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Global causal propagator for the Klein–Gordon equation on a class of supersymmetric AdS backgrounds

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Abstract

We analyze the Cauchy problem for the Klein–Gordon equation on the type IIB supergravity backgrounds $\mathrm{AdS}^5 \times Y^{p,q}$, where $Y^{p,q}$ is any of the Sasaki–Einstein 5-manifolds introduced by Gaunlett *et al.* (Adv. Theor. Math. Phys. **8** (2004) 711–734). Although these spaces are not globally hyperbolic, we prove that there exists a unique admissible propagator and derive an integral representation thereof using spectral-theoretic techniques.

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1 Introduction

Sasaki–Einstein manifolds, that is Einstein manifolds whose metric cone is Calabi–Yau, are of considerable interest in Physics because of their connections to the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [2]. More precisely, if Y is a Sasaki–Einstein 5-manifold, then the product manifold $\mathrm{AdS}^5 \times Y$ is a solution of type IIB supergravity that is conjectured to be dual to an $\mathcal{N} = 1$ superconformal field theory in four dimensions. In particular, the AdS/CFT correspondence implies that the asymptotic behavior at infinity of the Klein–Gordon propagator will carry information on the correlation functions of the dual superconformal field theory in four dimensions [2].

The first explicit examples of Sasaki–Einstein geometries in five dimensions (other than the round 5-sphere, the homogeneous metric $T^{1,1}$ on $S^2 \times S^3$ and some quotients thereof) were discovered by Gauntlett *et al.* in [9]. These manifolds, labeled by pairs of integers (p, q) and denoted by $Y^{p,q}$, are constructed as S^1-bundles over an axially squashed S^2-bundle over the 2-sphere. A description of the spaces $Y^{p,q}$ as cohomogeneity-1 manifolds has been given by Conti [6], while examples of Sasaki–Einstein manifolds in higher dimensions have been obtained in [7, 10]. The associated family of supergravity solutions $\mathrm{AdS}^5 \times Y^{p,q}$ includes both quasi-regular manifolds [5], which are dual to superconformal field theories with compact R-symmetry and rational central charges, and irregular manifolds, dual to field theories with noncompact R-symmetry and irrational central charges. A detailed construction of the dual superconformal quiver gauge theories was presented in [4].

In this paper, we aim to rigorously analyze the Cauchy problem for the Klein–Gordon equation on the manifolds $\mathrm{AdS}^5 \times Y^{p,q}$ through the construction of a global causal propagator. Although the spaces $\mathrm{AdS}^5 \times Y^{p,q}$ are not globally hyperbolic, we will see that under mild technical assumptions a unique propagator exists and can be constructed explicitly through spectral methods. Our main result (Theorem 4.1 and Corollary 4.1) is a spectral integral representation for this propagator.

The approach we take exploits the separability of the $\mathrm{AdS}^5 \times Y^{p,q}$ metrics to compute the eigenfunctions of the Laplace operator in $Y^{p,q}$ in quasi-closed form, by expressing them in terms of the eigenfunctions of the Friedrichs extension of a single second-order ordinary differential operator with seven regular singular points. The subtle geometry of the spaces $Y^{p,q}$ introduces additional complications in the analysis, since the ‘angular’ variables in which the metric of $Y^{p,q}$ separates are not defined globally. In order to circumvent this problem, we start by constructing a Fourier-type decomposition of the space of square-integrable functions on $Y^{p,q}$ adapted to the global structure of the manifold and to the action of the Laplacian. Once the eigenfunctions of the Laplacian in $Y^{p,q}$ have been computed, the analysis of the Klein–Gordon equation in $\mathrm{AdS}^5 \times Y^{p,q}$ can be reduced to that of a family of linear hyperbolic equations in anti-de Sitter space. We discuss the existence and uniqueness of causal propagators for these equations using Ishibashi and Wald's spectral-theoretic approach to wave equations on static space-times [11, 12, 19]. For our purpose, this presents several advantages over the classical method of Riesz transforms, since the latter method only yields local solutions to the Cauchy problem in the case in which the underlying space-time is not globally hyperbolic [3].

Our paper is organized as follows. In Section 2, we introduce a Fourier-type decomposition of $L^2(Y^{p,q})$ (Lemma 2.2), which is crucial to the rest of the paper. In Section 3, we use this Fourier decomposition to compute the eigenfunctions of the Laplacian in $Y^{p,q}$ (Theorem 3.1). The expression of these eigenfunctions is totally explicit and involves the spectral decomposition of a single ordinary differential operator, which we analyze in Lemma 3.2. Finally, in Section 4 we prove that there exists a unique physically admissible propagator for the Klein–Gordon equation in $\mathrm{AdS}^5 \times Y^{p,q}$ and derive an integral representation thereof for both the homogeneous and inhomogeneous Cauchy problem (Theorem 4.1 and Corollary 4.1). This paper concludes with an appendix where we recall the conditions that a propagator of a linear wave equation must satisfy in order to be physically admissible.

2 Fourier-type expansions in $Y^{p,q}$

After recalling some geometric facts about the family of cohomogeneity-1 Sasaki–Einstein 5-manifolds $Y^{p,q}$ recently discovered in [9], our goal for this section is to introduce a Fourier-type decomposition of the space of square-integrable functions on $Y^{p,q}$ which is adapted to the geometry of the manifold. This decomposition will be of crucial importance in the rest of the paper. Each manifold $Y^{p,q}$, labeled by two positive integers $p < q < 2p$, is an S^1-bundle over an axially squashed S^2-bundle B over a round 2-sphere. It should be noted that the integers p and q are not exactly the same as the ones labeling the spaces $Y^{p,q}$ in [9]; passing from one set of integers to the other is straightforward, but for our purposes it is slightly more convenient to define the labeling integers as we will do below. We will recall several results on the global geometry of the spaces $Y^{p,q}$ without further mention as we need them; proofs of these facts and further discussion can be found in [9, 15, 16].

We begin our discussion with the four-dimensional sphere bundle B over S^2. We start with the local metric

$$g_B := \frac{\mathrm{d}y^2}{w(y)\, r(y)} + \frac{r(y)}{9}(\mathrm{d}\psi - \cos\theta\, \mathrm{d}\phi)^2 + \frac{1-y}{6}(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2), \quad (2.1)$$

where

$$w(y) := \frac{2(a-y^2)}{1-y}, \quad r(y) := \frac{a-3y^2+2y^3}{a-y^2}, \quad (2.2)$$

and $0 < a < 1$ is a real constant. The Riemannian volume measure associated to g_B is thus given by

$$\mathrm{d}\mu_B := \rho_B(y)\, \mathrm{d}y\, \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\, \mathrm{d}\psi, \quad (2.3)$$

with

$$\rho_B(y) := \frac{1-y}{18\, w(y)^{1/2}}.$$

It was shown in [9] that the above local metric defines a unique complete 2-sphere bundle B over S^2, which is conformally Khler and diffeomorphic to $S^2 \times S^2$.

In the following lemma, we present a Fourier-type decomposition of the space of L^2 functions on B that is adapted to the above coordinate system and which will be used in turn to give a Fourier decomposition for the space of L^2 functions on $Y^{p,q}$. An explicit description of the bundle structure of B is required in order to obtain the desired decomposition, so in the proof of the lemma we indicate how B is defined globally as a complete manifold.

In order to state this lemma, we set some notation. The cubic equation

$$a - 3y^2 + 2y^3 = 0 \quad (2.4)$$

has three real roots for any $a \in (0, 1)$, one negative and two positive. In what follows, the negative root will be denoted by y_- and the smallest positive root by y_+ so that $y_- < 0 < y_+ < a$. We will also use the cover $\{V_1, V_2\}$ of S^2 given by

$$V_1 := \{\theta \in [0, \pi), \phi \in \mathbb{R}/2\pi\mathbb{Z}\}, \quad V_2 := \{\theta \in (0, \pi], \phi \in \mathbb{R}/2\pi\mathbb{Z}\},$$

so that V_1 (resp. V_2) stands for the sphere minus the north (resp. south) pole.

