k^*\mathcal{E}^\dagger \circ \mathcal{A}_k \circ \mathcal{T}^{-1} & \xRightarrow{\rho_k \circ \mathrm{id}} & \mathcal{E} \circ \mathcal{T}^{-1} \end{array}$$

that induces, via $\mathcal{B}un$, an isomorphism

$$\vartheta : R \otimes k^* E^* \longrightarrow E \tag{5.9}$$

of vector bundles over M. Finally, diagram (5.8) implies that this morphism is compatible with the equivariant structure ϕ on R in the sense that the diagram

$$\begin{array}{ccc} R \otimes k^*R^* \otimes E & \xrightarrow{\mathrm{id}\otimes k^*\vartheta^*} & R \otimes k^*E^* \\ & {\scriptstyle \phi\otimes\mathrm{id}} \searrow \quad \swarrow {\scriptstyle \vartheta} & \\ & E & \end{array}$$

of morphisms of vector bundles over M is commutative. Summarizing, every Jandl module for a trivialized Jandl gerbe gives rise to a vector bundle together with an isomorphism (5.9).

In Section 2, we described the descent theory of twisted-equivariant bundle gerbes as a way to obtain (all) Jandl gerbes over a smooth manifold M'. In the same way, we have

Proposition 5.1. *Let (Γ, ϵ) be an orientifold group for M with Γ_0 acting without fixed points, and let (Γ', ϵ') be the quotient orientifold group for the quotient $M' := M/\Gamma_0$. Then, there is a canonical bijection*

$$\left\{ \begin{array}{c} \textit{Equivalence classes of} \\ \textit{equivariant modules for} \\ (\Gamma,\epsilon)\textit{-equivariant} \\ \textit{bundle gerbes over } M \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \textit{Equivalence classes of equi-} \\ \textit{variant modules for } (\Gamma',\epsilon')\textit{-equi-} \\ \textit{variant bundle gerbes over } M' \end{array} \right\}.$$

Note that the latter proposition unites (as did Theorem 3.1 before) the two cases of $\Gamma_0 = \Gamma$ and $\Gamma/\Gamma_0 = \mathbb{Z}_2$.

Since a $(\mathcal{G}, \mathcal{J})$-module is nothing but an equivariant 1-morphism between $(\mathcal{G}, \mathcal{J})$ and $(\mathcal{I}_\omega, \mathcal{J}_\omega)$, we can apply the descent theory developed in Section 2 and pass to an associated quotient 1-morphism. The only thing to note is that the quotients are $(\mathcal{I}_\omega)' \cong \mathcal{I}_{\omega'}$ for the descended 2-form $\omega' \in \Omega^2(M')$, and $(\mathcal{J}_\omega)' \cong \mathcal{J}_{\omega'}$. But this is clear since all involved line bundles are the trivial ones, and all involved isomorphisms are identities. This way, it becomes obvious how the map in Proposition 5.1 is defined and that it is surjective. It remains to check that it is well-defined on equivalence classes and injective. For this purpose, we have to amend the discussion of Section 2 by providing a descent construction for equivariant 2-morphisms. Suppose that we have a $(\mathcal{G}^a, \mathcal{J}^a)$-module $(\mathcal{E}^a, \rho^a_\gamma)$ and an equivalent $(\mathcal{G}^b, \mathcal{J}^b)$-module $(\mathcal{E}^b, \rho^b_\gamma)$, i.e., there is an equivariant 1-isomorphism $(\mathcal{B}, \eta_\gamma) : (\mathcal{G}^a, \mathcal{J}^a) \longrightarrow (\mathcal{G}^b, \mathcal{J}^b)$ and an equivariant 2-isomorphism

$$\nu : (\mathcal{E}^b, \rho^b_\gamma) \circ (\mathcal{B}, \eta_\gamma) \Longrightarrow (\mathcal{E}^a, \rho^a_\gamma). \tag{5.10}$$

We have to construct an equivariant 2-isomorphism

$$\nu' : (\mathcal{E}^{b\prime}, \rho^{b\prime}_k) \circ (\mathcal{B}', \eta'_k) \Longrightarrow (\mathcal{E}^{a\prime}, \rho^{a\prime}_k), \tag{5.11}$$

which guarantees that the quotient Jandl modules are equivalent. Notice that the 1-morphism on the right side has the vector bundle E^a over Y^a, and the one on the left side has a vector bundle over the disjoint union of $Y^a \times_M Y^b \times_M Y^b_\gamma$ over all $\gamma \in \Gamma_0$, which is defined componentwise as $\pi_{12}^* B \otimes \pi_{23}^* A^b_\gamma \otimes \pi_3^* E^b$. This follows from the definition of quotient 1-morphisms and from Definition 2.4. Thus, the 2-morphism we have to construct has components $\nu'_\gamma : \pi_{12}^* B \otimes \pi_{23}^* A^b_\gamma \otimes \pi_3^* E^b \longrightarrow \pi_1^* E^a$, and we define them as

$$\pi_{12}^* B \otimes \pi_{23}^* A^b_\gamma \otimes \pi_3^* E^b \xrightarrow{\mathrm{id} \otimes \rho^b_\gamma} \pi_{12}^* B \otimes \pi_2^* E^b \xrightarrow{\nu} \pi_1^* E^a,$$

where ν comes from the given 2-morphism as in (5.6). It is straightforward to check that this, indeed, defines an equivariant 2-isomorphism.

Conversely, if an equivariant 1-isomorphism

$$(\mathcal{B}', \eta'_k) : (\mathcal{G}^{a\prime}, \mathcal{J}^{a\prime}) \longrightarrow (\mathcal{G}^{b\prime}, \mathcal{J}^{b\prime})$$

is given, every equivariant 2-isomorphism (5.11) immediately induces an equivariant 2-isomorphism (5.10) for $(\mathcal{B}, \eta_\gamma)$ the equivariant 1-isomorphism constructed on page 652. This shows that the map from Proposition 5.1 is injective.

6 Holonomy for unoriented surfaces

We show that a Jandl gerbe over a smooth manifold M together with Jandl-gerbe modules over submanifolds of M provides a well-defined notion of holonomy for unoriented surfaces with boundary, for example for the Möbius strip. This notion merges the holonomy for unoriented closed surfaces from [23] with that of the holonomy for oriented surfaces with boundary from [5, 11]. In the first subsection, we discuss its definition in terms of geometric structures, and then we develop expressions in terms of local data.

6.1 Geometrical definition

In short, holonomy arises by pulling back a bundle gerbe $\mathcal{G}$ along a smooth map $\phi : \Sigma \longrightarrow M$ to a surface Σ, where it becomes trivializable for dimensional reasons. For any choice of a trivialization $\mathcal{T} : \phi^*\mathcal{G} \longrightarrow \mathcal{I}_\rho$, there is a number

$$\mathrm{Hol}_{\mathcal{G}}(\phi, \Sigma) := \exp\left(\mathrm{i}\int_\Sigma \rho\right) \in U(1). \tag{6.1}$$

The integral requires Σ to be oriented, and its independence of the choice of $\mathcal{T}$ requires Σ to be closed.

If Σ has a boundary, the expression (6.1) is no longer well-defined since a boundary term emerges under a change of the trivialization. We shall assume for simplicity that the boundary has only one connected component. Compensating the boundary term then requires choices of a *$\mathcal{G}$-brane* [5, 10, 11], a submanifold $Q \subset M$ together with a $\mathcal{G}|_Q$-module $\mathcal{E} : \mathcal{G}|_Q \longrightarrow \mathcal{I}_\omega$. The maps $\phi : \Sigma \longrightarrow M$ which we take into account are now supposed to satisfy $\phi(\partial\Sigma) \subset Q$. If E is the vector bundle $\mathcal{B}un(\phi^*\mathcal{E} \circ \mathcal{T}^{-1})$ over $\partial\Sigma$ constructed in Section 5, the formula

$$\mathrm{Hol}_{\mathcal{G},\mathcal{E}}(\phi, \Sigma) := \exp\left(\mathrm{i}\int_\Sigma \rho\right) \cdot \mathrm{tr}\left(\mathrm{Hol}_E(\partial\Sigma)\right) \in \mathbb{C}, \tag{6.2}$$

written in terms of the vector bundle E of $\mathcal{E}$ is invariant under changes of the trivialization $\mathcal{T}$. If the boundary is empty, it reduces to (6.1). A generalization to several $\mathcal{G}$-branes in the case of more than one boundary component is straightforward.

