

Quantum 't Hooft operators and S -duality in $\mathcal{N} = 4$ super Yang–Mills

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Abstract

We provide a quantum path integral definition of an 't Hooft loop operator, which inserts a point-like monopole in a four-dimensional gauge theory. We explicitly compute the expectation value of the circular 't Hooft operators in $\mathcal{N} = 4$ super Yang–Mills with arbitrary gauge group G up to next to leading order in perturbation theory. We also compute in the strong coupling expansion the expectation value of the circular Wilson loop operators. The result of the computation of an 't Hooft loop operator in the weak coupling expansion exactly reproduces the strong coupling result of the conjectured dual Wilson loop operator under the action of S -duality. This paper demonstrates — for the first time — that correlation functions in $\mathcal{N} = 4$ super Yang–Mills admit the action of S -duality.

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1 Introduction

Electric–magnetic duality, also known as *S*-duality [1–3], is a remarkable conjectured equivalence relating $\mathcal{N} = 4$ super Yang–Mills at weak coupling to $\mathcal{N} = 4$ super Yang–Mills at strong coupling. Heuristically, this equivalence arises via a change of variables in the path integral, which identifies the two descriptions. This kind of duality transformation can be explicitly performed in certain statistical mechanics models such as the Ising model [4] as well as in electromagnetism, where electric fields are replaced by

magnetic fields. S -duality in $\mathcal{N} = 4$ super Yang–Mills conjecturally extends the electric–magnetic duality transformation in electromagnetism [5, 6] to a full-fledged interacting quantum field theory.

S -duality conjectures that $\mathcal{N} = 4$ super Yang–Mills with gauge group G and coupling constant τ is equivalent to $\mathcal{N} = 4$ super Yang–Mills with dual gauge group ${}^L G$ [7] and coupling constant ${}^L \tau$. The coupling constants of the two theories are related by

$${}^L \tau = -\frac{1}{n_{\mathfrak{g}} \tau},$$

where

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}, \quad {}^L \tau = \frac{{}^L \theta}{2\pi} + \frac{4\pi i}{({}^L g)^2}$$

and $n_{\mathfrak{g}} = 1, 2$ or 3 depending¹ on the choice of gauge group G . S -duality also acts on all gauge invariant operators of the theory and defines an operator isomorphism between the two theories

$$\mathcal{O} \longleftrightarrow {}^L \mathcal{O}.$$

Even though this map is rather poorly understood, progress in recent years has resulted in conjectures relating a large class of supersymmetric operators supported on various submanifolds in spacetime.

Since S -duality interchanges electric and magnetic charges, it exchanges a Wilson operator [8] with an 't Hooft operator [9]. These operators insert an electrically charged source and a magnetically charged source, respectively. Whereas a Wilson operator in the theory with gauge group G is labeled by a representation R of G , an 't Hooft operator is labeled [10] by a representation ${}^L R$ of the dual group ${}^L G$, and will be denoted by $W(R)$, $T({}^L R)$, respectively. Therefore, it is conjectured that under S -duality [10]

$$T({}^L R) \longleftrightarrow W({}^L R).$$

Explicit conjectures have also been made for the action of S -duality on chiral primary operators [11–13], surface operators [14, 15] and domain walls [16, 17] in $\mathcal{N} = 4$ super Yang–Mills.

The S -duality conjecture goes beyond the mapping of operators. It also predicts that the correlation functions of gauge invariant operators — which

¹Here $n_{\mathfrak{g}} = 1$ for simply laced algebras; $n_{\mathfrak{g}} = 2$ for $\mathfrak{so}(2N + 1)$, $\mathfrak{sp}(N)$ and \mathfrak{f}_4 ; and $n_{\mathfrak{g}} = 3$ for \mathfrak{g}_2 .

span the set of observables in $\mathcal{N} = 4$ super Yang–Mills — are related in the two theories by

$$\left\langle \prod_i \mathcal{O}_i \right\rangle_{G,\tau} = \left\langle \prod_i {}^L\mathcal{O}_i \right\rangle_{{}^L G, {}^L\tau}.$$

This aspect of the S -duality conjecture is a particularly challenging one to exhibit, as proving it necessarily requires understanding correlation functions at strong coupling, where no universal methods of computation are readily available.

In this paper we exhibit – for the first time – that the correlation function of dual operators are mapped into each other under the action of S -duality. We show that the weak coupling computation of the circular ’t Hooft operator $T({}^L R)$ in $\mathcal{N} = 4$ super Yang–Mills with gauge group G exactly reproduces the strong coupling computation of the expectation value of the circular Wilson operator $W({}^L R)$ in $\mathcal{N} = 4$ super Yang–Mills with gauge group ${}^L G$. We explicitly show that the prediction of S -duality

$$\langle T({}^L R) \rangle_{G,\tau} = \langle W({}^L R) \rangle_{{}^L G, {}^L\tau} \quad (1.1)$$

holds to next to leading order in the coupling constant expansion, which is weak for the ’t Hooft operator and strong for the dual Wilson operator.

Our computations verify in a quantitative manner the main prediction of S -duality for this class of observables. These results go beyond the previous tests of S -duality, which involve quantities for which the semiclassical approximation is exact or the theory is topologically twisted. Such tests include comparing the BPS spectra of particles [18] and operators [10–17], the effective action [19] in the Coulomb branch, and the partition function of the theory [20]. We note that the Wilson and ’t Hooft operators that we consider in this paper are different than the corresponding operators considered by Kapustin and Witten [21] in the topologically twisted $\mathcal{N} = 4$ super Yang–Mills theory relevant for the gauge theory approach to the geometric Langlands program. The Wilson and ’t Hooft operators considered in that theory are for arbitrary curves and have trivial expectation values.

Exhibiting S -duality for Wilson and ’t Hooft operators first requires defining and computing the expectation value of an ’t Hooft operator in $\mathcal{N} = 4$ super Yang–Mills. In Section 2 we provide a quantum definition of an ’t Hooft operator in four-dimensional gauge theory. It is defined in terms of a path integral where we integrate over all fields which have a prescribed singularity near the operator. Properly defining the ’t Hooft operator $T({}^L R)$ requires both renormalizing the operator as well as completely specifying

the measure of integration in the path integral. The classical singularity that we quantize is that of a singular monopole, which is characterized [10] by the highest weight B of the representation ${}^L R$ under which the 't Hooft operator $T({}^L R)$ transforms. Demanding that the path integral definition of the 't Hooft operator $T({}^L R)$ is gauge invariant requires integrating over the G -orbit of the classical singularity, which depends on B , and results in the inclusion of the measure of the adjoint orbit of B in the path integral measure. This quantum prescription applies to the computation of a general 't Hooft operator in an arbitrary gauge theory. We explicitly compute the expectation value of the circular 't Hooft operator $T({}^L R)$ in $\mathcal{N} = 4$ super Yang–Mills with arbitrary gauge group G up to one loop order. Given the path integral definition we provide, the computation of the expectation value of the 't Hooft operator $T({}^L R)$ can be extended to higher orders in perturbation theory by summing over the connected vacuum diagrams generated by the path integral.

In Section 3 we compute at strong coupling the expectation value of the circular Wilson loop operator $W(R)$ in $\mathcal{N} = 4$ super Yang–Mills with arbitrary gauge group G . It was conjectured in [22, 23] that the expectation value of the circular Wilson loop with $U(N)$ gauge group can be computed using a Gaussian matrix model, thereby reducing the complexity of the path integral of a four-dimensional field theory to a matrix integral. This result, extended to an arbitrary gauge group G , has been proven by Pestun [24], who, using localization techniques, has shown that the path integral over the four-dimensional fields reduces to an integral over a zero mode, which corresponds to the variable of integration in the matrix model integral. We use this result to evaluate the Wilson loop expectation value at strong coupling by performing the strong coupling expansion of the corresponding matrix integral.

In Section 4 we use the results of our computations of the 't Hooft operator at weak coupling and of the dual Wilson operator at strong coupling and explicitly show that these correlators transform precisely as conjectured by S -duality, and indeed verify equation (1.1). In Section 5 we argue that the subleading exponential corrections that appear in the Wilson loop computation can also be understood from the perturbative computation of the 't Hooft operator around extra saddle points. These saddle points arise due to the physics of monopole screening, whereby the charge of an 't Hooft operator is reduced/screened when a regular monopole configuration approaches the operator. Inclusion of these saddle points in the computation of the 't Hooft operator exactly reproduces the strong coupling result for the Wilson loop operator. Section 6 contains a summary and discussion of our results and future lines of inquiry. We have relegated to the appendices the details of some of our computations.

2 't Hooft loop expectation value

In this section we provide a quantum path integral definition of an 't Hooft operator in four-dimensional gauge theory, and explicitly compute the expectation value of the circular 't Hooft loop operator in $\mathcal{N} = 4$ super Yang–Mills up to one loop order. We begin by introducing basic facts regarding the classical field configuration produced by an 't Hooft operator and then proceed to its quantization.