Lemma 2.1. *Let $\mathcal{B}$ denote the complex Hilbert space*

$$\mathcal{B} := \{(u_{nm})_{n,m\in\mathbb{Z}} : u_{nm} \in L^2((y_-, y_+), \rho_B(y)\, \mathrm{d}y) \otimes L^2((0,\pi), \sin\theta\, \mathrm{d}\theta)\},$$

endowed with the norm defined by

$$\|(\Phi_{nm} \otimes \Theta_{nm})_{n,m\in\mathbb{Z}}\|_{\mathcal{B}}^2 := \sum_{n,m\in\mathbb{Z}} \left(\int_{y_-}^{y_+} |\Phi_{nm}(y)|^2\, \rho_B(y)\, \mathrm{d}y\right) \left(\int_0^\pi |\Theta_{nm}(\theta)|^2 \sin\theta\, \mathrm{d}\theta\right).$$

Then the map defined by

$$\mathcal{B} \ni (\Phi_{nm} \otimes \Theta_{nm})_{n,m\in\mathbb{Z}} \mapsto \sum_{n,m\in\mathbb{Z}} \Phi_{nm}(y)\, \Theta_{nm}(\theta)\, \frac{\mathrm{e}^{\mathrm{i}(n\phi+2m\psi)}}{2\pi} \in L^2(B), \tag{2.5}$$

defines an isomorphism between $\mathcal{B}$ and $L^2(B)$.

Proof. In order to show that the local metric (2.1) can be promoted to a complete metric on a 4-manifold, we choose $\theta \in [0,\pi]$ and $\phi \in \mathbb{R}/2\pi\mathbb{Z}$ so that, for each fixed $y \in (y_-, y_+)$, the last two terms in (2.1) yield the metric of a round 2-sphere.

The range of y is taken to be $[y_-, y_+]$. This ensures that w is strictly positive in this interval and $r \geqslant 0$, vanishing only at the endpoints $y_\pm$. If we identify ψ periodically, the part of g_B given by

$$\frac{\mathrm{d}y^2}{w(y)\, r(y)} + \frac{r(y)}{9} \mathrm{d}\psi^2$$

describes a circle fibered over the interval (y_-, y_+), the size of the circle shrinking to zero at the endpoints. Remarkably, the (y,ψ) fibers are free of conical singularities if the period of ψ is 2π, in which case the circles collapse smoothly and the (y,ψ) fibers are diffeomorphic to a 2-sphere.

One must now check that the 2-spheres described by the coordinates (y,ψ) fiber properly over the 2-spheres defined by (θ,ϕ). For that, it suffices to consider the circles associated to the coordinate ψ for fixed $y \in (y_-, y_+)$, so we will consider the corresponding S^1-bundles B_y over S^2. The curvature of the connection form $\frac{1}{2\pi} \cos\theta\, \mathrm{d}\phi$ defines an integral de Rham cohomology class in the base because

$$-\frac{1}{2\pi}\int_{S^2}\mathrm{d}(\cos\theta\,\mathrm{d}\phi)=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi=2. \tag{2.6}$$

Therefore, for all y, a well-known theorem of Kobayashi [13] then ensures that $-\cos\theta\,\mathrm{d}\phi$ defines a connection on a principal S^1-bundle B_y over S^2 isomorphic to $S^3/\mathbb{Z}_2$ (or to the 2-sphere's unit tangent bundle). Since $\{V_1, V_2\}$ is a trivializing cover of S^2, the S^1-bundle B_y is uniquely determined by this cover and the transition function $F_{12}: V_1\cap V_2\to\mathbb{R}/2\pi\mathbb{Z}$ given by $F_{12}:=\pi$.

The set $V_1\cap V_2$ is the sphere minus the north and south poles, so it has full measure in V_1 and V_2. Hence, by the definition of the transition function and the expression of the induced volume element in B_y, an L^2 function on B_y can be identified with a measurable function $f^y:(0,\pi)\times(\mathbb{R}/2\pi\mathbb{Z})\times(\mathbb{R}/2\pi\mathbb{Z})\to\mathbb{C}$ which satisfies

$$f^y(\theta,\phi,\psi)=f^y(\theta,\phi,\psi+F_{12}(\theta,\phi))=f^y(\theta,\phi,\psi+\pi)$$

a.e. and

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi}|f^y(\theta,\phi,\psi)|^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\psi<\infty$$

As usual, $\psi+\pi$ refers to the group operation in $\mathbb{R}/2\pi\mathbb{Z}$, so it is to be understood modulo 2π. This leads to the Fourier-type expansion

$$f^y(\theta,\phi,\psi)=\sum_{n,m\in\mathbb{Z}}f^y_{nm}(\theta)\,\mathrm{e}^{\mathrm{i}(n\phi+2m\psi)},$$

which converges in L^2 sense. Formula (2.5) immediately follows from the latter equation by taking into account the dependence on y and using (2.3) and Fubini's theorem to carry out the integration in ϕ and ψ. □

Remark 2.1. Since ϕ and ψ have period 2π, the difference between equation (2.5) and the ordinary Fourier decomposition for 2π-periodic functions of ϕ and ψ lies in the fact that only the even Fourier modes in ψ appear in (2.5). As we have seen, this reflects the fact that the bundle B is diffeomorphic but not isometric to the product of two round 2-spheres. Notice that F_{12} must be constant because $\partial/\partial\psi$ is a globally defined Killing vector.

We now consider Fourier decompositions of L^2 functions on the manifolds $Y^{p,q}$. We begin with the local metric given by

$$g := g_B + w(y)\,(\mathrm{d}\alpha + h(y)\,(\mathrm{d}\psi - \cos\theta\,\mathrm{d}\phi))^2, \tag{2.7}$$

where

$$h(y) := \frac{a - 2y + y^2}{6(a - y^2)}$$

and g_B and w are defined as in (2.1)–(2.2). It can be verified that g is (locally) Sasaki–Einstein, with $\mathrm{Ric} = 4g$, and that the corresponding Riemannian measure is

$$\mathrm{d}\mu := \rho(y)\,\sin\theta\,\mathrm{d}y\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\psi\,\mathrm{d}\alpha, \tag{2.8}$$

where $\rho(y) := (1 - y)/18$.

It was proved in [9] that, for any pair of positive integers p and q with $p < q < 2p$, one can choose a constant $a \in (0, 1)$ such that

$$\frac{h(y_+) - h(y_-)}{2\,h(y_+)} = \frac{p}{q}, \tag{2.9}$$

and that in this case there exists a unique complete, simply connected manifold $Y^{p,q}$ whose metric is locally given by (2.7). Furthermore, $Y^{p,q}$ is a Sasaki–Einstein principal S^1-bundle over B diffeomorphic to $S^2 \times S^3$. In the rest of the paper, we shall always assume that a has been chosen so that (2.9) is satisfied, with p and q coprime. Note that

$$h(y_\pm) = \frac{y_\pm - 1}{6y_\pm},$$

so $\mp h(y_\pm) > 0$.

The main result of this section is the following lemma, where we present a Fourier-type decomposition of $L^2(Y^{p,q})$ that which will be crucial for the rest of the paper. In order establish this decomposition, we introduce another open cover $\{U_-, U_+\}$ of the sphere, which will be used to describe the fibers of the sphere bundle B. These sets again correspond to the whole sphere minus a pole, and can be characterized in terms of y and ψ as

$$U_- := \{y \in [y_-, y_+),\ \psi \in \mathbb{R}/2\pi\mathbb{Z}\}, \quad U_+ := \{y \in (y_-, y_+],\ \psi \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

Moreover, let us call Σ_0 the S^2-fiber at any fixed point (θ_0, ϕ_0) in the base and let $\Sigma_\pm \simeq S^2$ be the submanifolds of B given by $y = y_\pm$. It is then easy to see that $\{\Sigma_0, \Sigma_+\}$ defines a basis of the homology group $H_2(B;\mathbb{Z})$. In the following lemma, we denote by $\operatorname{lcm}\{x_1,\dots,x_k\}$ the least common multiple of the positive integers $x_1,\dots,x_k$.

Lemma 2.2. *Let $\mathcal{Y}$ be the complex Hilbert space*

$$\mathcal{Y} := \{(u_{nml})_{n,m,l\in\mathbb{Z}} : u_{nml} \in L^2((y_-,y_+),\rho(y)\,\mathrm{d}y)\otimes L^2((0,\pi),\sin\theta\,\mathrm{d}\theta)\},$$

endowed with the norm

$$\|(\Phi_{nml}\otimes\Theta_{nml})_{n,m,l\in\mathbb{Z}}\|^2_{\mathcal{Y}} := \sum_{n,m,l\in\mathbb{Z}}\left(\int_{y_-}^{y_+}|\Phi_{nml}(y)|^2\rho(y)\,\mathrm{d}y\right)\left(\int_0^{2\pi}|\Theta_{nml}(\theta)|^2\sin\theta\,\mathrm{d}\theta\right),$$

and let us set

$$\tau := -2\,h(y_+)/q, \quad \sigma := \operatorname{lcm}\{2, pq, 2p-q\}. \tag{2.10}$$

Then the map defined by

$$\mathcal{Y}\ni(\Phi_{nml}\otimes\Theta_{nml})_{n,m,l\in\mathbb{Z}} \mapsto \sum_{n,m,l\in\mathbb{Z}}\Phi_{nml}(y)\,\Theta_{nml}(\theta)\,\frac{\mathrm{e}^{\mathrm{i}(n\phi+2m\psi+\sigma l\alpha/\tau)}}{(2\pi)^{3/2}}\in L^2(Y^{p,q}), \tag{2.11}$$

defines an isomorphism between $\mathcal{Y}$ and $L^2(Y^{p,q})$.