If Σ is unoriented, e.g., if it is unorientable, it is important to notice that there is a unique two-fold covering $\mathrm{pr} : \hat{\Sigma} \longrightarrow \Sigma$, called the *oriented double*, where $\hat{\Sigma}$ is oriented and equipped with an orientation-reversing involution $\sigma : \hat{\Sigma} \longrightarrow \hat{\Sigma}$ that permutes the sheets of $\hat{\Sigma}$ so that $\Sigma = \hat{\Sigma}/\sigma$. To obtain holonomy for unoriented surfaces, two changes in the above setup have to be made [23]. First, the bundle gerbe $\mathcal{G}$ has to be equipped with a Jandl structure, i.e., a twisted-equivariant structure with respect to an involution $k : M \longrightarrow M$. Second, the holonomy is taken for smooth maps $\hat{\phi} : \hat{\Sigma} \longrightarrow M$, which are equivariant in the sense that the diagram

$$\begin{array}{ccc} \hat{\Sigma} & \xrightarrow{\hat{\phi}} & M \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle k} \\ \hat{\Sigma} & \xrightarrow[\hat{\phi}]{} & M \end{array}$$

is commutative. This is just the stack-theoretic way to talk about a smooth map $\Sigma \longrightarrow M/k$ without requiring that the quotient M/k be a smooth manifold.

The pullback of the Jandl gerbe $(\mathcal{G}, \mathcal{J})$ along $\hat{\phi}$ is a Jandl gerbe over the surface $\hat{\Sigma}$, and hence trivializable. As discussed in Section 2.3, any choice of a trivialization $\mathcal{T} : \hat{\phi}^*\mathcal{G} \longrightarrow \mathcal{I}_\rho$ defines a σ-equivariant line bundle $(\hat{R}, \hat{\varphi})$ over $\hat{\Sigma}$ of curvature $-(\sigma^*\rho + \rho)$, which, in turn, descends to a line bundle R over Σ. To define the holonomy, we further need to choose a *fundamental domain* F of Σ in $\hat{\Sigma}$. This is a submanifold $F \subset \hat{\Sigma}$ with (possibly piecewise smooth) boundary such that

$$F \cap \sigma(F) \subset \partial F \quad \text{and} \quad F \cup \sigma(F) = \hat{\Sigma}, \tag{6.3}$$

see figure 1 for an example. In the case of a closed surface Σ, it is a key observation that the involution σ restricts to an orientation-*preserving* involution on the boundary ∂F, so that the quotient $\bar{F} := \partial F/\sigma$ is a closed oriented

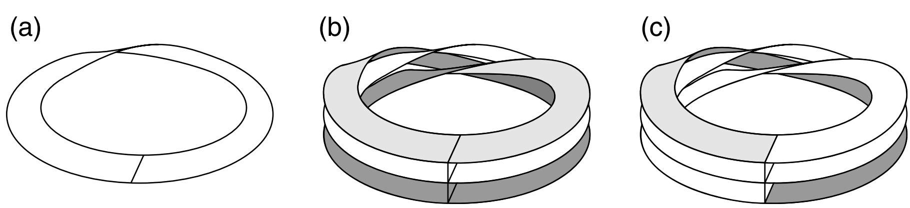

Figure 1: Panel (a) shows the Möbius strip. Panel (b) is a Möbius strip (in the middle layer) together with its oriented double. The latter is an ordinary strip with a bright and a dark side. Panel (c) shows a fundamental domain.

1D submanifold of Σ [23]. Then, the holonomy is defined by

$$\mathrm{Hol}_{\mathcal{G},\mathcal{J}}(\hat{\phi},\Sigma) := \exp\left(\mathrm{i}\int_F \rho\right)\cdot \mathrm{Hol}_R(\bar{F}). \tag{6.4}$$

This expression is invariant under changes of the trivialization $\mathcal{T}$ and of the fundamental domain F [23]. In the case when the surface Σ is orientable, the oriented double $\hat{\Sigma}$ has two global sections $s:\Sigma \longrightarrow \hat{\Sigma}$ intertwined by composition with the involution σ. The choice $F := s(\Sigma)$ with $\partial F = \emptyset$ satisfies $\mathrm{Hol}_{\mathcal{G},\mathcal{J}}(\hat{\phi},\Sigma) = \mathrm{Hol}_{\mathcal{G}}(\hat{\phi}\circ s,\Sigma)$, where on the right side Σ is taken with the orientation pulled back by s from $\hat{\Sigma}$.

Below, we introduce a simultaneous generalization of the formulæ (6.4) and (6.2) appropriate for unoriented surfaces *with* boundary. In addition to the choice of a Jandl structure on the bundle gerbe $\mathcal{G}$, the following new structure will be required.

Definition 6.1. Let $\mathcal{G}$ be a bundle gerbe over M and let $\mathcal{J}$ be a Jandl structure on $\mathcal{G}$ with involution $k:M \longrightarrow M$. A $(\mathcal{G},\mathcal{J})$-brane is a submanifold $Q\subset M$ such that $k(Q)=Q$, together with a $(\mathcal{G},\mathcal{J})|_Q$-module $(\mathcal{E},\rho_k)$.

We consider maps $\hat{\phi}:\hat{\Sigma}\longrightarrow M$ that satisfy the boundary condition $\hat{\phi}(\partial\hat{\Sigma})\subset Q$. As auxiliary data, we choose a trivialization $\mathcal{T}:\hat{\phi}^*\mathcal{G}\longrightarrow \mathcal{I}_\rho$ and obtain the associated σ-equivariant line bundle $(\hat{R},\hat{\varphi})$ over $\hat{\Sigma}$. The pullback of the Jandl module $(\mathcal{E},\rho_k)$ along $\hat{\phi}$ to $\partial\hat{\Sigma}$ yields a Jandl module for the trivialized Jandl gerbe: as discussed in Section 5, it induces a vector bundle E over $\partial\hat{\Sigma}$. A further auxiliary datum is, again, a fundamental domain F of Σ in $\hat{\Sigma}$. In order to account for the boundary, we need to choose a lift (a closed 1D submanifold) $\hat{\ell}\subset\partial\hat{\Sigma}$ of $\partial\Sigma$. It is easy to see that these lifts always exist.

Remark 6.1. If the boundary $\partial\Sigma$ consists of several components, one can choose a separate $(\mathcal{G},\mathcal{J})$-brane for each component ℓ. It is easy to generalize the subsequent discussion to this case.

We now define a 1D oriented closed submanifold $\bar{F}$ that generalizes the one used in the closed case. As a set, it is defined to be

$$\bar{F} := \mathrm{pr}(\partial F\setminus\hat{\ell}), \tag{6.5}$$

see figure 2. This space is equipped with the structure of an oriented 1D piecewise smooth manifold as follows. Let $U\subset\bar{F}$ be a small open neighbourhood. If $U\cap\partial\Sigma=\emptyset$, we have $U=\hat{U}/\sigma$ with $\hat{U}:=\mathrm{pr}^{-1}(U)\subset\partial F$ so that U

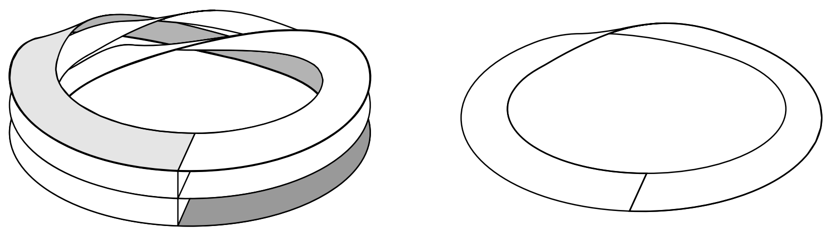

Figure 2: On the left, the Möbius strip with a fundamental domain as in Figure 1, together with a lift $\hat{\ell}$ of the boundary $\partial\Sigma$ (the thick line). On the right, the associated 1D oriented submanifold $\bar{F}$ of Σ.

is 1D and oriented like in the situation for a closed surface. Otherwise, there exists a unique continuous section $s : U \longrightarrow \partial F$ such that $s(U) \cap \hat{\ell} = \emptyset$. This section induces the structure of a 1D and oriented manifold on U. It is easy to see that the orientations coincide on intersections.