't Hooft originally defined [9] these operators by specifying a singular gauge transformation around an arbitrary curve that links the loop on which the 't Hooft operator is supported.² Therefore, in a gauge theory with gauge group G , these operators are labeled by $\pi_1(G)$, which measures the topological magnetic flux created by the operator.

Kapustin [10] — motivated by S -duality in $\mathcal{N} = 4$ super Yang–Mills — has further refined 't Hooft's original characterization of magnetic operators and has shown that 't Hooft operators³ in a gauge theory with gauge group G are labeled by a representation ${}^L R$ of the dual group ${}^L G$. Since $\pi_1(G) \simeq Z({}^L G)$, where $Z({}^L G)$ is the center of ${}^L G$, the topological magnetic flux created by an operator labeled by a representation ${}^L R$ is given by the charge $Z({}^L G) \subset {}^L G$ of the representation ${}^L R$ of ${}^L G$.⁴ Kapustin's classification is much finer, as there are (infinitely) many different operators for a given topological flux in $\pi_1(G)$.

Physically, an 't Hooft loop operator is an operator that inserts a probe point-like monopole whose worldline forms the loop in spacetime on which the 't Hooft operator is supported. The representation ${}^L R$ of ${}^L G$ which labels the operator characterizes the magnetic charge of the monopole [7]. This description parallels the more familiar discussion of a Wilson loop operator, which inserts a point-like electric charge, and is therefore labeled by a representation R of G . Unlike a Wilson operator, which can be described by the insertion of an operator made out of the fields appearing in the Lagrangian, an 't Hooft operator is defined by specifying a singularity along the loop for the microscopic fields that we integrate over in the path integral, and is therefore an example of a disorder operator [26].

The classical field configuration produced by an 't Hooft loop operator $T({}^L R)$ supported on an arbitrary curve $C \subset \mathbb{R}^4$ is obtained by specifying a singularity for the fields near each point in the loop. Near each point in the

²For a space-like curve, such a singular gauge transformation creates a magnetic flux tube along the loop. Thus an 't Hooft/Wilson loop can be interpreted as the operator that creates an infinitesimally thin magnetic/electric flux tube around the loop (see, e.g., [25]).

³We will name this broader class of operators also as 't Hooft operators.

⁴This charge is the conjugacy class of the representation, i.e., the highest weight modulo elements of the root lattice, which coincides with N -ality for $SU(N)$.

loop C , the local singularity is that associated to a straight line \mathbb{R} , and the singularities created by an 't Hooft loop operator supported on a general curve C can be constructed by patching together the local singularities for the 't Hooft operator supported on a straight line \mathbb{R} . In a given theory, an 't Hooft operator creates a codimension three singularity for the fields that appear in the classical action. The only restriction on the admissible codimension three singularities created by an 't Hooft operator is that they solve the equations of motion of the theory in $\mathbb{R}^4 \setminus C$.

In the rest of the paper we focus our attention on $\mathcal{N} = 4$ super Yang–Mills with gauge group G . The locally supersymmetric singularity created by an 't Hooft operator $T({}^L R)$ supported on a straight line $\mathbb{R} \subset \mathbb{R}^4$ and labeled by a representation ${}^L R$ of ${}^L G$ is given by [10]

$$F = \frac{B}{2} \text{vol}(S^2) + ig^2 \theta \frac{B}{16\pi^2} \frac{dt \wedge dr}{r^2}, \quad \phi = \frac{B}{2r} \frac{g^2}{4\pi} |\tau|. \quad (2.1)$$

The straight line \mathbb{R} is spanned by the coordinate t , r is the distance from the line and $\text{vol}(S^2)$ is the volume form on the two-sphere that surrounds the line \mathbb{R} . $B \equiv B^i H_i \in \mathfrak{t}$ takes values in the Cartan subalgebra of the Lie algebra \mathfrak{g} associated with the gauge group G . As shown in [7], the Dirac quantization condition $\exp(2\pi i B) = id_G$ implies that B can be identified with the highest weight of the representation ${}^L R$ of the dual group ${}^L G$, justifying the labeling of 't Hooft operators in terms of representations of the dual group [10]. The 't Hooft operator creates a magnetic field through the S^2 surrounding the monopole, and when $\theta \neq 0$ it also generates an electric field, as the monopole acquires electric charge via the Witten effect [27]. Unbroken supersymmetry at a point in the loop requires that a scalar field $\phi \equiv n^I \phi^I$ in the $\mathcal{N} = 4$ super Yang–Mills multiplet (here (n^I) is a unit vector in \mathbb{R}^6) acquires a pole near the loop with fixed residue.

We now consider 't Hooft operators that preserve maximal supersymmetry. Preservation of 16 supercharges everywhere in the loop C requires that the 't Hooft loop is supported on two possible curves — $C = \mathbb{R}$ or $C = S^1$ — which are related by a global conformal transformation. The symmetry preserved by the straight and circular 't Hooft operators is $OSp(4^*|4) \subset PSU(2, 2|4)$.⁵ The bosonic subgroup is $SO(4^*) \times USp(4)$, where $SO(4^*) \simeq SU(1, 1) \times SU(2)$ is the subgroup of the four-dimensional conformal group $SU(2, 2)$ preserving the curve \mathbb{R} or $S^1 \subset \mathbb{R}^4$ and $USp(4) \simeq SO(5) \subset SU(4)$ is left unbroken by the choice of the scalar field which develops a pole near the loop.

⁵This is the supergroup for a maximally supersymmetric 't Hooft loop in $\mathbb{R}^{1,3}$. Supersymmetric 't Hooft loops exist on both $\mathbb{R}^{1,3}$ and \mathbb{R}^4 . The corresponding symmetry group for the dual Wilson loop was exhibited in [28].

We are now ready to proceed with the quantum definition of the 't Hooft operator. The 't Hooft loop expectation value is specified by a path integral where one integrates over all fields which have the prescribed singularity (2.1) along the loop. In order to give a complete definition of the operator, the precise measure of integration needs to be determined. Before proceeding with the study of the measure, we first analyze the leading semiclassical result for the 't Hooft loop expectation value.

2.1 Semiclassical 't Hooft loop

The semiclassical evaluation of the path integral requires expanding the $\mathcal{N} = 4$ super Yang–Mills path integral around the monopole singularity

$$\begin{aligned} A &= A_0 + \hat{A}, \\ \phi^I &= \phi_0^I + \hat{\phi}^I, \end{aligned}$$

where (A_0, ϕ_0^I) is the classical singularity (2.1) corresponding to an 't Hooft operator $T(LR)$ and $(\hat{A}, \hat{\phi}^I)$ are the non-singular quantum fluctuations that we must integrate over in the path integral.

The $\mathcal{N} = 4$ super Yang–Mills action can be obtained by dimensional reduction of the ten-dimensional $\mathcal{N} = 1$ super Yang–Mills with the inclusion of the topological term⁶

$$S = \frac{1}{g^2} \int d^4x \sqrt{h} \text{tr} \left[\frac{1}{2} F^{MN} F_{MN} + i \bar{\psi} \Gamma^M D_M \psi \right] - i \frac{\theta}{8\pi^2} \int \text{tr}(F \wedge F), \quad (2.2)$$

where $\text{tr}(\cdot)$ is the invariant metric on the Lie algebra \mathfrak{g} associated with the gauge group G and (A_M, ψ) are the ten-dimensional gauge field and gaugino, respectively. The metric on the Lie algebra is normalized so that the short coroots of \mathfrak{g} have length squared equal to two. In this normalization the topological term equals $i\theta$ for the minimal instanton, θ has period 2π and the complexified coupling constant is given by

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}.$$

In terms of the four-dimensional fields in the $\mathcal{N} = 4$ super Yang–Mills multiplet $A_M = (A_\mu, \phi^I)$, where $\mu = 0, \dots, 3$ and $I = 4, \dots, 9$, the non-topological

⁶Here and throughout it is understood that derivatives with respect to $M = 4, 5, \dots, 9$ are trivial.

part of the action reads⁷

$$\frac{1}{g^2} \int d^4x \sqrt{h} \text{tr} \left[\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + D^\mu \phi^I D_\mu \phi^I + \frac{1}{2} [\phi^I, \phi^J]^2 + i \bar{\psi} \Gamma^\mu D_\mu \psi + \bar{\psi} \Gamma^I [\phi^I, \psi] \right].$$

In the leading semiclassical approximation the expectation value of the 't Hooft operator $T(^LR)$ is given by

$$\langle T(^LR) \rangle_{G,\tau} \simeq \exp(-S_{(0)}), \quad (2.3)$$

where $S_{(0)}$ is the $\mathcal{N} = 4$ super Yang–Mills action (2.2) evaluated on the classical singularity (2.1) created by the 't Hooft operator $T(^LR)$. In the leading semiclassical approximation the quantum fluctuations $(\hat{A}, \hat{\phi}^I)$ are neglected.