Proof. If we periodically identify α by making it take values in $\mathbb{R}/2\pi\tau\mathbb{Z}$, then for each fixed $y\in[y_-,y_+]$ the term $w(y)\,d\alpha^2$ in (2.7) describes a circle whose size does not shrink to zero. To see that the metric (2.7) corresponds to a complete compact manifold, notice that, by a theorem of Kobayashi [13], $A := h(y)(d\psi-\cos\theta\,d\phi)$ defines a connection in a principal S^1-bundle over B if and only if the (globally defined) curvature 2-form $dA/(2\pi\tau)$ defines an integral de Rham cohomology class in B.

An easy computation shows that

$$\int_{\Sigma_0}\frac{\mathrm{d}A}{2\pi\tau} = \frac{h(y_-)-h(y_+)}{\tau}, \quad \int_{\Sigma_+}\frac{\mathrm{d}A}{2\pi\tau} = \frac{2\,h(y_+)}{\tau}.$$

By (2.9), it follows that $\mathrm{d}A/(2\pi\tau)\in H^2(B;\mathbb{Z})$ if we set $\tau := -2\,h(y_+)/q$, in which case the periods of $\mathrm{d}A/(2\pi\tau)$ around Σ_0 and Σ_+ are respectively given by p and $-q$. Since p and q are coprime, it follows that $Y^{p,q}$ is simply connected.

The bundle $Y^{p,q}$ is completely determined by the cover $\{V_i \times U_\epsilon : i = 1, 2,\ \epsilon = \pm\}$ and the associated transition functions

$$F_{ij\epsilon\eta} : (V_i \cap V_j) \cap (U_\epsilon \cap U_\eta) \to \mathbb{R}/2\pi\tau\mathbb{Z}.$$

The transition functions F_{ii-+}, $i = 1, 2$, are easily derived using the fact that

$$\int_{\Sigma_0} \frac{dA}{2\pi\tau} = p$$

for any (θ_0, ϕ_0), since this means that $d\alpha + A$ defines on Σ_0 an S^1-bundle with winding number p. Hence this bundle is isomorphic to the lens space $S^3/\mathbb{Z}_p$ [18] and $F_{ii-+} = 2\pi\tau/p$. Similarly, $F_{12++} = -2\pi\tau/q$ because

$$\int_{\Sigma_+} \frac{\mathrm{d}A}{2\pi\tau} = -q.$$

To determine F_{12--}, it suffices to observe that

$$\int_{\Sigma_-} \frac{\mathrm{d}A}{2\pi\tau} = 2p - q$$

by (2.9). Note that $2p - q$ is granted to be strictly positive. In this case, $\mathrm{d}\alpha + A$ determines a connection on a principal S^1-bundle over Σ_- with winding number $2p - q$, so that $F_{12--} = 2\pi\tau/(2p - q)$. As

$$F_{ij\epsilon\eta} = F_{ii\epsilon\eta} + F_{ij\eta\eta},$$

the full set of transition functions is uniquely determined from F_{ii-+} and $F_{12\epsilon\epsilon}$.

Since $(V_i \cap V_j) \times (U_\epsilon \cap U_\eta)$ has full measure in $V_i \times U_\epsilon$, an L^2 function in $Y^{p,q}$ can now be identified with a measurable function

$$f : (0, \pi) \times (\mathbb{R}/2\pi\mathbb{Z}) \times (y_-, y_+) \times (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\tau\mathbb{Z}) \to \mathbb{C},$$

such that:

(i) $f(\theta, \phi, y, \psi, \alpha) = f(\theta, \phi, y, \psi, \alpha + 2\pi\tau/p) = f(\theta, \phi, y, \psi, \alpha - 2\pi\tau/q) = f(\theta, \phi, y, \psi, \alpha + 2\pi\tau/(2p - q))$ a.e., by definition of the transition functions $F_{ij\epsilon\eta}$.

(ii) $f(\theta,\phi,y,\psi,\alpha)=f(\theta,\phi,y,\psi+\pi,\alpha)=f(\theta,\phi,\psi,\alpha+\pi)$ a.e., because of the way the sets $V_i\times U_\epsilon$ are patched to yield the bundle B, as analyzed in Lemma 2.1 using the auxiliary S^1-bundles B_y.

(iii) The integral

$$\int |f(\theta,\phi,y,\psi,\alpha)|^2\,\rho(y)\,\mathrm{d}y\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\psi\,\mathrm{d}\alpha$$

is finite, by the expression of the Riemannian measure (2.8).

From (i) and (ii), we see that f must be $2\pi\tau/\sigma$-periodic in its last argument, with $\sigma:=\operatorname{lcm}\{pq,2p-q,2\}$ the least common multiple of 2, p, q and $2p-q$, and π-periodic in its fourth argument. This leads to the L^2 Fourier expansion

$$f(\theta,\phi,y,\psi,\alpha)=\sum_{n,m,l\in\mathbb{Z}} f_{nml}(y,\theta)\,\mathrm{e}^{\mathrm{i}(n\phi+2m\psi+l\sigma\alpha/\tau)},$$

which readily gives (2.11) after recalling equation (2.8) and carrying out the integrals in ϕ, ψ and α. It should be noticed that all the Fourier frequencies compatible with the above periodicity condition must appear in the decomposition formula due to the simple connectedness of $Y^{p,q}$. □

3 The Laplacian in $Y^{p,q}$

Our goal in this section is to derive a manageable formula for the spectral resolution associated to the Laplacian in $Y^{p,q}$. As we shall see, the computation of the spectral decomposition of the Laplacian actually boils down to the analysis of a single Fuchsian ordinary differential operator depending on three parameters.

It is well known that the Laplacian on $Y^{p,q}$, which we denote by Δ, defines a nonnegative, self-adjoint operator whose domain is the Sobolev space $H^2(Y^{p,q})$ of square-integrable functions with square-integrable second derivatives. The Laplacian is given in local coordinates as

$$\begin{aligned}\Delta:=g^{ij}\nabla_i\nabla_j&=\frac{1}{\rho(y)}\frac{\partial}{\partial y}\rho(y)\,w(y)\,r(y)\,\frac{\partial}{\partial y}+\frac{1}{w(y)}\frac{\partial^2}{\partial\alpha^2}+\frac{9}{r(y)}\left(\frac{\partial}{\partial\psi}-h(y)\,\frac{\partial}{\partial\alpha}\right)^2\\&+\frac{6}{1-y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\phi}+\cos\theta\frac{\partial}{\partial\psi}\right)^2\right].\end{aligned}$$

Therefore, it is not difficult to see that the action of the Laplacian on a function of the form $u(y,\theta)\,\mathrm{e}^{\mathrm{i}(n\phi+2m\psi+l\sigma\alpha/\tau)}$ (which are globally well defined, as discussed in Lemma 2.2) is given by

$$\Delta(u(y,\theta)\,\mathrm{e}^{\mathrm{i}(n\phi+2m\psi+l\sigma\alpha/\tau)}) = (\Delta_{nml}u(y,\theta))\,\mathrm{e}^{\mathrm{i}(n\phi+2m\psi+l\sigma\alpha/\tau)}, \tag{3.1}$$

where

$$\begin{aligned}\Delta_{nml} :=& \frac{1}{\rho(y)}\frac{\partial}{\partial y}\rho(y)\,w(y)\,r(y)\,\frac{\partial}{\partial y} - \frac{1}{w(y)}\left(\frac{\sigma l}{\tau}\right)^2 - \frac{9}{r(y)}\left(2m - h(y)\,\frac{\sigma l}{\tau}\right)^2\\ &+ \frac{6}{1-y}T_{nm}\end{aligned} \tag{3.2}$$

and

$$T_{nm} := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \left(\frac{n+2m\cos\theta}{\sin\theta}\right)^2. \tag{3.3}$$

Equation (3.1) suggests that the computation the spectral resolution of Δ should be equivalent to finding an appropriate orthonormal basis of eigenfunctions of the differential operators Δ_{nml} on $L^2((y_-,y_+),\rho(y)\,\mathrm{d}y)\otimes L^2((0,\pi),\sin\theta\,\mathrm{d}\theta)$. In the rest of this section we shall show how this can be accomplished. We begin by constructing an orthonormal basis of $L^2((0,\pi),$ $\sin\theta\,\mathrm{d}\theta)$ consisting of eigenfunctions of T_{nm}. If $I\subset\mathbb{R}$ is an open interval, we shall denote by $\mathcal{AC}^1(I)$ the set of continuously differentiable functions $v: I\to\mathbb{C}$ whose derivative is absolutely continuous on any compact subset of I.