Definition 6.2. Let $\mathcal{J}$ be a Jandl structure on a bundle gerbe $\mathcal{G}$ over M, let $(Q, \mathcal{E}, \rho)$ be a $(\mathcal{G}, \mathcal{J})$-brane and let $\hat{\phi} : \hat{\Sigma} \longrightarrow M$ be an equivariant smooth map with $\hat{\phi}(\partial\hat{\Sigma}) \subset Q$. Given a trivialization

$$\mathcal{T} : \hat{\phi}^*\mathcal{G} \longrightarrow \mathcal{I}_\rho,$$

let $\bar{R}$ be the induced line bundle over Σ, and let E be the pullback vector bundle over $\partial\hat{\Sigma}$. Choose, furthermore a fundamental domain F of Σ in its oriented double $\hat{\Sigma}$ and a lift $\hat{\ell}$ of the boundary of Σ. Then, the holonomy along $\hat{\phi}$ is defined as

$$\mathrm{Hol}_{\mathcal{G},\mathcal{J},\mathcal{E}}\big(\hat{\phi}, \Sigma\big) := \exp\left(\mathrm{i}\int_F \rho\right) \cdot \mathrm{Hol}_{\bar{R}}(\bar{F}) \cdot \mathrm{tr}\left(\mathrm{Hol}_E(\hat{\ell})\right).$$

Obviously, the holonomy formulae (6.4) and (6.2) are reproduced for an empty boundary or an oriented Σ, respectively. In particular, formula (6.1) is reproduced for an oriented closed surface.

Theorem 6.1. *Definition* 6.2 *depends neither on the choice of the trivialization* $\mathcal{T}$ *nor on the choice of the fundamental domain* F *nor on the choice of the lift* $\hat{\ell}$.

We give a complete proof of this theorem in the next section in terms of local data. Before we switch to local data, let us elaborate on those properties of the holonomy formula that can conveniently be discussed in the geometric setting.

For example, one can check that the holonomy of Definition 6.2 is independent of the choice of the trivialization. If two trivializations $\mathcal{T} : \hat{\phi}^*\mathcal{G} \longrightarrow \mathcal{I}_\rho$ and $\mathcal{T}' : \hat{\phi}^*\mathcal{G} \longrightarrow \mathcal{I}_{\rho'}$ are present, we form the composition

$$\mathcal{T}' \circ \mathcal{T}^{-1} : \mathcal{I}_\rho \longrightarrow \mathcal{I}_{\rho'}.$$

The functor $\mathcal{B}un$ induces a line bundle $T := \mathcal{B}un(\mathcal{T}' \circ \mathcal{T}^{-1})$ over $\hat{\Sigma}$ of curvature $\rho' - \rho$. The holonomy of T captures the difference that arises in the first factor:

$$\begin{aligned} \exp\left(\mathrm{i}\int_F \rho'\right) &= \exp\left(\mathrm{i}\int_F \rho\right) \cdot \exp\left(\mathrm{i}\int_F \rho' - \rho\right) \\ &= \exp\left(\mathrm{i}\int_F \rho\right) \cdot \mathrm{Hol}_T(\partial F). \end{aligned} \tag{6.6}$$

Notice that $\hat{Q} := \sigma^* T \otimes T$ is a line bundle over $\hat{\Sigma}$ with a canonical σ-equivariant structure given as the permutation of the two tensor factors. From the definition of $\bar{F}$, we find, for the holonomies of T and the descent line bundle Q,

$$\mathrm{Hol}_T(\partial F) = \mathrm{Hol}_Q(\bar{F}) \cdot \mathrm{Hol}_T(\hat{\ell}). \tag{6.7}$$

Let $\hat{R}$ and $\hat{R}'$ be the σ-equivariant line bundles associated to the trivializations $\mathcal{T}$ and $\mathcal{T}'$, respectively. We then obtain an isomorphism

$$\hat{R} \cong \hat{Q} \otimes \hat{R}'$$

of σ-equivariant line bundles over $\hat{\Sigma}$; see the discussion after Definition 10 in [26]. For the descent line bundles, this implies an isomorphism $R \cong Q \otimes R'$, so that

$$\mathrm{Hol}_Q(\bar{F}) \cdot \mathrm{Hol}_{R'}(\bar{F}) = \mathrm{Hol}_R(\bar{F}). \tag{6.8}$$

Concerning the vector bundles E and E', note that we have a 2-isomorphism

$$\mathcal{E} \circ \mathcal{T}'^{-1} \circ \mathcal{T}' \circ \mathcal{T}^{-1} \cong \mathcal{E} \circ \mathcal{T}^{-1},$$

which induces, via the functor $\mathcal{B}un$, an isomorphism $E' \otimes T \cong E$ of vector bundles over $\partial\hat{\Sigma}$. This shows that

$$\mathrm{Hol}_T(\hat{\ell}) \cdot \mathrm{tr}\left(\mathrm{Hol}_{E'}(\hat{\ell})\right) = \mathrm{tr}\left(\mathrm{Hol}_E(\hat{\ell})\right). \tag{6.9}$$

Formulae (6.6)–(6.9) prove that the holonomy in Definition 6.2 does not depend on the choice of the trivialization.

Another result on the holonomy is

Proposition 6.1. *The holonomy for the unoriented surface* Σ *determines a square root of the holonomy for the oriented double,*

$$\Big(\mathrm{Hol}_{\mathcal{G},\mathcal{J},\mathcal{E}}(\hat{\phi},\Sigma)\Big)^2 = \mathrm{Hol}_{\mathcal{G},\mathcal{E}}(\hat{\phi},\hat{\Sigma}).$$

Proof. To see this, one chooses a fundamental domain F and a lift $\hat{\ell}$ for the first of the two factors on the left-hand side, and makes the choices $F' := \sigma(F)$ and $\hat{\ell}' := \sigma(\hat{\ell})$ for the second factor. The square on the left-hand side consists, after reordering of the factors, of

$$\exp\left(\mathrm{i}\int_F \rho\right)\cdot\exp\left(\mathrm{i}\int_{F'}\rho\right) \overset{(6.3)}{=} \exp\left(\mathrm{i}\int_{\hat{\Sigma}}\rho\right)$$

and $\mathrm{Hol}_R(\bar{F})\cdot\mathrm{Hol}_R(\bar{F}') = 1$ (the latter identity follows from the fact that the submanifolds $\bar{F}$ and $\bar{F}'$ are the same sets, but with opposite orientations), as well as of $\mathrm{tr}(\mathrm{Hol}_E(\hat{\ell}))\cdot\mathrm{tr}(\mathrm{Hol}_E(\hat{\ell}')) = \mathrm{tr}(\mathrm{Hol}_E(\partial\hat{\Sigma}))$. Altogether, this reproduces the holonomy formula (6.2) for $\hat{\Sigma}$. $\square$

Finally, let us discuss what happens to the holonomy when we pass to equivalent background data.