In order to analyze the 't Hooft operators $T(^LR)$ supported on $C = \mathbb{R}$ and S^1 it is instructive to consider $\mathcal{N} = 4$ super Yang–Mills in $\text{AdS}_2 \times S^2$ instead of \mathbb{R}^4 . As already mentioned, these operators preserve an $\text{SU}(1, 1) \times \text{SU}(2)$ subgroup of the four-dimensional conformal group, and in $\text{AdS}_2 \times S^2$ these symmetries are manifest, since they act as isometries, while in \mathbb{R}^4 they act as conformal symmetries. We can go between \mathbb{R}^4 and $\text{AdS}_2 \times S^2$ by performing a Weyl transformation

$$ds_{\mathbb{R}^4}^2 = \Omega^2 ds_{\text{AdS}_2 \times S^2}^2,$$

which is a classical symmetry of $\mathcal{N} = 4$ super Yang–Mills. When considering the 't Hooft operator supported on $C = \mathbb{R}$ the metric on AdS_2 is the upper half-plane metric while when the operator is supported on $C = S^1$ the metric on AdS_2 is the metric on the Poincaré disk (see Appendix A for the explicit Weyl transformations). For both choices of curve C , the 't Hooft operator is supported at the conformal boundary of $\text{AdS}_2 \times S^2$, which is $C = \mathbb{R}$ for the upper half-plane metric and $C = S^1$ for the metric on the Poincaré disk.

Insertion of an 't Hooft loop operator $T(^LR)$ at the conformal boundary of $\text{AdS}_2 \times S^2$ creates the following field configuration:

$$F = \frac{B}{2} \text{vol}(S^2) + ig^2 \theta \frac{B}{16\pi^2} \text{vol}(\text{AdS}_2), \quad \phi = \frac{B}{2} \frac{g^2}{4\pi} |\tau|. \quad (2.4)$$

Since the S^2 is non-contractible and the scalar field is homogeneous in $\text{AdS}_2 \times S^2$, the field configuration created by the 't Hooft operator $T(^LR)$ in $\text{AdS}_2 \times S^2$ is non-singular.

⁷This expression is valid on an arbitrary curved background with metric h as long as we add to the action the conformal coupling of the scalars $\sqrt{h} R \text{tr}(\phi^I \phi^I)/6$.

We can now calculate the expectation value of the 't Hooft operator $T^{(LR)}$ by evaluating the $\mathcal{N} = 4$ super Yang–Mills action (2.2) in $\text{AdS}_2 \times S^2$ on the field configuration in equation (2.4) produced by the 't Hooft operator $T^{(LR)}$. In the action, h refers to the metric in $\text{AdS}_2 \times S^2$.⁸ Since the scalar field is homogeneous, the on-shell $\mathcal{N} = 4$ super Yang–Mills action is given by

$$S_{(0)} = \frac{1}{g^2} \int \text{tr}(F_0 \wedge *F_0) - i \frac{\theta}{8\pi^2} \int \text{tr}(F_0 \wedge F_0) = \text{tr}(B^2) \frac{g^2 |\tau|^2}{16\pi} \text{Vol}(\text{AdS}_2). \quad (2.5)$$

The on-shell action is divergent, being proportional to the volume of AdS_2 . This result is as expected, since the on-shell action measures the energy of an infinitely heavy point-like magnetic monopole.

In quantum field theory, the observables that are finite are the correlation functions of renormalized operators. Therefore, we must appropriately renormalize the 't Hooft operator $T^{(LR)}$, which we do as follows. We first parametrize the metric near the boundary of AdS_2 using the Fefferman–Graham gauge

$$ds_{\text{AdS}_2}^2 = \frac{dZ^2}{Z^2} + \frac{dX^2}{Z^2} (g_0(X) + Z^2 g_2(X) + \dots).$$

In this coordinate system the boundary is at $Z = 0$, and X parametrizes \mathbb{R} or S^1 for the upper half-plane metric and Poincaré disk metric, respectively.⁹ In order to define the renormalized 't Hooft operator we introduce a cutoff near the location of the operator, which is inserted at the boundary of $\text{AdS}_2 \times S^2$.¹⁰ This defines a three-dimensional hypersurface Σ located at $Z = \epsilon$. The renormalized 't Hooft operator is constructed by adding to the $\mathcal{N} = 4$ super Yang–Mills action (2.2) covariant counterterms supported on the hypersurface Σ

$$S \longrightarrow S + S_{\text{ct}}.$$

The explicit form of the covariant counterterms we use to define the renormalized 't Hooft operator are the boundary terms¹¹

$$S_{\text{ct}} = -\frac{1}{g^2} \int_{\Sigma} \text{tr}[F|_{\Sigma} \wedge \star_3 F|_{\Sigma} - f \wedge \star_3 f], \quad (2.6)$$

⁸Since the scalar curvature on $\text{AdS}_2 \times S^2$ vanishes, the conformal coupling vanishes.

⁹ $Z = 2e^{-\rho}$, $X = \psi$ for the Poincaré disk, and $Z = l$, $X = t$ for the upper half-plane (see equations (A.2) and (A.7) in Appendix A).

¹⁰The definition of the 't Hooft operator as the partition function of $\mathcal{N} = 4$ super Yang–Mills on $\text{AdS}_2 \times S^2$ is reminiscent of Sen's definition of the quantum entropy function [29–32] as the string theory path integral on AdS_2 , which encodes the macroscopic degeneracy of states of extremal black holes.

¹¹The boundary terms for surface operators [14] (see also [15]) were constructed in [33].

where $F|_{\Sigma}$ is the restriction of F to the hypersurface Σ , f is a one-form obtained by contracting F with the unit normal vector to Σ and \star_3 is the Hodge star operation on the three-dimensional hypersurface.

Taking into account the bulk action (2.5) and the boundary terms (2.6) in the semiclassical evaluation of the expectation value of the circular 't Hooft operator $T(^L R)$ we obtain that¹²

$$\langle T(^L R) \rangle_{G,\tau} = \exp \left(\frac{\text{tr}(B^2)}{8} g^2 |\tau|^2 \right). \quad (2.7)$$

When the 't Hooft loop is supported on $C = \mathbb{R}$ the expectation value is trivial, a result that follows from supersymmetry.

Exactly the same results for the semiclassical expectation value of the 't Hooft operator $T(^L R)$ are obtained when we consider the theory on \mathbb{R}^4 . Everything we have done can be translated into the \mathbb{R}^4 language by performing a Weyl transformation.¹³ We emphasize that we have presented the analysis on $\text{AdS}_2 \times S^2$ purely as a matter of convenience. We also note that our result for the expectation value applies to a general 't Hooft loop in any gauge theory where the matter fields are not excited by the operator or where adjoint matter fields have scale invariant singularities. We now proceed to the study of the quantum definition of the operator.

2.2 Quantum 't Hooft loop

The 't Hooft loop operator $T(^L R)$ is defined by integrating in the path integral over all fields which have a prescribed singularity near the loop. In order to evaluate the expectation value of the 't Hooft operator in the quantum theory, we must explicitly specify the measure of integration in the path integral.

The path integral for the 't Hooft operator $T(^L R)$ is performed by expanding the fields around the singularity

$$\begin{aligned} A &= A_0 + \hat{A}, \\ \phi^I &= \phi_0^I + \hat{\phi}^I, \end{aligned}$$

¹²The net effect of the boundary terms is to renormalize the volume of AdS_2 . For the metric on the upper half-plane the renormalized volume vanishes while the renormalized volume in the Poincaré disk is -2π , a well-known result from studies of Wilson loops in the AdS/CFT correspondence.

¹³The Weyl transformation from $\text{AdS}_2 \times S^2$ to \mathbb{R}^4 introduces a boundary term for the action in \mathbb{R}^4 proportional to $\text{tr}(\phi^I \phi^I)$.

where (A_0, ϕ_0^I) is the classical singularity corresponding to a 't Hooft loop $T(^LR)$ (2.1) and $(\hat{A}, \hat{\phi}^I)$ are the non-singular quantum fluctuations that we must integrate over in the path integral. In order to define the path integral and eliminate the gauge redundancies, we must specify a gauge fixing procedure. We quantize the theory in the background field gauge, where (A_0, ϕ_0^I) is the background about which the path integral is expanded. The gauge fixing condition we consider is the dimensional reduction to four dimensions of the covariant background field gauge fixing condition in 10-dimensional super Yang–Mills. It is given by

$$D_0^M \hat{A}_M = 0,$$

where

$$D_0^M = \partial^M - i[A_0^M, \cdot].$$

In terms of the fields in $\mathcal{N} = 4$ super Yang–Mills multiplet the gauge fixing condition takes the form

$$D_0^\mu \hat{A}_\mu - i[\phi_0^I, \hat{\phi}^I] = 0.$$

The gauge fixing procedure requires introducing Faddeev–Popov ghosts in the path integral as well as the addition of the following gauge fixing term and ghost action to the $\mathcal{N} = 4$ super Yang–Mills action

$$S_{\text{gf}} = \frac{1}{g^2} \int d^4x \sqrt{h} \operatorname{tr} \left[D_0^M \hat{A}_M D_0^N \hat{A}_N - \bar{c} D_0^M D_M c \right], \tag{2.8}$$

which in terms of four-dimensional fields reads

$$S_{\text{gf}} = \frac{1}{g^2} \int d^4x \sqrt{h} \operatorname{tr} \left[\left(D_0^\mu \hat{A}_\mu - i[\phi_0^I, \hat{\phi}^I] \right)^2 - \bar{c} D_0^\mu D_\mu c + \bar{c} [\phi_0^I, [\hat{\phi}^I, c]] \right].$$

From the gauge fixed path integral and by expanding around the background created by the 't Hooft operator $T(^LR)$, Feynman rules can be extracted and the expectation value of $T(^LR)$ can be computed to any desired order in perturbation theory. It is given by the sum over all connected vacuum diagrams.