Lemma 3.1. *Let us denote the Jacobi polynomials by* $P_j^{(\bar a,\bar b)}$ *and set*

$$C_{nmj} := \left(\frac{(2j+|n+2m|+|n-2m|+1)\,j!\,(j+|n+2m|+|n-2m|)!}{2\,(j+|n+2m|)!\,(j+|n-2m|)!}\right)^{1/2}.$$

Then the analytic functions $v_{nmj}:[0,\pi]\to\mathbb{R}$ *given by*

$$v_{nmj}(\theta) := C_{nmj}\,\sin^{|n+2m|}\tfrac{\theta}{2}\,\cos^{|n-2m|}\tfrac{\theta}{2}\,P_j^{(|n+2m|,|n-2m|)}(\cos\theta),\quad j\in\mathbb{N},$$

define an orthonormal basis of $L^2((0,\pi),\sin\theta\,d\theta)$ *and satisfy the eigenvalue equation* $T_{nm}v_{nmj} = -\Lambda_{nmj}v_{nmj}$, *with*

$$\begin{aligned}\Lambda_{nmj} :=& 2(2j(j+1)+(|n+2m|+|n-2m|)(2j+1)\\ &+|n+2m||n-2m|+2m^2+n^2).\end{aligned}$$

Proof. In terms of the variable $z := \sin^2 \frac{\theta}{2}$, the differential equation $T_{nm} v(\theta) = -\Lambda v(\theta)$ can be rewritten as

$$\widetilde{T}_{nm} \widetilde{v}(z) = -\Lambda \widetilde{v}(z), \tag{3.4}$$

where

$$\widetilde{T}_{nm} := z(1-z)\, \frac{\partial^2}{\partial z^2} + (1-2z)\, \frac{\partial}{\partial z} - \frac{(n+2m-4mz)^2}{4z(1-z)}, \tag{3.5}$$

and $\widetilde{v}(z)$ stands for the expression of the function $v(\theta)$ in the variable z. It is not difficult to show that this equation has three regular singular points, located at 0, 1 and ∞, whose characteristic exponents are respectively given by $\pm(\frac{n}{2}+m)$, $\pm(\frac{n}{2}-m)$ and $\frac{1}{2}[1 \pm (1+\Lambda+4m^2)^{1/2}]$.

It then follows that the symmetric operator on $L^2((0,1))$ defined by the action of (3.5) on $C_0^\infty((0,1))$ is in the limit point case at 0 (resp. at 1) if and only if $n \neq -2m$ (resp. $n \neq 2m$). When both conditions are satisfied, there exists a unique self-adjoint extension, whose domain is the set of $\widetilde{v} \in \mathcal{AC}^1((0,1))$ such that $\widetilde{T}_{nm} \widetilde{v} \in L^2((0,1))$. When $n \neq -2m$ (resp. $n \neq 2m$), we take the Friedrichs extension of the above operator, which is determined [14] by the boundary condition

$$\lim_{z \searrow 0} z\, \widetilde{v}'(z) = 0 \qquad \left(\text{resp.} \quad \lim_{z \nearrow 1} z\, \widetilde{v}'(z) = 0 \right).$$

We shall see that this choice of boundary conditions will preclude the appearance of logarithmic singularities. With a slight abuse of notation, we shall still denote by $\widetilde{T}_{nm}$ the self-adjoint operators under consideration.

If equation (3.4) holds, a simple calculation shows that

$$\hat{v}(z) := z^{-|n+2m|/2}\, (1-z)^{-|n-2m|/2}\, \widetilde{v}(z)$$

satisfies the hypergeometric equation

$$z(1-z)\, \hat{v}''(z) + (\bar{c} - (\bar{a}+\bar{b}+1)z)\, \hat{v}'(z) - \bar{a}\bar{b}\, \hat{v}(z) = 0, \tag{3.6}$$

with parameters

$$\begin{aligned} 2\bar{a} &= 1 + |n+2m| + |n-2m| - (\Lambda + 4m^2 + 1)^{1/2}, \\ 2\bar{b} &= 1 + |n+2m| + |n-2m| + (\Lambda + 4m^2 + 1)^{1/2}, \quad \bar{c} = |n+2m| + 1. \end{aligned}$$

The characteristic exponents of this equation at 0 and 1 are given respectively by $(0, -|n+2m|)$ and $(0, -|n-2m|)$.

It can be readily checked that $\widetilde{v}$ belongs to the domain of $\widetilde{T}_{nm}$ if and only if $\hat{v}$ is a bounded solution of equation (3.6), which implies [1, 15.3.6] that $\hat{v}$ is a polynomial. This only happens if $\bar{a}$ or $\bar{b}$ equals a nonpositive integer $-j$, that is, when

$$\begin{aligned}\Lambda = 2(2j(j+1) + (|n+2m| + |n-2m|)(2j+1) \\ + |n+2m||n-2m| + 2m^2 + n^2).\end{aligned}$$

In this case, $\hat{v}(z)$ is a constant multiple of the Jacobi polynomial $P_j^{(|n+2m|,|n-2m|)}(1-2z)$. The spectral theorem ensures that the corresponding eigenfunctions $\widetilde{v}$ of $\widetilde{T}_{nm}$ define an orthogonal basis of $L^2((0,1))$, while their squared norm $\int_0^1 |\widetilde{v}(z)|^2 \,\mathrm{d}z$ can be readily shown to be [1, 22.2.1]

$$\begin{aligned}&\int_0^1 z^{|n+2m|}(1-z)^{|n-2m|}\, P_j^{(|n+2m|,|n-2m|)}(1-2z)^2 \,\mathrm{d}z \\ &\quad = \frac{(j+|n+2m|)!\,(j+|n-2m|)!}{(2j+|n+2m|+|n-2m|+1)\,j!\,(j+|n+2m|+|n-2m|)!}.\end{aligned} \tag{3.7}$$

Since the change of variables $\theta \mapsto z$ defines a unitary isomorphism

$$L^2((0,\pi), \sin\theta\,\mathrm{d}\theta) \ni v \mapsto \widetilde{v} \in L^2((0,1), 2\,\mathrm{d}z),$$

the statement of the lemma readily follows by inverting this transformation and normalizing the eigenfunctions using (3.7). ☐

Let us now consider the differential operator

$$\begin{aligned}S_{ml}(\Lambda) := \frac{1}{\rho(y)}\frac{\partial}{\partial y}\rho(y)\,w(y)\,r(y)\,\frac{\partial}{\partial y} - \frac{1}{w(y)}\left(\frac{\sigma l}{\tau}\right)^2 - \frac{9}{r(y)}\left(2m - h(y)\,\frac{\sigma l}{\tau}\right)^2 \\ - \frac{6\Lambda}{1-y},\end{aligned} \tag{3.8}$$

arising from (3.2), which depends on a real parameter $\Lambda \geqslant 0$. It is clear that we cannot hope to express the solutions of the formal eigenvalue equation

$$S_{ml}(\Lambda)\, w = -\lambda w \tag{3.9}$$

in closed form using special functions because (3.9) is a Fuchsian differential equation with *seven* regular singular points, located at the three roots of the cubic (2.4), at 1, at $\pm a^{1/2}$ and at infinity. However, the information contained in the following lemma will suffice for our purposes:

Lemma 3.2. *For all $\Lambda \geqslant 0$, the differential operator* (3.8) *defines a nonnegative self-adjoint operator in $L^2((y_-, y_+), \rho(y)\,dy)$, which we also denote by $S_{ml}(\Lambda)$, whose domain consists of the functions $w \in \mathcal{AC}^1((y_-, y_+))$ such that $S_{ml}(\Lambda)w \in L^2((y_-, y_+))$ and*

$$\lim_{y \searrow y_-} y\, w'(y) = 0 \text{ if } m = (2p-q)\sigma l/4 \text{ and } \lim_{y \nearrow y_+} y\, w'(y) = 0 \text{ if } m = -q\sigma l/4.$$

Its spectrum consists of a decreasing sequence of eigenvalues $(-\ell_{mlk}(\Lambda))_{k\in\mathbb{N}} \searrow -\infty$ of multiplicity one whose associated normalized eigenfunctions $w_{mlk}(\Lambda)$ are $\mathcal{O}((y_+ - y)^{|m+q\sigma l/4|})$ as $y \nearrow y_+$ and $\mathcal{O}((y-y_-)^{|m+(q-2p)\sigma l/4|})$ as $y \searrow y_-$.