Proposition 6.2. *Suppose that* $(\mathcal{B},\eta_k) : (\mathcal{G}^a,\mathcal{J}^a) \longrightarrow (\mathcal{G}^b,\mathcal{J}^b)$ *is an equivariant 1-isomorphism between Jandl gerbes, that* $(\mathcal{E}^a,\rho^a)$ *and* $(\mathcal{E}^b,\rho^b)$ *are Jandl modules for* $(\mathcal{G}^a,\mathcal{J}^a)$ *and* $(\mathcal{G}^b,\mathcal{J}^b)$*, respectively, and that* $\nu : \mathcal{E}^b\circ\mathcal{B} \Longrightarrow \mathcal{E}^a$ *is a 2-isomorphism. Then,*

$$\mathrm{Hol}_{\mathcal{G}^a,\mathcal{J}^a,\mathcal{E}^a}(\hat{\phi},\Sigma) = \mathrm{Hol}_{\mathcal{G}^b,\mathcal{J}^b,\mathcal{E}^b}(\hat{\phi},\Sigma) \tag{6.10}$$

for any smooth equivariant map $\hat{\phi} : \hat{\Sigma} \longrightarrow M$*.*

Proof. We fix the choices of the fundamental domain F and of the lift $\hat{\ell}$ for both sides of (6.10). To compute the holonomy on the right-hand side, we choose a trivialization $\mathcal{T}^b : \hat{\phi}^*\mathcal{G}^b \longrightarrow \mathcal{I}_\rho$. It induces a trivialization $\mathcal{T}^a := \mathcal{T}^b\circ\mathcal{B}$ which we use to compute the left-hand side. Since $\mathcal{T}^a$ and $\mathcal{T}^b$ have the same 2-form ρ, the first factor of the holonomy formula from Definition 6.2 is the same on both sides of (6.10).

Associated to the trivialized Jandl gerbes $\hat{\phi}^*(\mathcal{G}^a, \mathcal{J}^a)$ and $\hat{\phi}^*(\mathcal{G}^b, \mathcal{J}^b)$, there are σ-equivariant line bundles over $\hat{\Sigma}$, as discussed in Section 2.3. By Lemma 2.2, these line bundles are isomorphic as equivariant line bundles and hence induce isomorphic line bundles R^a and R^b over Σ. Isomorphic line bundles have equal holonomies, therefore also the second factor of the holonomy formula from Definition 6.2 coincides on both sides of (6.10).

Finally, we induce a 2-isomorphism

$$\mathcal{E}^b \circ (\mathcal{T}^b)^{-1} \Longrightarrow \mathcal{E}^b \circ \mathcal{B} \circ (\mathcal{T}^b \circ \mathcal{B})^{-1} \xRightarrow{\nu \circ \mathrm{id}} \mathcal{E}^a \circ (\mathcal{T}^a)^{-1} \ ,$$

whose image under the functor $\mathcal{B}un$ yields an isomorphism $E^b \longrightarrow E^a$ of vector bundles over $\partial\hat{\Sigma}$. Again, these vector bundles have equal holonomies, so that also the third factor coincides on both sides. □

Of course, it follows that equivalent Jandl modules have equal holonomies. We remark, however, that the 2-isomorphism ν in Proposition 6.2 does not have to be equivariant. In other words, the holonomy from Definition 6.2 cannot distinguish all equivalence classes of Jandl modules.

6.2 Local-data counterpart

Here, we rewrite the holonomy for unoriented surfaces (with boundary) from Definition 6.2 in terms of local data. Thus, let $\mathfrak{O} = \{O_i\}_{i\in I}$ be an open cover of M, with $k(O_i) = O_{ki}$, that permits to extract local data, namely the data $c = (B_i, A_{ij}, g_{ijk})$ of the bundle gerbe $\mathcal{G}$, the data $b = (\Pi_i, \chi_{ij})$ and $a = (f_i)$ of the Jandl structure $\mathcal{J}$ (see Section 4.1), and the data $\beta = (\Lambda_i, G_{ij})$ and $\phi = (H_i)$ of the $(\mathcal{G}, \mathcal{J})|_Q$-module $(Q, \mathcal{E}, \rho)$, see Section 5. The local data of the bundle gerbe satisfy relations (4.1)–(4.3). For reader's convenience, let us recall the relations between the local data of the Jandl structure and those of the gerbe module, specialized to the present case of the orientifold group $(\mathbb{Z}_2, \mathrm{id})$. Concerning the Jandl structure, these are (4.6) to (4.8), namely

$$\begin{aligned} &-k^*B_{ki} - B_i = \mathrm{d}\Pi_i, \quad -k^*A_{ki\,kj} - A_{ij} = \Pi_j - \Pi_i - \mathrm{i}\chi_{ij}^{-1}\mathrm{d}\chi_{ij} \quad \text{and} \\ &\quad k^*g_{ki\,kj\,kl}^{-1} \cdot g_{ijl}^{-1} = \chi_{ij}^{-1} \cdot \chi_{il} \cdot \chi_{jl}^{-1}, \end{aligned} \tag{6.11}$$

as well as (4.9) to (4.11), namely

$$\begin{aligned} &-k^*\Pi_{ki} + \Pi_i = \mathrm{i}f_i^{-1}\mathrm{d}f_i, \quad k^*\chi_{ki\,kj}^{-1} \cdot \chi_{ij} = f_i^{-1} \cdot f_j \quad \text{and} \\ &\quad k^*f_{ki}^{-1} \cdot f_i^{-1} = 1. \end{aligned} \tag{6.12}$$

Concerning the gerbe module, these are (5.7),

$$\begin{aligned}
-k^*\overline{\Lambda_{ki}} &= H_i^{-1}\cdot\Lambda_i\cdot H_i + \mathrm{i}H_i^{-1}\mathrm{d}H_i - \Pi_i,\\
k^*\overline{G_{ki\,kj}} &= H_i^{-1}\cdot G_{ij}\cdot H_j\cdot\chi_{ij}^{-1} \quad\text{and}\quad 1 = H_i\cdot k^*\overline{H_{ki}}\cdot f_i^{-1}.
\end{aligned} \tag{6.13}$$

From the open cover $\mathfrak{O}$ of M, an equivariant smooth map $\hat{\phi}:\hat{\Sigma}\longrightarrow M$ induces an open cover of $\hat{\Sigma}$ with open sets $\hat{V}_i := \hat{\phi}^{-1}(O_i)$. Let T be a triangulation of Σ which is subordinate to this cover in the following sense. The preimage of each triangular face $t\in T$ in $\hat{\Sigma}$ is supposed to have two connected components, and we require that if $\hat{t}$ is one of these components, there exists an index $i(\hat{t})\in I$ such that $\hat{t}\subset\hat{V}_{i(\hat{t})}$. The indices may be chosen such that

$$i(\sigma(\hat{t})) = ki(\hat{t}).$$

For the edges e and the vertices v, we make similar choices of indices.

According to the prescription from Definition 6.2, we have to choose a fundamental domain. As described in [23], this can be done by selecting one of the two components of the preimage of each face $t\in T$, to be denoted by $\underline{\hat{t}}$. For a sufficiently well-behaved triangulation (e.g., one with trivalent vertices),

$$F := \bigcup_{t\in T}\underline{\hat{t}} \tag{6.14}$$

is a smooth submanifold with piecewise smooth boundary, as required. The subsequent discussion does not use this assumption. Next, we have to choose a trivialization $\mathcal{T}:\hat{\phi}^*\mathcal{G}\longrightarrow\mathcal{I}_\rho$. With respect to the cover $\hat{V}_i$, it has local data $\theta = (\Theta_i,\tau_{ij})$ with

$$(\rho,0,1) = \phi^*c + D_1\theta. \tag{6.15}$$

Finally, we choose a lift $\hat{\ell}$ of $\partial\Sigma$.

Equipped with these choices of F, $\mathcal{T}$ and $\hat{\ell}$, we start to translate the formula of Definition 6.2 into the language of local data. The first factor is

$$\mathcal{F}_1 := \exp\left(\mathrm{i}\int_F\rho\right) = \exp\left(\mathrm{i}\sum_{t\in T}\int_{\underline{\hat{t}}}\rho\right) = \exp\left(\mathrm{i}\sum_{t\in T}\int_{\underline{\hat{t}}}\hat{\phi}^*B_{i(\underline{\hat{t}})} + \mathrm{d}\Theta_{i(\underline{\hat{t}})}\right).$$

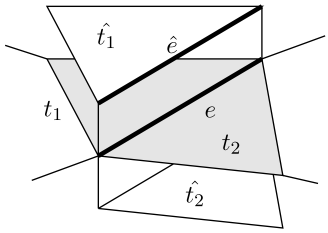

Figure 3: An orientation-reversing edge e between two faces t_1 and t_2 in Σ, which are lifted to different sheets. The edge e itself is, here, lifted to the same sheet as t_1.