The definition given thus far for the 't Hooft operator $T(^LR)$ is, however, not gauge invariant. The singularity produced by the operator $T(^LR)$

$$F = \frac{B}{2} \operatorname{vol}(S^2) + ig^2 \theta \frac{B}{16\pi^2} \frac{dt \wedge dr}{r^2} \quad \phi = \frac{B}{2r} \frac{g^2}{4\pi} |\tau|,$$

breaks the G -invariance of the $\mathcal{N} = 4$ super Yang–Mills action to invariance under a stability subgroup $H \subset G$. The choice of $B \in \mathfrak{t}$, which characterizes the strength of the singularity, determines the unbroken gauge group H . This is generated by the generators $T \subset \mathfrak{g}$ for which

$$[B, T] = 0. \quad (2.9)$$

In order to have a path integral definition of the 't Hooft operator $T^{(LR)}$ which is gauge invariant, we must integrate over all the G -orbits of $B \in \mathfrak{t}$ along the loop. This integration, which we include in our definition of the path integral, restores G -invariance. The integral we must perform is over the adjoint orbit of B

$$O(B) \equiv \{B_{\mathfrak{g}} = \mathfrak{g}B\mathfrak{g}^{-1}, \mathfrak{g} \in G\}, \quad (2.10)$$

which is diffeomorphic to the coset space G/H . The integration over the adjoint orbit of B is reminiscent of the integration over collective coordinates around a soliton in quantum field theory. In the context of quantization of the 't Hooft operator, integration over $O(B)$ follows from demanding that the path integral is gauge invariant. In the computation of a general 't Hooft operator in an arbitrary gauge theory we must also include this measure factor.

Having an explicit definition of the quantum 't Hooft operator $T^{(LR)}$ we now proceed to calculate the expectation value of $T^{(LR)}$ to one loop order. Integrating out the quantum fluctuations to one loop requires expanding the complete gauge fixed $\mathcal{N} = 4$ super Yang–Mills action obtained by combining (2.2) and (2.8) to quadratic order in the fluctuations. The quadratic action is given by the dimensional reduction to four dimensions of

$$S_{(2)} = \frac{1}{g^2} \int d^4x \sqrt{h} \operatorname{tr} \left[\hat{A}_M (-\delta^{MN} D_0^2 + 2iF_0^{MN}) \hat{A}_N + i\bar{\psi} \Gamma^M D_{0M} \psi - \bar{c} D_0^2 c \right],$$

where we are packaging the $\mathcal{N} = 4$ super Yang–Mills fields into ten-dimensional fields. Therefore, up to one loop order the expectation value of the circular 't Hooft loop operator $T^{(LR)}$ is given by¹⁴

$$\begin{aligned} \langle T^{(LR)} \rangle_{G, \tau} &= \exp \left(\frac{\operatorname{tr}(B^2)}{8} g^2 |\tau|^2 \right) \frac{[\det_f(i\Gamma^M D_{0M})]^{1/4} \det_g(-D_0^2)}{[\det_b(-\delta^{MN} D_0^2 + 2iF_0^{MN})]^{1/2}} \\ &\times \int d\mu_{O(B)}. \end{aligned} \quad (2.11)$$

¹⁴In Lorentzian signature the fermions are Majorana–Weyl. In Euclidean signature, the fermions are chiral and complex, but ψ and $\bar{\psi}$ are not independent, resulting in the exponent of $1/4$ for the fermionic determinant.

The first factor arises, as we have seen in (2.7), from the renormalized on-shell action evaluated on the classical singularity produced by $T({}^L R)$, the second one from integrating out the fluctuations of the bosons, fermions and ghost fields, and $\int d\mu_{O(B)}$ is the integration over the adjoint orbit of B required by gauge invariance.

In Appendix B we show that the one loop determinants all cancel among themselves. The reason behind this cancellation is that the background for an 't Hooft loop operator (2.4) is invariant under half of the supersymmetries of the theory. Moreover, the background is self-dual if we package the three components of the gauge field and the scalar field ϕ sourced by the loop as a four component gauge field.¹⁵ The cancellation of the determinants is then quite analogous to the cancellation of the corresponding determinants of $\mathcal{N} = 4$ super Yang–Mills around an instanton background.

We now have to construct the metric on the adjoint orbit of B , $O(B)$.¹⁶ This is obtained by computing

$$\text{tr}(dB_{\mathfrak{g}}^2),$$

where $\mathfrak{g} \in G$. This yields

$$\text{tr}(dB_{\mathfrak{g}}^2) = \text{tr}([B, \mathfrak{g}^{-1}d\mathfrak{g}]^2). \quad (2.12)$$

We write the Lie algebra \mathfrak{g} in the Cartan basis $\{H_i, E_\alpha\}$. The generators H_i span the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and E_α are ladder operators associated to roots α of the Lie algebra \mathfrak{g} . We can decompose the Maurer–Cartan form of the group G in terms of the generators of \mathfrak{g}

$$\mathfrak{g}^{-1}d\mathfrak{g} = i \left(\sum_i d\xi^i H_i + \sum_\alpha d\xi^\alpha E_\alpha \right). \quad (2.13)$$

In order to explicitly determine the physical metric in $O(B)$ we must specify the overall normalization. We fix the normalization of the metric from the quadratic form defined by the on-shell action (2.7) of the 't Hooft operator. Therefore, by evaluating (2.12) the physical metric on the adjoint orbit of B is given by

$$ds_{O(B)}^2 = \frac{g^2 |\tau|^2}{4} \sum_{\substack{\alpha > 0 \\ \alpha(B) \neq 0}} \alpha(B)^2 2 \text{tr}(E_\alpha E_{-\alpha}) |d\xi^\alpha|^2,$$

¹⁵This is the familiar statement that the monopole equations arise by dimensional reduction to one lower dimension from the self-duality equations, where the scalar field in the monopole equations arises from the fourth component of the gauge field.

¹⁶While $O(B)$ is diffeomorphic to G/H , their metrics as a submanifold and as a quotient are different.

where the sum is over all the positive roots α that do not annihilate B , and we have used that $[X, E_\alpha] = \alpha(X)E_\alpha$ for any $X \in \mathfrak{t}$. This implies that¹⁷

$$\int d\mu_{O(B)} = \left(\frac{g^2|\tau|^2}{8\pi}\right)^{\dim(G/H)/2} \text{Vol}(G/H) \prod_{\substack{\alpha>0 \\ \alpha(B)\neq 0}} \alpha(B)^2, \tag{2.14}$$

since

$$\sum_{\substack{\alpha>0 \\ \alpha(B)\neq 0}} 2 \text{tr}(E_\alpha E_{-\alpha}) |d\xi^\alpha|^2 = ds_{G/H}^2.$$

The complete one loop result for the expectation value of a circular 't Hooft operator $T({}^L R)$ in $\mathcal{N} = 4$ super Yang–Mills with arbitrary gauge group G is then

$$\begin{aligned} \langle T({}^L R) \rangle_{G,\tau} &= \exp\left(\frac{\text{tr}(B^2)}{8} g^2 |\tau|^2\right) \left(\frac{g^2|\tau|^2}{8\pi}\right)^{\dim(G/H)/2} \text{Vol}(G/H) \\ &\times \prod_{\substack{\alpha>0 \\ \alpha(B)\neq 0}} \alpha(B)^2, \end{aligned} \tag{2.15}$$

where we recall that B is identified with the highest weight ${}^L w$ of the representation ${}^L R$ of ${}^L G$, which labels the operator.

Since we have given a complete definition of the path integral measure and have an explicit gauge fixed action, the expectation value of the 't Hooft operator $T({}^L R)$ can now be computed to any desired higher order in perturbation theory. It is given by the sum over all connected vacuum graphs around the singularity created by $T({}^L R)$.

2.2.1 Examples

The discussion thus far has been very general, applying to an arbitrary circular 't Hooft operator $T({}^L R)$ in $\mathcal{N} = 4$ super Yang–Mills with gauge group G . Such an operator is labeled by a representation ${}^L R$ of ${}^L G$. In order to make the discussion a bit less abstract, here we present the relevant formulas for various elementary gauge groups.

- $G = \text{SU}(2)$ and $\text{SO}(3)$: 't Hooft operators in this theory are labeled by a highest weight of the dual group, which are ${}^L G = \text{SO}(3)$ and

¹⁷As usual in path integrals, a factor of $1/\sqrt{2\pi}$ multiplies each integration variable $d\xi^\alpha$, which guarantees that the path integral for the Gaussian model is normalized to 1.