Proof. Let y_ϵ be one of the endpoints of the interval (y_-, y_+) and set $\zeta := y - y_\epsilon$. An easy computation shows that

$$a - 3y^2 + 2y^3 = -6y_\epsilon(1-y_\epsilon)\,\zeta + \mathcal{O}(\zeta^2), \quad r(y) = -\frac{\zeta}{3y_\epsilon} + \mathcal{O}(\zeta^2),$$

as $y \to y_\epsilon$, which shows that the differential equation (3.9) can be asymptotically written as

$$-(12y_\epsilon\,\zeta + \mathcal{O}(\zeta^2))\,\widetilde{w}''(\zeta) - (12y_\epsilon + \mathcal{O}(\zeta))\,\widetilde{w}'(\zeta) + \left[\frac{3y_\epsilon}{\zeta}\left(2m - h(y_\epsilon)\frac{\sigma l}{\tau}\right)^2 + \mathcal{O}(1)\right]\widetilde{w}(\zeta) = 0,$$

with $\widetilde{w}(\zeta) := w(\zeta + y_\epsilon)$ standing for the expression of the function $w(y)$ in the new variable ζ.

It then follows that the characteristic exponents of the equation (3.9) at y_ϵ are $\pm\nu_\epsilon$, with $\nu_\epsilon := |m - h(y_\epsilon)\sigma l/(2\tau)|$. Using (2.9) and (2.10), one can immediately derive the more manageable formula

$$\nu_+ = |m + q\sigma l/4|, \quad \nu_- = |m + (q-2p)\sigma l/4|. \tag{3.10}$$

Since σ is even by Lemma 2.2, it stems from the latter equation that $2\,\nu_\epsilon$ is a nonnegative integer. Therefore, it is standard that the symmetric operator defined by (3.8) on $C_0^\infty((y_-, y_+))$ is in the limit point case at y_ϵ if and only if $\nu_\epsilon \neq 0$. If $\nu_+\nu_- \neq 0$, the latter operator is then essentially self-adjoint on

$C_0^\infty((y_-,y_+))$, and has a unique self-adjoint extension of domain [8]

$$\mathcal{D} := \{w \in \mathcal{AC}^1((y_-,y_+)) : S_{ml}(\Lambda)\, w \in L^2((y_-,y_+))\}.$$

When $\nu_+\nu_- = 0$, the above symmetric operator is not essentially self-adjoint. In this case, in order to rule out logarithmic singularities we shall choose its Friedrichs extension [14], whose domain consists of the functions $w \in \mathcal{D}$ such that

$$\lim_{y\searrow y_-} y\, w'(y) = 0 \quad \text{if } \nu_- = 0 \quad \text{and} \quad \lim_{y\nearrow y_+} y\, w'(y) = 0 \text{ if } \nu_+ = 0.$$

It is well known [8] that $S_{ml}(\Lambda)$ is then a nonnegative operator with compact resolvent and that its eigenvalues are nondegenerate. □

Remark 3.1. It should be noticed that the critical exponents (3.10) are half-integers rather than integers because in $|y - y_\epsilon|$ is proportional to the square of the distance to the pole y_ϵ, as discussed in [9].

Lemmas 3.1 and 3.2 provide us with all the information we need in order to derive the following eigenfunction expansion for the Laplacian, which is the main result of this section:

Theorem 3.1. *Let $u_{nmlkj} : Y^{p,q} \to \mathbb{C}$ be the analytic functions on $Y^{p,q}$ given by*

$$u_{nmlkj} := v_{nmj}(\theta)\, w_{mlk}(\Lambda_{nmj})(y)\, \frac{\mathrm{e}^{\mathrm{i}(n\phi+2m\psi+\sigma l\alpha/\tau)}}{(2\pi)^{3/2}} \tag{3.11}$$

and set $\lambda_{nmlkj} := \ell_{mlk}(\Lambda_{nmj})$. Then $\{u_{nmlkj} : j,k \in \mathbb{N},\ l,m,n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(Y^{p,q})$ and

$$\Delta u_{nmlkj} = -\lambda_{nmlkj}\, u_{nmlkj}. \tag{3.12}$$

Proof. We know that, by construction, $\{v_{nmj} \otimes w_{mlk} : j,k \in \mathbb{N}\}$ is a basis of $L^2((y_-,y_+), \rho(y)\, dy) \otimes L^2((0,\pi), \sin\theta\, \mathrm{d}\theta)$ for each $n,m,l \in \mathbb{Z}$. By Lemma 2.2, this implies that $\{u_{nmlkj} : j,k \in \mathbb{N},\ l,m,n \in \mathbb{Z}\}$ is then a basis of $L^2(Y^{p,q})$. In the light of Lemma 3.1, or after a short computation, it is apparent that the functions u_{nmlkj} are analytic, and therefore lie in the domain $H^2(Y^{p,q})$ of the Laplacian. From equations (3.1)–(3.3) and (3.8) and from Lemmas 3.1 and 3.2 we subsequently infer that the Laplace operator in diagonal in this basis and equation (3.12) holds, as claimed. □

Remark 3.2. It should be observed that the proof fails if one takes v_{nmj} or $w_{mlk}(\Lambda)$ to be the eigenfunction basis corresponding to a different self-adjoint extension of (3.3) or (3.8), since in this case the above eigenfunctions will present logarithmic singularities which will prevent some of the elements u_{nmlkj} of the basis of $L^2(Y^{p,q})$ from being in the domain of the Laplacian. This is due to the fact that the action of the Laplacian on these logarithmically divergent functions will give rise to Dirac-type singularities.

To conclude, some remarks on the form of the eigenfunctions (3.11) are in order. First, one should observe that the angular dependence of the eigenfunctions (in α and ψ) is quite different from, e.g., that of axisymmetric eigenfunctions in Euclidean space. This is a consequence of the considerable geometric complexity of the manifold $Y^{p,q}$ and its fibration structure, and indeed one can gain some additional insight into its geometry by fixing $y = y_\pm$ or $(\theta, \phi) = (\theta_0, \phi_0)$ and studying the expression of the eigenfunctions. From an analytic point of view, the main advantage of Theorem 3.1 is that most of the properties of the Laplacian in $Y^{p,q}$ can be analyzed through the simpler one-dimensional Sturm–Liouville operator $S_{ml}(\Lambda)$. Theorem 3.1 will be crucial in our analysis of the wave equation in M, which is the main result of this paper.

4 The Klein–Gordon equation in $\mathrm{AdS}^5 \times Y^{p,q}$

We will denote by AdS^5 the simply connected Lorentzian 5-manifold of constant sectional curvature $-\kappa$, for fixed $\kappa > 0$. It is well known that AdS^5 is diffeomorphic to $\mathbb{R}^5$. If $\vartheta \equiv (\vartheta^1, \vartheta^2, \vartheta^3) \in [0,\pi] \times [0,\pi] \times \mathbb{R}/2\pi\mathbb{Z}$ are the standard coordinates on the 3-sphere, the metric on AdS^5 can be globally written as

$$g_\kappa := \frac{-\mathrm{d}t^2 + \mathrm{d}x^2 + \cos^2 x\, g_{S^3}}{\kappa \sin^2 x},$$

where $t \in \mathbb{R}$, $x \in (0, \pi/2]$ and

$$g_{S^3} := (\mathrm{d}\vartheta^1)^2 + \sin^2\vartheta^1\, (\mathrm{d}\vartheta^2)^2 + \sin^2\vartheta^1\, \sin^2\vartheta^2\, (\mathrm{d}\vartheta^3)^2$$

is the canonical metric on S^3. The Laplacian and the Riemannian volume measure on S^3 will be respectively denoted by Δ_{S^3} and

$$d\omega := \sin^2\vartheta^1\, \sin\vartheta^2\, \mathrm{d}\vartheta^1\, \mathrm{d}\vartheta^2\, \mathrm{d}\vartheta^3.$$

Let us now focus of the 10-dimensional Lorentzian manifold $\mathrm{AdS}^5 \times Y^{p,q}$, endowed with the metric $\bar{g} := g_\kappa \oplus g$. It is apparent that there is no loss of generality in assuming that the Ricci curvature of $Y^{p,q}$ is $4g$, as in Section 2. Its associated wave operator will be denoted by $\overline{\Box} := \bar{g}^{ij}\,\overline{\nabla}_i\,\overline{\nabla}_j$, and we shall call Σ^T the spacelike hypersurface in $\mathrm{AdS}^5 \times Y^{p,q}$ defined by $t = T$. We will occasionally use the above coordinates to naturally identify each Σ^T with a fixed time slice Σ and define the measure $\mathrm{d}\nu\,\mathrm{d}\omega\,\mathrm{d}\mu$ on Σ or Σ^T, with $\mathrm{d}\nu := 2\cot^3 x\,dx$. Notice that $\mathrm{d}\nu\,\mathrm{d}\omega\,\mathrm{d}\mu$ is not the hypersurface measure on Σ^T.