Here, the orientation on $\underline{\hat{t}}$ is the one induced from $\hat{\Sigma}$. Using Stokes' Theorem and (6.15), we obtain

$$\mathcal{F}_1 = \prod_{t\in T} \exp\left(\mathrm{i}\int_{\underline{\hat{t}}} \hat{\phi}^* B_{i(\underline{\hat{t}})}\right) \prod_{\hat{e}\in\partial\underline{\hat{t}}} \exp\left(\mathrm{i}\int_{\hat{e}} \hat{\phi}^* A_{i(\underline{\hat{t}})i(\hat{e})} + \Theta_{i(\hat{e})}\right)$$
$$\cdot \prod_{\hat{v}\in\partial\hat{e}} \hat{\phi}^* g^{\epsilon(\hat{v},\hat{e})}_{i(\underline{\hat{t}})i(\hat{e})i(\hat{v})}(\hat{v}) \cdot \tau^{-\epsilon(\hat{v},\hat{e})}_{i(\hat{e})i(\hat{v})}(\hat{v}) \cdot \tau^{\epsilon(\hat{v},\hat{e})}_{i(\underline{\hat{t}})i(\hat{v})}(\hat{v}).$$

Here, the edge $\hat{e}$ has the orientation induced from the boundary of $\underline{\hat{t}}$, and $\epsilon(\hat{v},\hat{e}) = \pm 1$ is negative if, in this orientation, $\hat{v}$ is the starting point of $\hat{e}$, and positive otherwise. We make two manipulations in this formula. First, the very last factor can be dropped since every vertex in a fixed face $\underline{\hat{t}}$ appears twice, each time with a different sign of ϵ. Second, many edges $\hat{e}$ appear twice and with different orientations, so that the corresponding integrals of $\Theta_{i(\hat{e})}$ cancel. More precisely, the edges which appear only once can be of two types. If e is a common edge of two faces t_1 and t_2, we call e *orientation-reversing* whenever $\underline{\hat{t_1}}$ and $\underline{\hat{t_2}}$ have no common edge (see figure 3), and we denote by E the set of orientation-reversing edges. For each orientation-reversing edge $e \in E$, we choose a lift $\underline{\hat{e}}$. The second type of edges that appear only once are those on the boundary of Σ. We denote the set of these edges by B. Any $e \in B$ belongs to a unique face t, and the lift $\underline{\hat{t}}$ determines a lift $\underline{\hat{e}}$ such that $\underline{\hat{e}} \in \partial\underline{\hat{t}}$. We can now write the above formula as

$$\mathcal{F}_1 = \prod_{t\in T} \exp\left(\mathrm{i}\int_{\underline{\hat{t}}} \hat{\phi}^* B_{i(\underline{\hat{t}})}\right) \prod_{\hat{e}\in\partial\underline{\hat{t}}} \exp\left(\mathrm{i}\int_{\hat{e}} \hat{\phi}^* A_{i(\underline{\hat{t}})i(\hat{e})}\right) \prod_{\hat{v}\in\partial\hat{e}} \hat{\phi}^* g^{\epsilon(\hat{v},\hat{e})}_{i(\underline{\hat{t}})i(\hat{e})i(\hat{v})}(\hat{v})$$
$$\times \prod_{e\in E} \exp\left(\mathrm{i}\int_{\underline{\hat{e}}} \Theta_{i(\underline{\hat{e}})} + \sigma^*\Theta_{ki(\underline{\hat{e}})}\right) \prod_{\hat{v}\in\partial\underline{\hat{e}}} \tau^{-\epsilon(\hat{v},\underline{\hat{e}})}_{i(\underline{\hat{e}})i(\hat{v})}(\hat{v}) \cdot \sigma^*\tau^{-\epsilon(\hat{v},\underline{\hat{e}})}_{ki(\underline{\hat{e}})\,ki(\hat{v})}(\hat{v})$$
$$\times \prod_{e\in B} \exp\left(\mathrm{i}\int_{\underline{\hat{e}}} \Theta_{i(\underline{\hat{e}})}\right) \prod_{\hat{v}\in\partial\underline{\hat{e}}} \tau^{-\epsilon(\hat{v},\underline{\hat{e}})}_{i(\underline{\hat{e}})i(\hat{v})}(\hat{v}).$$

The second factor $\mathcal{F}_2 := \mathrm{Hol}_R(\bar{F})$ is more complicated because the line bundle R over Σ whose holonomy we need is only given abstractly as a descended σ-equivariant line bundle $(\hat{R}, \hat{\varphi})$ over $\hat{\Sigma}$. As described in [23], we can compute its holonomy by integrating the local data of $\hat{R}$ along piecewise lifts of $\bar{F}$, and then use the local data of its equivariant structure $\hat{\varphi}$ at points where the lift has a jump. According to the definition of $\hat{R}$, its local data can be determined from the one of the trivialization $\mathcal{T}$ and the Jandl structure by the formula $-\theta + \hat{\phi}^*b + \sigma^*\theta$. Thus, the line bundle $\hat{R}$ has local connection 1-forms

$$\Psi_i := -\Theta_i + \hat{\phi}^*\Pi_i - \sigma^*\Theta_{ki}$$

on $\hat{V}_i$, and transition functions

$$\psi_{ij} := \tau_{ij}^{-1} \cdot \hat{\phi}^*\chi_{ij} \cdot \sigma^*\tau_{ki\,kj}^{-1}$$

on $\hat{V}_i \cap \hat{V}_j$. Its equivariant structure $\hat{\varphi}$ has local data f_i. Mimicking the definition (6.5) of $\bar{F}$, we fix the subset $\bar{B} \subset B$ consisting of those edges e for which $\underline{\hat{e}}$ is *not* contained in the lift $\hat{\ell}$ of $\partial\Sigma$. Note that $\bar{F}$ is simply covered by the lifts $\underline{\hat{e}}$ of edges in $E \cup \bar{B}$ we chose before. As remarked above, these lifts typically do not patch together, i.e., there exist vertices $v \in e_1 \cap e_2$ between pairs of edges $e_1, e_2 \in E \cup \bar{B}$ such that $\underline{\hat{e_1}} \cap \underline{\hat{e_2}} = \emptyset$. In order to take care of these, let us choose, for *all* vertices v, a lift $\underline{\hat{v}}$. Then, the second factor is

$$\mathcal{F}_2 = \prod_{e \in E \cup \bar{B}} \exp\left(\mathrm{i} \int_{\underline{\hat{e}}} \Psi_{i(\underline{\hat{e}})}\right) \prod_{\hat{v} \in \partial\underline{\hat{e}}} \psi_{i(\underline{\hat{e}})i(\hat{v})}^{-\epsilon(\underline{\hat{e}},\hat{v})}(\hat{v}) \cdot \prod_{\underline{\hat{v}} \in \partial\underline{\hat{e}}} \hat{\phi}^* f_{i(\underline{\hat{v}})}^{\epsilon(\underline{\hat{e}},\underline{\hat{v}})}(\underline{\hat{v}}). \tag{6.16}$$

Here, the sign in the exponent of the transition functions ψ_{ij} is due to our conventions for the relation between connection 1-forms and transition functions of a line bundle. Note that a vertex v contributes to the last factor of (6.16) only if it belongs to the adjacent edges in $E \cup \bar{B}$ whose lifts do *not* patch together.

As for the third factor, local data of the vector bundle E are provided by the expression $\hat{\phi}^*\beta - \theta$. For each edge $e \in B$, we denote by $\hat{e}_{\hat{\ell}}$ the corresponding lift such that $\hat{e}_{\hat{\ell}} \subset \hat{\ell}$. Then,

$$\mathcal{F}_3 := \mathrm{tr}\left(\mathrm{Hol}_E(\hat{\ell})\right) = \mathrm{tr}\, \mathcal{P} \prod_{e \in B} \exp\left(\mathrm{i} \int_{\hat{e}_{\hat{\ell}}} \hat{\phi}^*\Lambda_{i(\hat{e}_{\hat{\ell}})} - \Theta_{i(\hat{e}_{\hat{\ell}})}\right)$$
$$\times \prod_{\hat{v} \in \partial\hat{e}_{\hat{\ell}}} \hat{\phi}^* G_{i(\hat{e}_{\hat{\ell}})i(\hat{v})}^{-\epsilon(\hat{v},\hat{e}_{\hat{\ell}})}(\hat{v}) \cdot \tau_{i(\hat{e}_{\hat{\ell}})i(\hat{v})}^{\epsilon(\hat{v},\hat{e}_{\hat{\ell}})}(\hat{v}),$$

where the symbol $\mathcal{P}$ indicates that the edges e have to be ordered according to the orientation on $\hat{\ell}$ (which is the one induced from $\partial\hat{\Sigma}$). Since we take the trace, it does not matter at which vertex one starts.