${}^L G = \text{SU}(2)$, respectively. A highest weight of $\text{SO}(3)$ can be labeled in terms of a spin $j \in \mathbb{Z}^+$ while for $\text{SU}(2)$ $j \in (1/2)\mathbb{Z}^+$. For an 't Hooft operator with $j \neq 0$, the broken symmetry near the loop is $H = U(1)$ (for $j = 0$, we just get the identity operator). In this case

$$\langle T(j) \rangle_{G,\tau} = \exp\left(\frac{j^2}{4} g^2 |\tau|^2\right) j^2 g^2 |\tau|^2. \tag{2.16}$$

- $G = U(N)$: 't Hooft operators in this theory are labeled by a highest weight of the dual group, which is also ${}^L G = U(N)$. A highest weight of $U(N)$ can be labeled by a set of integers ${}^L w = [m_1, m_2, \dots, m_N]$ with $m_1 \geq m_2 \geq \dots \geq m_N$.¹⁸ The corresponding data characterizing the monopole singularity (2.1) is given by

$$B = \begin{pmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & m_N \end{pmatrix} \in \mathfrak{t} \simeq \mathfrak{u}(1)^N.$$

Let us now consider various representations of $U(N)$:

- ${}^L w = [k, 0, \dots, 0]$: This corresponds to the rank- k symmetric representation. The stability group in this case is $H = U(1) \times U(N - 1)$ and

$$\langle T([k, 0, \dots, 0]) \rangle_{G,\tau} = \exp\left(\frac{k^2}{8} g^2 |\tau|^2\right) \left(\frac{g^2 |\tau|^2 k^2}{4}\right)^{N-1} \frac{1}{(N - 1)!}, \tag{2.17}$$

where we have used that $\text{Vol}(U(N)) = (2\pi)^{N(N+1)/2} / \prod_{n=1}^{N-1} n!$ [34].

- ${}^L w = [\overbrace{1, \dots, 1}^{k \text{ times}}, 0, \dots, 0]$. This corresponds to the rank- k antisymmetric representation. The stability group in this case is $H = U(k) \times U(N - k)$ and

$$\begin{aligned} &\langle T([1, \dots, 1, 0, \dots, 0]) \rangle_{G,\tau} \\ &= \exp\left(\frac{k}{8} g^2 |\tau|^2\right) \left(\frac{g^2 |\tau|^2}{4}\right)^{k(N-k)} \frac{\prod_{n=1}^{k-1} n!}{\prod_{n=1}^k (N - n)!}. \end{aligned} \tag{2.18}$$

¹⁸A highest weight of $U(N)$ with $m_N \geq 0$ is in one-to-one correspondence with a Young diagram containing m_l boxes in the l th row.

- $Lw = [m_1, m_2, \dots, m_N]$ with $m_1 > m_2 \dots > m_N$: The stability group in this case is $H = U(1)^N$ and

$$\langle T([m_1, m_2, \dots, m_N]) \rangle_{G, \tau} = \exp\left(\frac{\sum_i m_i^2}{8} g^2 |\tau|^2\right) \left(\frac{g^2 |\tau|^2}{4}\right)^{N(N-1)/2} \times \frac{1}{\prod_{n=1}^{N-1} n!} \prod_{i < j} (m_i - m_j)^2. \tag{2.19}$$

3 Wilson loop expectation value

The aim of this section is to compute the expectation value of the circular Wilson loop in $\mathcal{N} = 4$ super Yang–Mills at strong coupling. The ultimate goal is to show that our result (2.15) for the expectation value of the circular 't Hooft operator at weak coupling maps in the dual theory to the expectation value of the Wilson loop at strong coupling, thereby exhibiting S -duality in $\mathcal{N} = 4$ super Yang–Mills for correlation functions.

The supersymmetric circular Wilson loop in $\mathcal{N} = 4$ super Yang–Mills with gauge group G is labeled by a representation R of G . It is given by [35, 36]

$$W(R) \equiv \text{Tr}_R P \exp \oint (iA + \phi),$$

where $\phi \equiv \phi^I n^I$ and (n^I) is a unit vector in \mathbb{R}^6 .

A remarkable property of the supersymmetric circular Wilson loop $W(R)$ is that its expectation value can be computed in terms of a matrix model, thereby reducing the complexity of the path integral of a four-dimensional field theory to a matrix integral. This result was first conjectured in [22, 23], and was based on computations of the Wilson loop in perturbation theory. This remarkable result has been proven in an elegant paper by Pestun [24], who, using localization techniques, has shown that the path integral over the four-dimensional fields reduces to an integral over a zero mode, which corresponds to the variable of integration in the matrix model integral.

The expectation value of the supersymmetric circular Wilson loop $W(R)$ transforming in a representation R of G is given by the matrix integral [22–24]

$$\langle W(R) \rangle_{G, \tau} = \frac{1}{\mathcal{Z}} \int_{\mathfrak{g}} [dM] \exp\left(-\frac{2}{g^2} \text{tr}(M^2)\right) \text{Tr}_R e^M. \tag{3.1}$$

M is an element in the Lie algebra \mathfrak{g} corresponding to G and \mathcal{Z} is the matrix model partition function. As in the previous section, $\text{tr}(\cdot, \cdot)$ is the invariant metric on the Lie algebra \mathfrak{g} , and is normalized so that the length-squared of the short coroots is two. This normalization fixes the measure $[dM]$, which is the volume element on the Lie algebra \mathfrak{g} . The normalization factor is

$$\mathcal{Z} = \int_{\mathfrak{g}} [dM] \exp\left(-\frac{2}{g^2} \text{tr}(M^2)\right) = \left(\frac{\pi g^2}{2}\right)^{\dim(G)/2}. \tag{3.2}$$

We now “gauge fix” and reduce the integral over \mathfrak{g} to integration over the Cartan subalgebra \mathfrak{t} . Any $M \in \mathfrak{g}$ is conjugate to an element X in the maximal torus T of G , and the Lie algebra \mathfrak{g} decomposes into orbits of the G -action, with the generic orbit being diffeomorphic to G/T . In formulas

$$\forall M \in \mathfrak{g}, \exists X \in T, \mathfrak{g} \in G/T : \quad M = \mathfrak{g}X\mathfrak{g}^{-1},$$

and the metric in \mathfrak{g} is given by

$$\text{tr}(dM^2) = \text{tr}(dX^2) + \text{tr}[X, \mathfrak{g}^{-1}d\mathfrak{g}]^2.$$

Using the decomposition of the Maurer–Cartan form of G in (2.13) we find that

$$\text{tr}(dM^2) = \text{tr}(dX^2) + \sum_{\substack{\alpha > 0 \\ \alpha(X) \neq 0}} \alpha(X)^2 2 \text{tr}(E_{\alpha}E_{-\alpha}) |d\xi^{\alpha}|^2,$$

where α are the roots of \mathfrak{g} .

Generically, there is more than one X in the maximal torus T associated with a given $M \in \mathfrak{g}$, but these are related to each other by the action of the Weyl group \mathcal{W} of \mathfrak{g} . Correspondingly, the orbits of the G -action are parametrized by $X \in T$ up to the action of \mathcal{W} . Therefore, integration over the orbit yields

$$\int_{\mathfrak{g}} [dM] e^{-\frac{2}{g^2} \text{tr}(M^2)} \text{Tr}_R e^M = \frac{\text{Vol}(G/T)}{|\mathcal{W}|} \int_{\mathfrak{t}} [dX] \Delta(X)^2 e^{-\frac{2}{g^2} \langle X, X \rangle} \text{Tr}_R e^X, \tag{3.3}$$

where

$$\Delta(X)^2 = \prod_{\alpha} |\alpha(X)| = \prod_{\alpha > 0} \alpha(X)^2,$$

and $\langle \cdot, \cdot \rangle$ is the metric on the Cartan subalgebra \mathfrak{t} . The factor of $\Delta(X)^2$ plays the role of the Vandermonde determinant in Hermitian matrix models.

It is convenient to write the insertion of the group character in (3.3) as the sum over all the weights $v \in \Omega(R)$ in the representation R

$$\mathrm{Tr}_R e^X = \sum_{v \in \Omega(R)} n(v) e^{v(X)},$$

where $n(v)$ is the multiplicity of the weight v and $\Omega(R)$ is the set of all weights in the representation R . By completing squares in the exponential, we obtain

$$\begin{aligned} \langle W(R) \rangle_{G,\tau} &= \frac{\mathrm{Vol}(G/T)}{|\mathcal{W}|\mathcal{Z}} \sum_{v \in \Omega(R)} n(v) e^{\frac{g^2}{8} \langle v, v \rangle} \\ &\times \int_{\mathfrak{t}} [dX] e^{-\frac{2}{g^2} \langle X, X \rangle} \prod_{\alpha > 0} \left(\alpha(X) + \frac{g^2}{4} \langle \alpha, v \rangle \right)^2. \end{aligned} \quad (3.4)$$

For each weight $v \in \Omega(R)$, we obtain the expectation value of a polynomial in the X 's, which can be evaluated using Wick contractions, yielding a polynomial in the coupling constant g .