In this section, we shall show that, under appropriate assumptions to be made precise below, the Klein–Gordon equation in $\mathrm{AdS}^5 \times Y^{p,q}$,

$$(\overline{\Box} - M^2)\varphi = 0, \tag{4.1a}$$

with Cauchy data

$$\varphi\Big|_{\Sigma^0} = \varphi^0, \quad \frac{\partial\varphi}{\partial t}\Big|_{\Sigma^0} = \varphi^1, \tag{4.1b}$$

admits a unique propagator, for any value of the constant $M \geqslant 0$. By a *propagator* we mean a bilinear map

$$\mathcal{R} : C_0^\infty(\Sigma) \times C_0^\infty(\Sigma) \to C^\infty(\mathrm{AdS}^5 \times Y^{p,q}) \cap C^\infty(\mathbb{R}, L^2(\Sigma, \mathrm{d}\nu\,\mathrm{d}\omega\,\mathrm{d}\mu)),$$

which maps each Cauchy data $(\varphi^0, \varphi^1) \in C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$ to a smooth solution $\varphi \equiv \mathcal{R}(\varphi^0, \varphi^1)$ of the Cauchy problem (4.1).

The analysis of the Cauchy problem (4.1) presents some additional difficulties related to the fact that $\mathrm{AdS}^5 \times Y^{p,q}$ is not globally hyperbolic. Generally speaking, the existence or uniqueness of global solutions to the Cauchy problem for the Klein–Gordon equation is not granted on a manifold which fails to be globally hyperbolic. The case of static, nonglobally hyperbolic manifolds such as $\mathrm{AdS}(\kappa) \times Y^{p,q}$ is quite special, however, and propagators on these spaces can be analyzed using results of Wald [19] and Ishibashi and Wald [11, 12].

For various linear hyperbolic equations, Ishibashi and Wald discuss the existence and uniqueness of propagators satisfying three essential assumptions [11, Section 2] which that are required for the propagation to be physically sensible. Roughly speaking, these conditions mean that the propagation in causal, invariant under time translation and reflection, and that it preserves an appropriate energy functional. For the reader's convenience, we present these assumptions in the Appendix. Propagators for the Cauchy problem (4.1) which satisfy Assumptions A.1–A.3 in the appendix will be called *admissible*. Our main result is Theorem 4.1, where we prove that the Cauchy problem (4.1) has a unique admissible propagator and derive a manageable integral spectral representation for the solution.

Before stating this theorem, we need to introduce some further notation. Let us consider the linear differential operators defined by

$$L(s,\lambda) := -\frac{\partial^2}{\partial x^2} + 3(\tan x + \cot x)\,\frac{\partial}{\partial x} + \frac{s(s+2)}{\cos^2 x} + \frac{M^2+\lambda}{\kappa\,\sin^2 x}, \tag{4.2}$$

where $s \in \mathbb{N}$ and $\lambda \geqslant 0$ are constants. We shall denote by $\mathcal{L}^0(s,\lambda)$ the positive symmetric operator in $L^2((0,\frac{\pi}{2}),\mathrm{d}\nu)$ with domain $C_0^\infty((0,\frac{\pi}{2}))$ defined by (4.2).

An orthonormal basis of $L^2(S^3,\mathrm{d}\omega)$ is given by the spherical harmonics (cf. e.g. [17])

$$Y^{s_1s_2s_3}(\vartheta) := N_{s_1s_2s_3}\,\sin^{s_2}\vartheta^1\, C^{(s_2+1)}_{s_1-s_2}(\cos\vartheta^1)\, P^{s_3}_{s_2}(\cos\vartheta^2)\,\mathrm{e}^{\mathrm{i}s_3\vartheta^3}, \tag{4.3}$$

where $C^{(l)}_r$ and P^m_l, respectively, denote the Gegenbauer polynomials and the associated Legendre functions, $(s_1,s_2,s_3) \in \mathbb{N}^2 \times \mathbb{Z}$ with $s_1 \geqslant s_2 \geqslant |s_3|$, and the normalization constants are

$$N_{s_1s_2s_3} := \left(\frac{2^{2s_2-1}(s_1+1)\,(2s_2+1)\,(s_1-s_2)!\,(s_2-s_3)!\,(s_2!)^2}{\pi^2(s_1+s_2+1)!\,(s_2+s_3)!}\right)^{1/2}.$$

The spherical harmonics satisfy the differential equation

$$\Delta_{S^3} Y^{s_1s_2s_3} = -s_1(s_1+2)\,Y^{s_1s_2s_3}. \tag{4.4}$$

For the ease of notation, we will consider the set of multi-indices

$$B := \{\beta \equiv (\beta_1,\dots,\beta_8) : \beta_1,\beta_2,\beta_7,\beta_8 \in \mathbb{N},\ \beta_3,\beta_4\beta_5,\beta_6 \in \mathbb{Z},\ \beta_1 \geqslant \beta_2 \geqslant |\beta_3|\}$$

and denote by $\Psi_\beta : S^3 \times Y^{p,q} \to \mathbb{C}$ the smooth functions given by

$$\Psi_\beta(\vartheta,\eta) := Y^{\beta_1\beta_2\beta_3}(\vartheta)\, u_{\beta_4\beta_5\beta_6\beta_7\beta_8}(\eta),$$

where u_{nmlkj} was defined in (3.11). We shall also use the notation

$$\lambda_\beta := \lambda_{\beta_4\beta_5\beta_6\beta_7\beta_8} \quad \text{and} \quad c_\beta := \left(4 + \frac{M^2+\lambda_\beta}{\kappa}\right)^{1/2}. \tag{4.5}$$

A multi-index β should be thought of as an ordered 8-uple consisting of the spectral parameters $(s_1, s_2, s_3, n, m, l, k, j)$ used in (3.11) and (4.3).

Lemma 4.1. *For each $\beta \in B$, the operator $\mathcal{L}^0(\beta_1, \lambda_\beta)$ is essentially self-adjoint. If $\mathcal{L}_\beta$ denotes its closure and and $F : [0, \infty) \to \mathbb{C}$ is a bounded continuous function, then*

$$F(\mathcal{L}_\beta)\, f(x) = \sum_{i\in\mathbb{N}} F(\Omega_i^\beta) \left(\int_0^{\frac{\pi}{2}} f(x')\, \overline{f_i^\beta(x')}\, \mathrm{d}\nu(x') \right) f_i^\beta(x) \tag{4.6}$$

a.e. for all $f \in L^2((0, \frac{\pi}{2}), d\nu)$, where for each $i \in \mathbb{N}$ the function

$$\begin{aligned} f_i^\beta(x) := \left(\frac{2(2i + s + c_\beta + 2)\, i!\, \Gamma(i + s + c_\beta + 2)}{(i + s + 1)!\, \Gamma(i + c_\beta + 1)} \right)^{1/2} \cos^{\beta_1} x \, \sin^{2+c_\beta} x \\ \times P_i^{(\beta_1+1, c_\beta)}(-\cos 2x) \end{aligned} \tag{4.7}$$

is a normalized eigenfunction of $\mathcal{L}_\beta$ with eigenvalue

$$\Omega_i^\beta := (2i + s + c_\beta + 2)^2. \tag{4.8}$$

Proof. Let us consider the change of variables $\xi := \cos^2 x$, which maps $L^2((0, \frac{\pi}{2}), \mathrm{d}\nu)$ onto $L^2((0,1), \xi(1-\xi)^{-2} d\xi)$ and transforms the ordinary differential equation $L(\beta_1, \lambda_\beta) f = \Omega f$ into

$$4\xi(1-\xi)\, \widetilde{f}''(\xi) + 4(2-\xi)\, \widetilde{f}'(\xi) + \left(\Omega - \frac{\beta_1(\beta_1+2)}{\xi} - \frac{c_\beta^2}{1-\xi} \right) \widetilde{f}(\xi) = 0, \tag{4.9}$$

where $\widetilde{f}$ stands for the expression of the function f under the above change of variables. This is a Fuchsian differential equation with three regular singular points: 0 (with characteristic exponents $\beta_1/2$ and $-1 - \beta_1/2$), 1 (with characteristic exponents $1 + c_\beta/2$ and $1 - c_\beta/2$) and infinity (with characteristic exponents $\pm\Omega^{1/2}/2$). From the expression for the exponents and equation (4.5), it is manifest that equation (4.9) does not admit any solutions in $L^2((0,1), \xi(1-\xi)^{-2}\mathrm{d}\xi)$ when $\operatorname{Im}\Omega \neq 0$, which implies that the only self-adjoint extension of $\mathcal{L}^0(\beta_1, \lambda_\beta)$ is its closure, $\mathcal{L}_\beta$.