We may now compute the product $\mathcal{F}_1 \cdot \mathcal{F}_2 \cdot \mathcal{F}_3$, which is, by construction, the holonomy of Definition 6.2. We claim that all occurrences of Θ_i and τ_{ij} drop out: First of all, for each edge $e \in E$, the contributions from $\mathcal{F}_1$ and $\mathcal{F}_2$ are

$$\exp\left(\mathrm{i}\int_{\underline{\hat{e}}} \Theta_{i(\underline{\hat{e}})} + \sigma^*\Theta_{ki(\underline{\hat{e}})}\right) \cdot \exp\left(\mathrm{i}\int_{\underline{\hat{e}}} -\Theta_{i(\underline{\hat{e}})} - \sigma^*\Theta_{ki(\underline{\hat{e}})}\right) \\ \times \prod_{\hat{v}\in\partial\underline{\hat{e}}} \tau^{-\epsilon(\hat{v},\underline{\hat{e}})}_{i(\underline{\hat{e}})i(\hat{v})}(\hat{v}) \cdot \sigma^*\tau^{-\epsilon(\hat{v},\underline{\hat{e}})}_{ki(\underline{\hat{e}})\,ki(\hat{v})}(\hat{v}) \cdot \tau^{\epsilon(\hat{v},\underline{\hat{e}})}_{i(\underline{\hat{e}})i(\hat{v})}(\hat{v}) \cdot \sigma^*\tau^{\epsilon(\hat{v},\underline{\hat{e}})}_{ki(\underline{\hat{e}})\,ki(\hat{v})}(\hat{v}),$$

which is obviously equal to 1. For each edge $e \in \bar{B}$, we have contributions from all three factors, namely

$$\exp\left(\mathrm{i}\int_{\underline{\hat{e}}} \Theta_{i(\underline{\hat{e}})}\right) \cdot \exp\left(\mathrm{i}\int_{\underline{\hat{e}}} -\Theta_{i(\underline{\hat{e}})} - \sigma^*\Theta_{ki(\underline{\hat{e}})}\right) \cdot \exp\left(\mathrm{i}\int_{\hat{e}_{\hat{\ell}}} -\Theta_{i(\hat{e}_{\hat{\ell}})}\right) \\ \times \prod_{\hat{v}\in\partial\underline{\hat{e}}} \tau^{-\epsilon(\hat{v},\underline{\hat{e}})}_{i(\underline{\hat{e}})i(\hat{v})}(\hat{v}) \cdot \tau^{\epsilon(\hat{v},\underline{\hat{e}})}_{i(\underline{\hat{e}})i(\hat{v})}(\hat{v}) \cdot \sigma^*\tau^{\epsilon(\hat{v},\underline{\hat{e}})}_{ki(\underline{\hat{e}})\,ki(\hat{v})}(\hat{v}) \cdot \prod_{\hat{v}\in\partial\hat{e}_{\hat{\ell}}} \tau^{\epsilon(\hat{v},\hat{e}_{\hat{\ell}})}_{i(\hat{e}_{\hat{\ell}})i(\hat{v})}(\hat{v}).$$

By the definition of $\bar{B}$, we have $\underline{\hat{e}} = \sigma(\hat{e}_{\hat{\ell}})$, with opposite orientations. Hence, these contributions also cancel out. For the remaining edges $e \in B \setminus \bar{B}$, we only have contributions from $\mathcal{F}_1$ and $\mathcal{F}_3$, namely

$$\exp\left(\mathrm{i}\int_{\underline{\hat{e}}} \Theta_{i(\underline{\hat{e}})}\right) \prod_{\hat{v}\in\partial\underline{\hat{e}}} \tau^{-\epsilon(\hat{v},\underline{\hat{e}})}_{i(\underline{\hat{e}})i(\hat{v})}(\hat{v}) \cdot \exp\left(\mathrm{i}\int_{\hat{e}_{\hat{\ell}}} -\Theta_{i(\hat{e}_{\hat{\ell}})}\right) \prod_{\hat{v}\in\partial\hat{e}_{\hat{\ell}}} \tau^{\epsilon(\hat{v},\hat{e}_{\hat{\ell}})}_{i(\hat{e}_{\hat{\ell}})i(\hat{v})}(\hat{v}).$$

Here, we know that $\underline{\hat{e}} = \hat{e}_{\hat{\ell}}$, so that these terms cancel out, too. Finally, we end up with the following *local holonomy formula*:

$$\mathrm{Hol}_{\mathcal{G},\mathcal{J},\mathcal{E}}(\hat{\phi},\Sigma) = \prod_{t\in T} \exp\left(\mathrm{i}\int_{\underline{\hat{t}}} \hat{\phi}^* B_{i(\underline{\hat{t}})}\right) \prod_{\hat{e}\in\partial\underline{\hat{t}}} \exp\left(\mathrm{i}\int_{\hat{e}} \hat{\phi}^* A_{i(\underline{\hat{t}})i(\hat{e})}\right) \\ \times \prod_{\hat{v}\in\partial\hat{e}} \hat{\phi}^* g^{\epsilon(\hat{v},\hat{e})}_{i(\underline{\hat{t}})i(\hat{e})i(\hat{v})}(\hat{v}) \prod_{e\in E\cup\bar{B}} \exp\left(\mathrm{i}\int_{\underline{\hat{e}}} \hat{\phi}^*\Pi_{i(\underline{\hat{e}})}\right)$$

$$\times \prod_{\hat{v}\in\partial\underline{\hat{e}}} \hat{\phi}^*\chi_{i(\underline{\hat{e}})i(\hat{v})}^{-\epsilon(\hat{v},\underline{\hat{e}})}(\hat{v}) \cdot \prod_{\underline{\hat{v}}\in\partial\underline{\hat{e}}} \hat{\phi}^* f_{i(\underline{\hat{v}})}^{\epsilon(\underline{\hat{v}},\underline{\hat{e}})}(\underline{\hat{v}})$$

$$\times \prod_{e\in B} \mathrm{tr}\, \mathcal{P}\exp\left(\mathrm{i}\int_{\hat{e}_{\hat{\ell}}} \hat{\phi}^*\Lambda_{i(\hat{e}_{\hat{\ell}})}\right) \prod_{\hat{v}\in\partial\hat{e}_{\hat{\ell}}} \hat{\phi}^* G_{i(\hat{e}_{\hat{\ell}})i(\hat{v})}^{-\epsilon(\hat{v},\hat{e}_{\hat{\ell}})}(\hat{v}). \quad (6.17)$$

Let us briefly review where the particular terms come from. The first line pairs up the local data of the bundle gerbe with the triangulation of the fundamental domain. The second line takes care of the orientation-reversing edges. It pairs up the local data of the Jandl structure with lifts of these edges, and compensates inconsistent lifts. The third line pairs up the local data of the gerbe module with $\hat{\ell}$.

If the boundary is empty, $B = \emptyset$, then formula (6.17) reduces exactly to the one given in [23] for the holonomy of closed unoriented surfaces written in terms of local data. If the surface is oriented, we can make choices such that $E = \bar{B} = \emptyset$, so that the formula reduces to the one given in [10] for the holonomy of oriented surfaces with boundary. Finally, if the surface is oriented and closed, only the first line survives and reduces to the formula found in [1, 9].