Since we are interested in understanding the action of S -duality on our perturbative computation of the 't Hooft operator (2.15), we need to solve the matrix model for the Wilson loop at strong coupling. The large g behavior of the Wilson loop (3.4) is controlled by the exponential prefactor. At large g , the leading contribution arises from the terms in the sum over weights involving the longest weights in the representation R . It is for these weights that the length of the weight — given by $\langle v, v \rangle$ — is maximal.

The longest weights $v \in \Omega(R)$ are related to the highest weight¹⁹ in the representation R — which we denote by w — by the action of the Weyl group \mathcal{W} . However, there is an invariant subgroup $H \subset G$ that leaves that highest weight w invariant, and the Weyl group of H — which we denote by $\mathcal{W}(H)$ — acts trivially on w .

The strong coupling limit of the circular Wilson loop operator is thus given by

$$\langle W(R) \rangle_{G,\tau} = \frac{\mathrm{Vol}(G/T)}{|\mathcal{W}(H)|\mathcal{Z}} e^{\frac{g^2}{8} \langle w, w \rangle} \int_{\mathfrak{t}} [dX] e^{-\frac{2}{g^2} \langle X, X \rangle} \prod_{\alpha > 0} \left(\alpha(X) + \frac{g^2}{4} \langle \alpha, w \rangle \right)^2.$$

The leading contribution at strong coupling is obtained by factoring out $\langle \alpha, w \rangle$ from the integral for the roots α in the Lie algebra \mathfrak{g} for which

¹⁹The highest weight appears with multiplicity one in the set $\Omega(R)$ of all possible weights in the representation, as otherwise the representation would be reducible.

$\langle \alpha, w \rangle \neq 0$. This yields

$$\begin{aligned} \langle W(R) \rangle_{G,\tau} &= \frac{\text{Vol}(G/T)}{|\mathcal{W}(H)|\mathcal{Z}} e^{\frac{g^2}{8}\langle w,w \rangle} \left(\prod_{\alpha>0, \langle \alpha,w \rangle \neq 0} \frac{g^2}{4} \langle \alpha, w \rangle \right)^2 \\ &\times \int_{\mathfrak{t}} [dX] e^{-\frac{2}{g^2}\langle X,X \rangle} \prod_{\substack{\alpha>0 \\ \langle \alpha,w \rangle = 0}} \alpha(X)^2. \end{aligned} \tag{3.5}$$

We can now perform the integral over the Cartan subalgebra elements X , which is proportional to the inverse of $\text{Vol}(H/T)$ (see Appendix C for details). The expectation value of the circular Wilson loop $W(R)$ in $\mathcal{N} = 4$ super Yang–Mills at strong coupling is then given by

$$\langle W(R) \rangle_{G,\tau} = \exp\left(\frac{\langle w, w \rangle}{8} g^2\right) \left(\frac{g^2}{8\pi}\right)^{\dim(G/H)/2} \text{Vol}(G/H) \prod_{\alpha>0, \langle \alpha,w \rangle \neq 0} \langle \alpha, w \rangle^2, \tag{3.6}$$

where w is the highest weight of the representation R of G .²⁰

4 S-duality for loop operators

$\mathcal{N} = 4$ super Yang–Mills with gauge group G is conjectured to have a symmetry group $\Gamma \subset \text{SL}(2, \mathbb{R})$, which acts on all the gauge invariant operators in the theory as well as on the complexified coupling constant

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2},$$

on which it acts by fractional linear transformations

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}).$$

The symmetry group Γ has two generators, usually denoted by T and S . T generates the classical symmetry

$$T : \tau \rightarrow \tau + 1,$$

which follows by inspecting the $\mathcal{N} = 4$ super Yang–Mills path integral. S conjecturally generates a quantum symmetry which exchanges the gauge

²⁰We evaluate (3.6) for some sample representations and gauge groups in Appendix D.

group G with the dual group ${}^L G$ and inverts the coupling constant

$$S : \tau \rightarrow -\frac{1}{n_{\mathfrak{g}}\tau}, \tag{4.1}$$

where $n_{\mathfrak{g}}$ is the ratio $|\text{long root}|^2/|\text{short root}|^2$ for \mathfrak{g} (see footnote 1). When the Lie algebra is simply laced $\Gamma = \text{SL}(2, \mathbb{Z})$.

In $\mathcal{N} = 4$ super Yang–Mills with gauge group G , a Wilson operator is labeled by a representation R of G while an 't Hooft operator is labeled by a representation ${}^L R$ of the dual group ${}^L G$. Under the action of S -duality a Wilson operator in the theory with gauge group G maps to an 't Hooft operator in the theory with the dual group ${}^L G$ and vice versa

$$\begin{array}{ccc} G & & {}^L G \\ W(R) & \longleftrightarrow & T(R) \\ T({}^L R) & \longleftrightarrow & W({}^L R) \end{array}$$

Non-trivial evidence for S -duality was presented by Kapustin [10], where it was shown that given a Wilson operator in the theory with gauge group G that one can construct the classical singularity of an 't Hooft operator for the theory with dual gauge group ${}^L G$ with precisely the same quantum numbers as the original Wilson operator. The S -duality conjecture further predicts that the correlation functions of dual operators are the same. In particular, S -duality predicts that the expectation value of an 't Hooft operator gets mapped to the expectation value of a Wilson operator in the dual theory

$$\langle T({}^L R) \rangle_{G, \tau} = \langle W({}^L R) \rangle_{{}^L G, L\tau}. \tag{4.2}$$

We now use our computation of the semiclassical 't Hooft operator expectation value, and of the expectation value of the Wilson operator strong coupling to exhibit that correlation functions in $\mathcal{N} = 4$ super Yang–Mills transform precisely as predicted by S -duality. We recall that up to one loop order, the expectation value of a circular 't Hooft loop operator in $\mathcal{N} = 4$ super Yang–Mills with gauge group G is given by (2.15)

$$\begin{aligned} \langle T({}^L R) \rangle_{G, \tau} &= \exp\left(\frac{\text{tr}(B^2)}{8} g^2 |\tau|^2\right) \left(\frac{g^2 |\tau|^2}{8\pi}\right)^{\dim(G/H)/2} \\ &\times \text{Vol}(G/H) \prod_{\substack{\alpha > 0 \\ \alpha(B) \neq 0}} \alpha(B)^2, \end{aligned} \tag{4.3}$$

where B is the highest weight of the representation ${}^L R$ of ${}^L G$. The dual operator is a circular Wilson loop in $\mathcal{N} = 4$ super Yang–Mills with gauge

group ${}^L G$, whose expectation value at strong coupling is given by²¹

$$\begin{aligned} \langle W({}^L R) \rangle_{L_G, L_\tau} &= \exp\left(\frac{\langle {}^L w, {}^L w \rangle} 8 Lg^2\right) \left(\frac{Lg^2}{8\pi}\right)^{\dim({}^L G/{}^L H)/2} \\ &\times \text{Vol}({}^L G/{}^L H) \prod_{\substack{L\alpha > 0 \\ \langle L\alpha, {}^L w \rangle \neq 0}} \langle {}^L \alpha, {}^L w \rangle^2, \end{aligned} \tag{4.4}$$

where ${}^L w$ is the highest weight of the representation ${}^L R$ of ${}^L G$.

In order to study the prediction of S -duality for these correlators, we first note that the action of S -duality on the coupling constant (4.1) implies that

$$({}^L g)^2 = n_{\mathfrak{g}} g^2 |\tau|^2. \tag{4.5}$$

We should also pay attention to the difference between $\text{tr}(B^2)$ and $\langle {}^L w, {}^L w \rangle$ when comparing the correlators, since they are constructed in terms of the metric defined on \mathfrak{t} and ${}^L \mathfrak{t}^*$, respectively. These metrics are in turn induced from the metrics on \mathfrak{g} and ${}^L \mathfrak{g}$. In our computations, we have normalized the Lie algebra metrics that appear in the Lagrangians so that short coroots have length-squared equal to two, or equivalently long roots have length-squared equal to two. However, roots in G are identified with coroots of the dual group ${}^L G$, so a long root in G is identified with a long coroot in ${}^L G$. It follows that when we identify \mathfrak{t} with ${}^L \mathfrak{t}^*$, the metric on \mathfrak{t} is $n_{\mathfrak{g}}$ times the metric on ${}^L \mathfrak{t}^*$. Therefore, the norms of B and ${}^L w$ are related by²²

$$\text{tr}(B^2) = n_{\mathfrak{g}} \langle {}^L w, {}^L w \rangle. \tag{4.6}$$

Finally, we can relate $\text{Vol}(G/H)$ and $\text{Vol}({}^L G/{}^L H)$ (see Appendix C):

$$\text{Vol}(G/H) \prod_{\substack{\alpha > 0 \\ \alpha(B) \neq 0}} \alpha(B)^2 = n_{\mathfrak{g}}^{\dim(G/H)/2} \text{Vol}({}^L G/{}^L H) \prod_{\substack{L\alpha > 0 \\ \langle L\alpha, {}^L w \rangle \neq 0}} \langle L\alpha, {}^L w \rangle^2. \tag{4.7}$$

Inserting equations (4.5–4.7) into the formula for the expectation value of the Wilson loop at strong coupling (4.4), we find precisely the same result we obtained for the expectation value of the 't Hooft operator (4.3) in the dual theory.