An easy computation shows that $\hat{f}(\xi) := \xi^{-\beta_1/2}(1-\xi)^{-1-c_\beta/2}\, \widetilde{f}(\xi)$ satisfies the hypergeometric equation

$$\xi(1-\xi)\, \hat{f}''(\xi) + (\bar{c} - (\bar{a} + \bar{b} + 1)\xi)\, \hat{f}'(\xi) - \bar{a}\bar{b}\, \hat{f}(\xi) = 0,$$

with parameters

$$\bar{a} := \frac{2 + \beta_1 + c_\beta - \Omega^{1/2}}{2}, \quad \bar{b} := \frac{2 + \beta_1 + c_\beta + \Omega^{1/2}}{2}, \quad \bar{c} := \beta_1 + 2.$$

It is standard (cf. e.g. [1, 15.3.6]) that a solution $\widetilde{f}$ to the above equation lies in $L^2((0,1), \xi(1-\xi)^{-2}\mathrm{d}\xi)$ if and only if $\hat{f}$ is a polynomial. In turn, this is equivalent to say that $\bar{a} = -i$ with $i \in \mathbb{N}$, i.e., that

$$\Omega = (2i + s + c_\beta + 2)^2,$$

which implies that $\hat{f}(\xi)$ is proportional to the polynomial $P_i^{(\beta_1+1,c_\beta)}(1-2\xi)$. When rewritten in terms of the variable x, this readily yields the expressions (4.7) and (4.8) for the eigenvalues and eigenfunctions of $\mathcal{L}_\beta$, the normalization constant being easily read off from the formula [1, 22.2.1]

$$\begin{aligned}\int_0^1 \xi^{\beta_1+1}(1-\xi)^{c_\beta} P_i^{(\beta_1+1.c_\beta)}(1-2\xi)^2 \,\mathrm{d}\xi \\ = \frac{(i+\beta_1+1)!\,\Gamma(i+c_\beta+1)}{2(2i+\beta_1+c_\beta+2)\,i!\,\Gamma(i+\beta_1+c_\beta+2)},\end{aligned}$$

with Γ being Euler's Gamma function. Equation (4.6) now follows through continuous functional calculus [8]. □

Theorem 4.1. *There exists a unique admissible propagator for the Cauchy problem* (4.1)*, which is given by*

$$\varphi(t,x,\vartheta,\eta) := \sum_{\beta\in B}\left(\cos(t\,\mathcal{L}_\beta^{1/2})\,\varphi_\beta^0(x) + \frac{\sin(t\,\mathcal{L}_\beta^{1/2})}{\mathcal{L}_\beta^{1/2}}\,\varphi_\beta^1(x)\right)\,\Psi_\beta(\vartheta,\eta). \tag{4.10}$$

Here

$$\varphi_\beta^j(x) := \int_{S^3\times Y^{p,q}} \varphi^j(x,\vartheta,\eta)\,\overline{\Psi_\beta(\vartheta,\eta)}\,\mathrm{d}\omega(\vartheta)\,\mathrm{d}\mu(\eta), \quad j = 0,1,$$

and $\cos(t\,\mathcal{L}_\beta^{1/2})\,\varphi_\beta^j$ *and* $\mathcal{L}_\beta^{-1/2}\sin(t\,\mathcal{L}_\beta^{1/2})\,\varphi_\beta^j$ *are defined through the formula* (4.6)*.*

Proof. The wave operator in $\mathrm{AdS}^5 \times Y^{p,q}$ reads

$$\overline{\square} = \kappa \left(- \sin^2 x \, \frac{\partial^2}{\partial t^2} + \sin^2 x \, \tan^3 x \, \frac{\partial}{\partial x} \cot^3 x \, \frac{\partial}{\partial x} + \tan^2 x \, \Delta_{S^3} \right) + \Delta. \tag{4.11}$$

Let us assume that $\mathcal{R}$ is a propagator for the Cauchy problem (4.1) and set $\varphi := \mathcal{R}(\varphi^0, \varphi^1)$, so that $\varphi|_{\Sigma^T} \in L^2(\Sigma, \mathrm{d}\nu \, \mathrm{d}\omega \, \mathrm{d}\mu)$ for almost every $T \in \mathbb{R}$. Let us decompose φ as

$$\varphi(t, x, \vartheta, \eta) = \sum_{\beta \in B} \varphi_\beta(t, x) \, \Psi_\beta(\vartheta, \eta),$$

with

$$\varphi_\beta(t, x) := \int_{S^3 \times Y^{p,q}} \varphi(t, x, \vartheta, \eta) \, \overline{\Psi_\beta(\vartheta, \eta)} \, \mathrm{d}\omega(\vartheta) \, \mathrm{d}\mu(\eta).$$

From equation (4.11) it then follows that φ_β satisfies the Cauchy problem

$$\frac{\partial^2 \varphi_\beta}{\partial t^2} + L(\beta_1, c_\beta) \varphi_\beta = 0, \quad \varphi_\beta(0, \cdot) = \varphi_\beta^0, \quad \frac{\partial \varphi_\beta}{\partial t}(0, \cdot) = \varphi_\beta^1, \tag{4.12}$$

where, by hypothesis, $\varphi_\beta^j \in C_0^\infty((0, \frac{\pi}{2}]) \subset L^2((0, \frac{\pi}{2}), \mathrm{d}\nu)$ for $j = 0, 1$. If we additionally assume that $\varphi_\beta^j \in C_0^\infty((0, \frac{\pi}{2}))$, a theorem of Ishibashi and Wald [11] ensures that the only admissible solutions of (4.12) such that $\varphi_\beta(T, \cdot) \in L^2(\Sigma, \mathrm{d}\nu \, \mathrm{d}\omega \, \mathrm{d}\mu)$ for a.e. $T \in \mathbb{R}$ are given by

$$\varphi_\beta(t, x) = \cos(t \, \widetilde{\mathcal{L}}_\beta^{1/2}) \, \varphi_\beta^0(x) + \frac{\sin(t \, \widetilde{\mathcal{L}}_\beta^{1/2})}{\widetilde{L}_\beta^{1/2}} \, \varphi_\beta^1(x), \tag{4.13}$$

where $\widetilde{\mathcal{L}}_\beta$ is a positive self-adjoint extension of $\mathcal{L}^0(\beta_1, \lambda_\beta)$. As $\mathcal{L}^0(\beta_1, \lambda_\beta)$ is essentially self-adjoint by Lemma 4.1, $\widetilde{\mathcal{L}}_\beta$ necessarily coincides with the Friedrichs extension $\mathcal{L}_\beta$, which proves Theorem 4.1 under the additional hypothesis that the support of (φ^0, φ^1) does not contain the point of Σ^0 given by $x = 0$. To remove this hypothesis, it suffices to observe that (4.13) still solves the Cauchy problem (4.12) for arbitrary $\varphi_\beta^j \in C_0^\infty((0, \frac{\pi}{2}])$, thus ensuring the validity of (4.10) for arbitrary Cauchy data $(\varphi^0, \varphi^1) \in C_0^\infty(\Sigma^0) \times C_0^\infty(\Sigma^0)$, which is what we had to prove. □

It is clear that Theorem 4.1 and its proof remain valid when the Cauchy data are not assumed smooth and compactly supported but taken in a