We remark that the local data θ of the trivialization have vanished completely from the formula (6.17). This reflects the independence of Definition 6.2 of the choice of trivializations demonstrated in the previous section. Similarly, Proposition 6.2 implies that (6.17) is independent of the choice of local data of the bundle gerbe, the Jandl structure and the equivariant gerbe module. It is a good exercise to check this directly by showing that the local holonomy formula (6.17) is independent of the choices of the subordinated indices i, the lifts $\underline{\hat{e}}$ and $\underline{\hat{v}}$, and that it remains unaltered when passing to a finer triangulation or to cohomologically equivalent local data. In order to complete the proof of Theorem 6.1, it is then enough to show that the local expression (6.17) is independent of the choice of the lifts $\underline{\hat{t}}$ of the faces of the triangulation T and of that of the lift $\hat{\ell}$ of the boundary ℓ.

Let us first suppose that we replace a triangle $\underline{\hat{t}}$ by $\underline{\hat{t}}'$ differing from $\sigma(\underline{\hat{t}})$ by the orientation or, in short, $\underline{\hat{t}}' = -\sigma(\underline{\hat{t}})$. We write the first line of the local holonomy formula (6.17) as

$$\prod_{t\in T} H(\underline{\hat{t}}),$$

where $H(\hat{\underline{t}})$ is the contribution of the face t with the choice $\hat{\underline{t}}$ of the lift. One obtains after simple algebra employing relations (6.11):

$$H(\hat{\underline{t}}') = H(\hat{\underline{t}}) \cdot \prod_{\hat{e} \in \partial\hat{\underline{t}}} I(\hat{e}) \quad \text{with}$$

$$I(\hat{e}) := \exp\left(\mathrm{i} \int_{\hat{e}} \hat{\phi}^* \Pi_{i(\hat{e})} \right) \cdot \prod_{\hat{v} \in \partial\hat{e}} \hat{\phi}^* \chi_{i(\hat{e}) i(\hat{v})}^{-\epsilon(\hat{v},\hat{e})}(\hat{v}).$$

To compute the changes in the second line, let $E_t := \partial t \cap (E \cup \bar{B})$ be the set of those edges of t that are either orientation-reversing or located on the boundary. We may assume that the edges $\hat{\underline{e}}$ chosen for them satisfy $\hat{\underline{e}} \in \partial\hat{\underline{t}}$. Under the replacement of $\hat{\underline{t}}$ by $\hat{\underline{t}}'$, the set E_t changes to the complementary set of edges of t and we may assume that the edges $\hat{\underline{e}}'$ chosen for them satisfy $\hat{\underline{e}}' \in \partial\hat{\underline{t}}'$. We have using relations (6.12):

$$\prod_{e \in \partial t \setminus E_t} I(\hat{\underline{e}}') = \prod_{e \in \partial t \setminus E_t} I(\hat{e})^{-1} \cdot \prod_{\hat{v} \in \partial\hat{e}} \hat{\phi}^* f_{i(\hat{v})}^{-\epsilon(\hat{v},\hat{e})}(\hat{v}), \tag{6.18}$$

where $\hat{e}$ denotes the edge in $\hat{\underline{t}}$ projecting to $e \subset t$ (with the orientation induced from $\hat{\underline{t}}$) and where we have used the fact that, for $e \not\subset E_t$, $\hat{\underline{e}}'$ and $\sigma(\hat{e})$ differ only by the orientation, i.e., $\hat{\underline{e}}' = -\sigma(\hat{e})$. Note that

$$\prod_{\hat{e} \in \partial\hat{\underline{t}}} I(\hat{e}) \ \cdot \prod_{e \in \partial t \setminus E_t} I(\hat{e})^{-1} = \prod_{e \in E_t} I(\hat{e}),$$

which is a needed expression, a part of the original second line. The remaining factors

$$\prod_{e \in \partial t \setminus E_t} \prod_{\hat{v} \in \partial\hat{e}} \hat{\phi}^* f_{i(\hat{v})}^{-\epsilon(\hat{v},\hat{e})}(\hat{v})$$

from (6.18) compensate the remaining changes in the second line. Indeed, again with $\hat{e} \in \partial\hat{\underline{t}}$ and $\hat{\underline{v}}' = \sigma(\hat{\underline{v}})$,

$$\begin{aligned}
&\prod_{e \in \partial t \setminus E_t} \cdot \prod_{\hat{\underline{v}} \in \partial\hat{\underline{e}}'} \hat{\phi}^* f_{i(\hat{\underline{v}})}^{\epsilon(\hat{\underline{v}},\hat{\underline{e}}')}(\hat{v}) \cdot \prod_{e \in \partial t \setminus E_t} \cdot \prod_{\hat{v} \in \partial\hat{e}} \hat{\phi}^* f_{i(\hat{v})}^{-\epsilon(\hat{v},\hat{e})}(\hat{v}) \\
&= \prod_{e \in \partial t \setminus E_t} \cdot \prod_{\hat{\underline{v}}' \in \partial\hat{e}} \hat{\phi}^* f_{i(\hat{\underline{v}}')}^{\epsilon(\hat{\underline{v}}',\hat{e})}(\hat{v}') \cdot \prod_{e \in \partial t \setminus E_t} \cdot \prod_{\hat{v} \in \partial\hat{e}} \hat{\phi}^* f_{i(\hat{v})}^{-\epsilon(\hat{v},\hat{e})}(\hat{v}) \\
&= \prod_{e \in \partial t \setminus E_t} \cdot \prod_{\hat{\underline{v}} \in \partial\hat{e}} \hat{\phi}^* f_{i(\hat{\underline{v}})}^{-\epsilon(\hat{\underline{v}},\hat{e})}(\hat{\underline{v}}) \\
&= \prod_{e \in E_t} \cdot \prod_{\hat{\underline{v}} \in \partial\hat{\underline{e}}} \hat{\phi}^* f_{i(\hat{\underline{v}})}^{\epsilon(\hat{\underline{v}},\hat{\underline{e}})}(\hat{\underline{v}}),
\end{aligned}$$

where the last equality follows from the identity

$$\prod_{\hat{e}\in\partial\hat{\underline{t}}} \cdot \prod_{\hat{\underline{v}}\in\partial\hat{e}} \hat{\phi}^* f_{i(\hat{\underline{v}})}^{\epsilon(\hat{\underline{v}},\hat{e})}(\hat{\underline{v}}) = 1.$$

We infer that the local holonomy formula does not change when $\hat{\underline{t}}$ is replaced by $\hat{\underline{t}}'$.

Next, we want to analyze the effect of the change of the lift of ℓ from $\hat{\ell}$ to $\hat{\ell}' = -\sigma(\hat{\ell})$. Only the lines two and three of the local holonomy formula (6.17) change in this case. In the third line L_3, we find a change of the form

$$L_3' = L_3 \cdot \prod_{e\in B} I(\hat{e}_\ell)^{-1}.$$

Here we have used the relations (6.13) and the identities

$$\mathrm{tr}(\overline{G}) = \mathrm{tr}(G^{-1}) \quad \text{and} \quad \mathrm{tr}\left(\mathrm{e}^{\mathrm{i}\,\overline{\Lambda}}\right) = \mathrm{tr}\left(\mathrm{e}^{\mathrm{i}\Lambda}\right)$$

valid for $G \in U(n)$ and $\Lambda \in \mathfrak{u}(n)$. In the second line L_2, we find

$$L_2' = \prod_{e\in B\setminus\bar{B}} I(\hat{e}_{\hat{\ell}}) \cdot \prod_{\hat{\underline{v}}\in\partial\hat{e}_{\hat{\ell}}} \hat{\phi}^* f_{i(\hat{\underline{v}})}^{\epsilon(\hat{\underline{v}},\hat{e}_{\hat{\ell}})}(\hat{\underline{v}})$$

where $\hat{e}_{\hat{\ell}} \subset \hat{\ell}$ is taken with the orientation inherited from $\hat{\ell}$. Multiplying both lines together, we obtain

$$L_3' \cdot L_2' = L_3 \cdot \prod_{e\in\bar{B}} I(\hat{e}_{\hat{\ell}})^{-1} \cdot \prod_{e\in B\setminus\bar{B}} \prod_{\hat{\underline{v}}\in\partial\hat{e}_{\hat{\ell}}} \hat{\phi}^* f_{i(\hat{\underline{v}})}^{\epsilon(\hat{\underline{v}},\hat{e}_{\hat{\ell}})}(\hat{\underline{v}})\,.$$