²¹Note that in Section 3 we calculated the Wilson loop for gauge group G , while here we need the result for gauge group ${}^L G$. This explains the appearance of the dual group, dual coupling and so on.

²²More precisely, the isomorphism $\varphi : {}^L \mathfrak{t}^* \rightarrow \mathfrak{t}$ satisfies $B = \varphi({}^L w)$, $\langle \varphi({}^L w), \varphi({}^L w) \rangle = n_{\mathfrak{g}} \langle {}^L w, {}^L w \rangle$.

To summarize, we have shown up to the order we have computed that the expectation value of a Wilson operator is exchanged under S -duality with the expectation value of an 't Hooft operator, thus exhibiting that these correlation functions in $\mathcal{N} = 4$ super Yang–Mills transform precisely as predicted by the S -duality conjecture.

5 Saddle points from monopole screening

In the computation of the expectation value of the circular Wilson loop operator $W({}^L R)$ in $\mathcal{N} = 4$ super Yang–Mills with gauge group ${}^L G$, we have represented²³ the insertion of the group character in the matrix integral in terms of the sum over all weights ${}^L v \in \Omega({}^L R)$ in the representation ${}^L R$ of ${}^L G$

$$\text{Tr}_{{}^L R} e^X = \sum_{{}^L v \in \Omega({}^L R)} n({}^L v) e^{{}^L v(X)},$$

where $n({}^L v)$ is the multiplicity of the weight ${}^L v$ and $\Omega({}^L R)$ is the set of all weights in the representation ${}^L R$.

We have noted that the leading contribution at strong coupling ${}^L g \gg 1$ arises from the longest weights, those with maximal norm $\langle {}^L v, {}^L v \rangle$. These in turn are obtained from the highest weight of the representation ${}^L w \in \Omega({}^L R)$ by the action of the Weyl group.

It is instructive to also consider the effect of the non-longest weights in the representation ${}^L v \in \Omega({}^L R)$ to the expectation value of the Wilson loop. The leading contribution of a non-longest weight ${}^L v \in \Omega({}^L R)$ at strong coupling is proportional to

$$\begin{aligned} & \exp\left(\frac{\langle {}^L v, {}^L v \rangle}{{}^L g^2}\right) \left(\frac{{}^L g^2}{8\pi}\right)^{\frac{1}{2} \dim({}^L G / {}^L H({}^L v))} \text{Vol}({}^L G / {}^L H({}^L v)) \\ & \times \prod_{\substack{{}^L \alpha > 0 \\ \langle {}^L \alpha, {}^L v \rangle \neq 0}} \langle {}^L \alpha, {}^L v \rangle^2, \end{aligned} \tag{5.1}$$

where ${}^L H({}^L v)$ is the subgroup of ${}^L G$ that leaves the weight ${}^L v$ invariant. In the strong coupling limit, this contribution is exponentially suppressed with

²³Note that we are considering a Wilson loop in the theory with gauge group ${}^L G$, which explains the appearance of the dual representation, dual coupling and so on when compared to Section 3.

respect to the contribution from the longest weights, which is proportional to $\exp[\langle Lw, Lw \rangle Lg^2/8] \gg \exp[\langle Lv, Lv \rangle Lg^2/8]$.

S -duality predicts that these subleading contributions to the Wilson loop at strong coupling should also arise in the semiclassical computation of the expectation value of the 't Hooft operator. We argue that these subleading exponentials appear in the 't Hooft loop correlator via the physics of monopole screening.

The physics of screening of an 't Hooft operator by a regular monopole is S -dual to the more familiar screening of an electric source by dynamical gluons. The non-abelian charge inserted by a Wilson loop in a representation ${}^L R$ of ${}^L G$ can be screened by gluons, which also carry non-abelian charge and are constantly appearing and disappearing from the vacuum due to quantum fluctuations. The gluons are, however, uncharged under the center $Z({}^L G) \subset {}^L G$. Therefore, the quantum number that cannot be screened is the charge of the representation ${}^L R$ of the Wilson loop under $Z({}^L G)$. On the other hand, since $\pi_1(G) \simeq Z({}^L G)$, the quantum number associated with the S -dual 't Hooft operator $T({}^L R)$ that cannot be screened by regular monopoles is an element of $\pi_1(G)$, which is precisely the topological charge carried by an 't Hooft operator [9].

In the path integral definition of a 't Hooft loop $T({}^L R)$ we have quantized the singularity produced by $T({}^L R)$ in the background field gauge. The singularity determining the field configuration near the loop is specified by the highest weight ${}^L w$, which gets identified with B . The norm of B — $\text{tr}(B^2)$ — determines the strength of the singularity. The subleading exponentials predicted by S -duality arise from boundaries of the region of integration in field space where the singularity produced by the 't Hooft operator $T({}^L R)$ is weaker, and is controlled by a non-longest weight ${}^L v$ of the representation ${}^L R$. The necessity to include the less singular configurations was noticed in [21], where the phenomenon was dubbed “monopole bubbling.” This weaker singularity arises physically from the physics of monopole screening, whereby a regular 't Hooft-Polyakov monopole approaches a singular monopole (an 't Hooft operator), and screens the charge of the 't Hooft operator.

The charges of regular 't Hooft-Polyakov monopoles are spanned by the simple coroots of the Lie algebra \mathfrak{g} , which generate the coroot lattice. When we bring a regular monopole — labeled by a coroot — near an 't Hooft operator $T({}^L R)$, the charge of the 't Hooft operator is screened. The resulting effective charge is obtained by the action of lowering operator associated with the coroot labeling the regular monopole on the highest weight ${}^L w$ that characterizes the singularity of the 't Hooft operator $T({}^L R)$. The action of

the ladder operators associated with the regular monopoles on the highest weight Lw generates all the weights in the representation LR [21].

This was explicitly realized in [37] by constructing a classical solution to the equations of motion that contains a regular monopole in the presence of a singular monopole in the cases $G = \text{SU}(2)$ and $G = \text{SO}(3)$. When the gauge group is $G = \text{SU}(2)$, the dual group is ${}^LG = \text{SO}(3)$, and the minimal 't Hooft operator $T(1)$ carries spin one with respect to ${}^LG = \text{SO}(3)$. In the spin one representation a state with vanishing weight can be obtained from the highest weight state by applying the lowering operator. This can be translated into the language of monopole screening by noting that $T(1)$ can be completely screened by an 't Hooft–Polyakov monopole. Indeed the singularity in the solution disappears when the regular monopole approaches the 't Hooft operator. If on the other hand $G = \text{SO}(3)$, the minimal 't Hooft operator $T(1/2)$ carries spin one-half with respect to ${}^LG = \text{SU}(2)$. The spin one-half representation has no state with vanishing weight and so the 't Hooft operator cannot be screened. In the solution, the size of the regular monopole remains finite as it approaches the singularity, and the strength of the singularity remains intact/unscreened.

The subleading saddle points corresponding to the non-longest weights in the representation should then be included in the path integral and can be computed in the same manner as we have done in Section 2. Instead of quantizing the singularity of the 't Hooft operator $T({}^LR)$ associated with the highest weight of LR , we quantize the singularity produced by the weaker singularities that appear at the boundaries of the region of integration of field space, which are labeled by the weights of LR . These saddle points are then in one-to-one correspondence with the subleading contributions to the expectation value of the Wilson loop in (5.1), and reproduce the Wilson loop result including the prefactor.

6 Discussion

We conclude by summarizing our results and describing several interesting lines of inquiry stemming from this work. First we have defined the renormalized 't Hooft operator in terms of a path integral quantized in the background field gauge around a certain codimension three singularity created by the operator. We have shown that an important ingredient that goes into the definition of the operator is the measure of integration in the path integral, which is dictated by gauge invariance, and which requires integrating over the adjoint orbit of the classical singularity produced by the 't Hooft operator. This measure factor should be included in the computation

of a general 't Hooft operator in an arbitrary gauge theory. We have then explicitly computed the expectation value of the circular 't Hooft loop operator in $\mathcal{N} = 4$ super Yang–Mills up to one loop order. Computations to higher orders in perturbation theory can now be carried out by summing over all connected vacuum diagrams generated by the path integral.

By solving for the expectation value of the S -dual Wilson operator at strong coupling we have been able to exhibit that correlation functions in $\mathcal{N} = 4$ super Yang–Mills transform properly under S -duality. We have shown that the perturbative result for the expectation value of the 't Hooft operator up to one loop order exactly reproduces the strong coupling expansion of the S -dual Wilson loop in the dual theory. Unlike most of the previous studies of S -duality, the matching goes beyond comparing “topological” features like the spectra, quantum numbers and so on, but it rather tests the quantum dynamics underlying S -duality.²⁴

We have also argued that the subleading exponential corrections to the expectation value of the Wilson loop at strong coupling can be identified with the weaker singularities that appear near an 't Hooft loop due to monopole screening. It would be interesting to understand in more detail the contribution from these subleading saddle points. In this respect, it would be illuminating to try to evaluate the path integral for the circular 't Hooft loop using localization techniques, extending to monopole operators the work by Pestun [24] for Wilson loops. We expect that the subleading saddle points arise in this context as solutions to the localization equations. Furthermore, S -duality predicts that the expectation value of the circular 't Hooft operator in $\mathcal{N} = 4$ super Yang–Mills is also described by a matrix model (see equation (3.4)), so it would be desirable to give a direct derivation of this matrix model from the 't Hooft loop path integral.