Sobolev space of sufficiently high order, but we shall not pursue this generalization here. It is also well known that solutions to the inhomogeneous Cauchy problem can be constructed through time integration of solutions to (4.1) using Duhamel's principle. In this case, the propagator is a trilinear map

$$\begin{aligned}\widehat{\mathcal{R}} : &C_0^\infty(\mathrm{AdS}^5 \times Y^{p,q}) \times C_0^\infty(\Sigma^0) \times C_0^\infty(\Sigma^0) \\ &\to C^\infty(\mathrm{AdS}^5 \times Y^{p,q}) \cap C^\infty(\mathbb{R}, L^2(\Sigma, \mathrm{d}\nu\, \mathrm{d}\omega\, \mathrm{d}\mu))\end{aligned}$$

mapping $(\Theta, \varphi^0, \varphi^1)$ to a solution $\Phi \equiv \widehat{\mathcal{R}}(\Theta, \varphi^0, \varphi^1)$ of the PDE

$$(\overline{\Box} - M^2)\Phi = \Theta, \quad \Phi\bigg|_{\Sigma^0} = \varphi^0, \quad \frac{\partial \Phi}{\partial t}\bigg|_{\Sigma^0} = \varphi^1. \tag{4.14}$$

For completeness we state the analog of Theorem 4.1 for the inhomogeneous equation, which follows using the same reasoning as in Theorem 4.1. It should be noticed that the admissibility conditions presented in the appendix can be readily extended to the case of inhomogeneous equations, mutatis mutandis.

Corollary 4.1. *There exists a unique admissible propagator $\widehat{\mathcal{R}}$ for the inhomogeneous Cauchy problem* (4.14). *The propagator is given by $\Phi \equiv \widehat{\mathcal{R}}(\Theta, \varphi^0, \varphi^1) = \varphi + \widetilde{\varphi}$, with φ as in* (4.10) *and $\widetilde{\varphi}$ defined by*

$$\widetilde{\varphi}(t, x, \vartheta, \eta) := \sum_{\beta \in B} \left(\int_0^t \frac{\sin[(t - T)\mathcal{L}_\beta^{1/2}]}{\mathcal{L}_\beta^{1/2}} \Theta_\beta(T, x)\, \mathrm{d}T \right) \Psi_\beta(\vartheta, \eta), \tag{4.15}$$

where

$$\Theta_\beta(t, x) := \int_{S^3 \times Y^{p,q}} \Theta(t, x, \vartheta, \eta)\, \overline{\Psi_\beta(\vartheta, \eta)}\, \mathrm{d}\omega(\vartheta)\, \mathrm{d}\mu(\eta)$$

and the integral in (4.15) *is defined through the formula* (4.6).

Appendix

In this appendix, we will state and briefly discuss Ishibashi and Wald's assumptions [11], which must be satisfied by any physically meaningful propagator $\mathcal{R}$ of the Klein–Gordon equation (4.1). To this end, we will denote by $\varphi \equiv \mathcal{R}(\varphi^0, \varphi^1) : \mathrm{AdS}^5 \times Y^{p,q} \to \mathbb{C}$ the solution of the Cauchy problem (4.1) determined by $\mathcal{R}$.

Let us start by defining the time translation and reflection operator on $C^\infty(\mathrm{AdS}^5 \times Y^{p,q})$:

$$(\mathcal{T}_T\Phi)(t,x,\theta,\eta) := \Phi(t+T,x,\theta,\eta), \quad (\mathcal{P}\Phi)(t,x,\theta,\eta) := \Phi(-t,x,\theta,\eta),$$

with $T \in \mathbb{R}$. The causal future (resp. past) of a set $U \subset \mathrm{AdS}^5 \times Y^{p,q}$ will be denoted by $J^+(U)$ (resp. $J^-(U)$). The first condition we must impose on the propagator is that a solution φ of (4.1) must be compatible with causality:

Assumption A.1. The support of φ is contained in $J^+(\operatorname{supp}(|\varphi^0| + |\varphi^1|) \cup J^-(\operatorname{supp}(|\varphi^0| + |\varphi^1|))$.

The second assumption is that the propagation is compatible with the time translation and reflection symmetries. In order to state this condition, we need to recall [11, Lemma 2.1] that Assumption A.1 ensures that for any initial conditions $(\varphi^0, \varphi^1) \in C_0^\infty(\Sigma^0) \times C_0^\infty(\Sigma^0)$ there exists some $\epsilon(\varphi^0, \varphi^1) > 0$ such that the function $\varphi_T(x, \vartheta, \eta) := \varphi(T, x, \vartheta, \eta)$ is smooth and compactly supported for all $|T| < \epsilon(\varphi^0, \varphi^1)$. (Of course, the function φ_T would be compactly supported for all T in a globally hyperbolic manifold.)

Assumption A.2. Let φ be the solution associated with the Cauchy data (φ^0, φ^1) and $|T| \leqslant \epsilon(\varphi^0, \varphi^1)$. Then the solution associated with the Cauchy data $(\varphi_T, (\partial\varphi/\partial t)_T)$ (resp. $(\varphi^0, -\varphi^1)$) is $\mathcal{T}_{-T}\varphi$ (resp. $\mathcal{P}\varphi$).

To state the third assumption, let us introduce the vector space

$$\mathcal{V} := \left\{ \Phi \equiv \sum_{i=1}^N \mathcal{T}_{T_i}\mathcal{R}(\varphi_i^0, \varphi_i^1) : N \in \mathbb{N},\ T_i \in \mathbb{R}, \right.$$
$$\left. (\varphi_i^0, \varphi_i^1) \in C_0^\infty(\Sigma^0) \times C_0^\infty(\Sigma^0) \right\}$$

of all finite linear combinations of solutions of the form $\mathcal{T}_T\mathcal{R}(\varphi^0, \varphi^1)$. In the case of a globally hyperbolic manifold, this would simply be the space $\mathcal{R}(C_0^\infty(\Sigma^0) \times C_0^\infty(\Sigma^0))$ of solutions to the Cauchy problem (4.1) with data in C_0^∞.

Assumption A.3 consists of three related conditions ensuring the existence of a well-defined conserved energy functional E. The first one asserts that this energy is invariant under time translations and reflections. By equation (4.11), the equation (4.1a) can be rewritten as

$$\left(-\frac{\partial^2}{\partial t^2}+\tan^3 x\,\frac{\partial}{\partial x}\cot^3 x\,\frac{\partial}{\partial x}+\frac{\Delta_{S^3}}{\cos^2 x}+\frac{\Delta-M^2}{\kappa\,\sin^2 x}\right)\varphi=0, \tag{A.1}$$

so the second condition guarantees that the E reduces to the 'naive' energy functional for equation (A.1) in certain cases. Finally, the third condition requires that the topology defined by the energy functional be compatible with the weak C^∞ topology on the Cauchy data.

Assumption A.3. There exists an inner product $E:\mathcal{V}\times\mathcal{V}\to\mathbb{C}$ such that the following properties hold:

(i) For all $\Phi_1,\Phi_2\in\mathcal{V}$ and $T\in\mathbb{R}$,

$$E(\Phi_1,\Phi_2)=E(\mathcal{P}\Phi_1,\mathcal{P}\Phi_2)=E(\mathcal{T}_T\Phi_1,\mathcal{T}_T\Phi_2).$$

(ii) For all $\Phi\in\mathcal{V}$,

$$E(\Phi,\varphi)=\int_{\Sigma^0}\left[\frac{\overline{\partial\Phi}}{\partial t}\,\varphi^1-\overline{\Phi}\left(\frac{\partial^2}{\partial x^2}+3(\tan x+\cot x)\,\frac{\partial}{\partial x}+\frac{\Delta_{S^3}}{\sin^2 x}\right.\right.$$
$$\left.\left.+\ \frac{\Delta-M^2}{\kappa\,\sin^2 x}\right)\varphi^0\right]\,\mathrm{d}\nu\,\mathrm{d}\omega\,\mathrm{d}\mu.$$

(iii) Let $(\Phi_n)_{n\in\mathbb{N}}\subset\mathcal{V}$ be a Cauchy sequence with respect to the norm $\|\cdot\|_E$ defined by E and suppose that there exists $\Phi\in\mathcal{V}$ such that the sequences $(\Phi_n|_{\Sigma^0})_{n\in\mathbb{N}}$ and $(\partial\Phi_n/\partial t|_{\Sigma^0})_{n\in\mathbb{N}}$ respectively tend to $\Phi|_{\Sigma^0}$ and $\partial\Phi/\partial t|_{\Sigma^0}$ in the weak $C^\infty(\Sigma^0)$ topology. Then

$$\lim_{n\to\infty}\|\Phi_n-\Phi\|_E=0.$$

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