On the right-hand side, we may pass back from $\hat{e}_{\hat{\ell}}$ to $\hat{e}_{\hat{\ell}'} = -\sigma(\hat{e}_{\hat{\ell}})$ using (6.12): to obtain

$$\prod_{e\in\bar{B}} I(\hat{e}_{\hat{\ell}})^{-1} = \prod_{e\in\bar{B}} I(\hat{e}_{\hat{\ell}'}) \prod_{\hat{v}\in\partial\hat{e}_{\hat{\ell}'}} \hat{\phi}^* f_{i(\hat{v})}^{\epsilon(\hat{v},\hat{e}_{\hat{\ell}'})}(\hat{v})$$

and, for $\hat{\underline{v}}' = \sigma(\hat{\underline{v}})$,

$$\begin{aligned}\prod_{e\in B\setminus\bar{B}} \prod_{\hat{\underline{v}}\in\partial\hat{e}_{\hat{\ell}}} \hat{\phi}^* f_{i(\hat{\underline{v}})}^{\epsilon(\hat{\underline{v}},\hat{e}_{\hat{\ell}})}(\hat{\underline{v}}) &= \prod_{e\in B\setminus\bar{B}} \prod_{\hat{\underline{v}}'\in\partial\hat{e}_{\hat{\ell}'}} \hat{\phi}^* f_{i(\hat{\underline{v}}')}^{\epsilon(\hat{\underline{v}}',\hat{e}_{\hat{\ell}'})}(\hat{\underline{v}}') \\ &= \prod_{e\in\bar{B}} \prod_{\hat{\underline{v}}'\in\partial\hat{e}_{\hat{\ell}'}} \hat{\phi}^* f_{i(\hat{\underline{v}}')}^{-\epsilon(\hat{\underline{v}}',\hat{e}_{\hat{\ell}'})}(\hat{\underline{v}}')\,,\end{aligned}$$

where the last equality follows from the obvious identity

$$\prod_{e\in B}\ \prod_{\underline{\hat{v}}'\in\partial\hat{e}_{\hat{\ell}'}} f_{i(\underline{\hat{v}}')}^{\epsilon(\underline{\hat{v}}',\hat{e}_{\hat{\ell}'})}(\underline{\hat{v}}') \;=\; 1\,.$$

Upon performing all these transformations, we arrive at the formula:

$$L_3'\cdot L_2' = L_3\,\cdot \prod_{e\in\bar{B}} I(\hat{e}_{\hat{\ell}'})\cdot \prod_{\underline{\hat{v}}\in\partial\hat{e}_{\hat{\ell}'}} \hat{\phi}^* f_{i(\underline{\hat{v}})}^{\epsilon(\underline{\hat{v}},\hat{e}_{\hat{\ell}'})}(\underline{\hat{v}})\,.$$

Noting that $\hat{e}_{\hat{\ell}'} = \underline{\hat{e}}$ for $e\in\bar{B}$, we identify the right-hand side with $L_3\cdot L_2$. This ends the proof of the independence of the local expression (6.17) of the lift $\hat{\ell}$.

Thus, altogether, the local holonomy formula (6.17) is independent of all the arbitrary choices made. Accordingly, also the geometric holonomy formula from Definition 6.2 is manifestly associated only to the bundle gerbe, its Jandl structure and its modules, and, of course to the equivariant map $\hat{\phi}:\hat{\Sigma}\to M$. This proves Theorem 6.1.

7 Conclusions

We considered in this paper manifolds M equipped with a closed 3-form H and an orientifold-group action. The latter is an action of a finite group Γ on M such that, for $\gamma\in\Gamma$, one has $\gamma^*H=\epsilon(\gamma)H$ for a homomorphism $\epsilon:\Gamma\to\{\pm1\}$. We introduced the notion of a (Γ,ϵ)-equivariant (or twisted-equivariant) structure on a gerbe $\mathcal{G}$ over M with curvature H. This notion extends that of a so-called Jandl structure introduced in [23], to which it reduces for $\Gamma=\{\pm1\}$ and $\epsilon(\pm1)=\pm1$.

In the case of $\Gamma_0=\ker(\epsilon)$ acting on M without fixed points, equivalence classes of (Γ,ϵ)-equivariant gerbes over M were shown to descend to equivalence classes of gerbes over $M'=M/\Gamma_0$, with (Γ',ϵ')-equivariant structures for $\Gamma'=\Gamma/\Gamma_0$ and ϵ' induced from ϵ. For $\Gamma_0=\Gamma$, this gives a way to construct gerbes over M' from gerbes over M, and for $\Gamma/\Gamma_0=\mathbb{Z}_2$, it enables to construct gerbes with Jandl structure over M'. Working with local data, we showed that equivalence classes of (Γ,ϵ)-equivariant gerbes can be identified with classes of the 2^{nd} hypercohomology group of a double complex of chains on Γ with values in the (real) Deligne complex in degree 2. This identification permitted to study the obstructions to the existence of (Γ,ϵ)-equivariant structures on a given gerbe $\mathcal{G}$ with curvature H. In the case of

2-connected manifolds, the unique obstruction takes values in the cohomology group $H^3(\Gamma, U(1)_\epsilon)$, where the coefficient group $U(1)$ is taken with the action $(\gamma, u) \longmapsto u^{\epsilon(\gamma)}$ of Γ. If this obstruction vanishes, equivalence classes of (Γ, ϵ)-equivariant structures on $\mathcal{G}$ are parameterized by cohomology classes in $H^2(\Gamma, U(1)_\epsilon)$. This agrees with the purely local analysis of [14].

In [14], these results were applied to the case of gerbes $\mathcal{G}_k$ with curvature $H = \frac{k}{12\pi}\,\mathrm{tr}\, g^{-1}dg^{\wedge 3}$ over simple simply-connected compact Lie groups G, for integer k. We considered there the orientifold groups $\Gamma = \mathbb{Z}_2 \ltimes Z$ with Z a subgroup of the center $Z(G)$ of G acting on G by multiplication, and the nontrivial element of $\mathbb{Z}_2$ sending $g \in G$ to $(\zeta g)^{-1}$ for $\zeta \in Z(G)$. In that paper, we also computed the classes $[u] \in H^3(\Gamma, U(1)_\epsilon)$ obstructing the existence of (Γ, ϵ)-equivariant structures on the gerbes $\mathcal{G}_k$ and found the trivializing chains v such that $u = \delta v$ whenever $[u]$ vanishes. These data enter an explicit construction of (Γ, ϵ)-equivariant structures on the gerbes $\mathcal{G}_k$ that will be described in [13] in analogy to the construction of [12] for the orbifold group $\Gamma = Z$ with trivial ϵ. Such structures on $\mathcal{G}_k$ permit to construct orientifold WZW models for closed surfaces.

With applications to the boundary field theories in view, we discussed above twisted-equivariant gerbe modules, and their equivalence, as well as the descent theory for them. These results will be used to construct boundary orientifold WZW models. The construction, extending the one of [10] for the orbifold case, is postponed to [13]. We also plan to compare in [13] our geometric approach to WZW orientifolds to the algebraic ones of [3, 8].

The (Γ, ϵ)-equivariant structures on gerbes and gerbe modules are used to define the contribution of the H-flux to the Feynman amplitudes of the orientifold sigma models. Such contributions describe the gerbe holonomy along surfaces in M defined by classical fields, with contributions from gerbe modules in the case of surfaces with boundary. We discussed above the holonomy for surfaces in the particular case of Jandl structures in both geometric and local terms, extending the discussion of [23] to the boundary case. In [13], we shall relate the surface holonomy to the more standard loop-holonomy of connections on line and vector bundles with (Γ, ϵ)-equivariant structures over spaces of closed and open curves ("strings") in M. Such structures will be obtained from twisted-equivariant gerbes and gerbe modules by transgression, see [11] for the discussion of the orbifold case. They play an important role in the geometric quantization of orientifold sigma models where the equivariant sections of the bundles over the spaces of curves represent quantum states of the theory. This is the approach that we will adopt in [13] for the orientifolds of boundary WZW models.

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