A worthwhile future direction is to extend our computation for the 't Hooft loop expectation value to other gauge theories, in particular to finite $\mathcal{N} = 2$ theories. An interesting class of $\mathcal{N} = 2$ superconformal field theories are those that cannot be obtained by quotienting $\mathcal{N} = 4$ super Yang–Mills, such as $\mathcal{N} = 2$ $SU(N)$ super Yang–Mills coupled to $2N$ fundamental hypermultiplets [40, 41], which are conjectured to be invariant under an S -duality group $\Gamma \subset SL(2, \mathbb{Z})$ [40, 42] and to exhibit rich duality relations at strong coupling [43]. $\mathcal{N} = 2$ orbifolds [44] of $\mathcal{N} = 4$ super Yang–Mills are also of interest, and Wilson and 't Hooft operators in these theories are relevant probes for the very rich S -duality groups in these theories, where for instance, the

²⁴Note that our test of S -duality is purely in field theory. The tests that are based on AdS/CFT and the identification of S -dualities in $\mathcal{N} = 4$ super Yang–Mills and type IIB superstring theory include [23, 38, 39].

S -duality group of the \hat{A}_{n-1} quiver gauge theory is conjectured to be the mapping class group of a torus with n punctures [45]. Very recently, similar duality relations were conjectured for a larger class of $\mathcal{N} = 2$ conformal theories with more than one gauge group [46]. Moreover gravity duals of such $\mathcal{N} = 2$ theories have been proposed [47]. Computing correlators of 't Hooft operators and Wilson operators provide a useful framework to explore the conjectured S -duality maps as well as the holographic correspondences.

It is also of interest to go beyond the computation of the expectation value, and determine whether the perturbative correlators of 't Hooft operators with local operators get mapped to the corresponding strong coupling correlators of the S -dual operators in the S -dual theory [48], by generalizing the large N results in [39] to finite rank. One can also extend the computations in the present paper to correlators of disorder operators in three-dimensional $\mathcal{N} = 6$ superconformal Chern–Simons theories [49], such as monopole operators [50, 51] and vortex loop operators [52]. Another rich class of operators that deserve further study are the mixed Wilson-'t Hooft loop operators in $\mathcal{N} = 4$ super Yang–Mills [10]. These operators insert dyonic probe particles and have interesting conjectured transformation properties under S -duality [10]. Giving a quantum definition of these operators and studying their correlation functions opens a novel arena in which to probe the quantum dynamics underlying S -duality.

Wilson and 't Hooft operators exhibit the area law in the confining and Higgs phases respectively, and are order parameters for these phases. Is there a similar interpretation for the tree-level result (2.7), which applies to the 't Hooft loop of any gauge theory in the (Abelian or non-Abelian) Coulomb phase? A notable feature is its dependence on the theta angle θ . Therefore it can be used to distinguish phases of a theory that have different values of θ . Precisely, such phases for gauge theories with $U(1)$ gauge group have been discussed recently in the condensed matter literature. The orbital motion of electrons has been shown to generate non-zero θ [53]. The so-called \mathbb{Z}_2 topological insulators in $3 + 1$ dimensions are particularly interesting examples, where time reversal symmetry sets $\theta = \pi$ [54, 55]. Thus the expectation value (2.7) distinguishes the topologically non-trivial phase at $\theta = \pi$ from the vacuum at $\theta = 0$.²⁵

Explicitly showing that the vacuum expectation value of 't Hooft operators are exchanged under duality with the correlation function of Wilson operators

²⁵For a Wilson-'t Hooft operator $WT_{m,n}$ carrying electric and magnetic charges (m, n) , the expectation value in the $\theta = 0$ and π phases are, respectively,

$$\langle WT_{m,n} \rangle_{\theta=0} = \exp\left(\frac{g^2}{8}m^2 + \frac{2\pi^2}{g^2}n^2\right), \text{ and } \langle WT_{m,n} \rangle_{\theta=\pi} = \exp\left(\frac{g^2}{8}\left(m + \frac{n}{2}\right)^2 + \frac{2\pi^2}{g^2}n^2\right).$$

is a step in the right direction towards the goal of finding the electric–magnetic duality transformation that relates the two dual descriptions. In the past, there have been attempts to formulate Yang–Mills theories directly in terms of gauge invariant variables, i.e., Wilson variables. In this formulation of the theory, the non-perturbative information of the theory is encoded in the loop equation, which describes the dynamics of Wilson loop operators in loop space. Constructing the loop equation for the ’t Hooft loop variables and studying how it maps to the Wilson loop equation, may provide a non-perturbative framework in which to study the transformation between gauge invariant electric and magnetic variables that underlie S -duality in $\mathcal{N} = 4$ super Yang–Mills.

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Appendix A Weyl transforms between metrics

In this appendix we discuss the two Weyl transformations relating \mathbb{R}^4 and $\text{AdS}_2 \times S^2$, which we have used in Section 2. The first transformation is relevant for the circular ’t Hooft loop computation, and the second one for the straight line.

Let us parametrize \mathbb{R}^4 using two sets of polar coordinates so that

$$ds_{\mathbb{R}^4}^2 = dl^2 + l^2 d\psi^2 + dL^2 + L^2 d\phi^2. \tag{A.1}$$

These coordinates are relevant for a circular loop, which we take to be located at $l = a$ and $L = 0$. By making the following change of coordinates

$$\begin{aligned} \Omega^2 &= \frac{(l^2 + L^2 - a^2)^2 + 4a^2 L^2}{4a^2} = \frac{a^2}{(\cosh \rho - \cos \theta)^2}, \\ l &= \Omega \sinh \rho, \quad L = \Omega \sin \theta, \end{aligned} \tag{A.2}$$

we find that the metric becomes

$$ds_{\mathbb{R}^4}^2 = \Omega^2 (ds_{\text{AdS}_2}^2 + d\theta^2 + \sin^2 \theta d\phi^2), \tag{A.3}$$

where

$$ds_{\text{AdS}_2}^2 = d\rho^2 + \sinh^2 \rho d\psi^2 \tag{A.4}$$

is the metric on the AdS_2 Poincaré disk in global coordinates. Thus \mathbb{R}^4 is conformal to $\text{AdS}_2 \times S^2$. Note that the loop, which was located at $l = a, L = 0$ in \mathbb{R}^4 , gets mapped to the conformal boundary of the Poincaré disk.

The metric for \mathbb{R}^4 can also be written as

$$ds_{\mathbb{R}^4}^2 = dt^2 + dr^2 + r^2 d\Omega_2^2, \tag{A.5}$$

where $d\Omega_2^2$ is the S^2 metric. We place the straight line at $r = 0$. In this case the Weyl transformation to $\text{AdS}_2 \times S^2$ produces the hyperbolic metric on the upper half-plane

$$ds_{\mathbb{R}^4}^2 = r^2 (ds_{\text{AdS}_2}^{\prime 2} + d\Omega_2^2), \tag{A.6}$$

where

$$ds_{\text{AdS}_2}^{\prime 2} = \frac{dt^2 + dr^2}{r^2} \tag{A.7}$$

is the AdS_2 metric in Poincaré coordinates.

Appendix B Cancellation of non-zero modes

In this appendix, we show the cancellation of the one loop determinants in (2.11). This will be first done for the straight line and $\theta = 0$ using self-duality of the background, and then we will generalize it to non-zero θ and the circular loop.

Let us package the four-dimensional bosonic fields into the 10-dimensional gauge field in the order

$$(A_M) \equiv (A_1, \dots, A_3, \phi^1, \dots, \phi^6, A_0), \quad M = 1, 2, \dots, 10. \quad (\text{B.1})$$

Without loss of generality we can take $\phi = \phi^1 = A_4$ in (2.4). Then the non-zero components of the 10-dimensional background field strength are given by

$$F_{ij} = \frac{B}{2r^3} \epsilon_{ijk} x^k, \quad F_{4i} = \frac{B}{2r^3} x^i, \quad i, j = 1, 2, 3. \quad (\text{B.2})$$

If we let the index μ take values $\mu = 1, \dots, 4$, the four-dimensional field strength $F_{\mu\nu}$ is anti-self-dual. To exploit this, we represent the 10-dimensional gamma matrices in terms of four- and six-dimensional ones as

$$\Gamma^\mu = \gamma^\mu \otimes 1, \quad \Gamma^m = \gamma \otimes \gamma^m, \quad (m = 5, 6, \dots, 10) \quad (\text{B.3})$$

and further decompose γ^μ as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\delta^{\mu\nu} \quad (\text{B.4})$$

using $\sigma^4 = i = -\bar{\sigma}^4$, $\sigma^j = \bar{\sigma}^j$.

We begin with the fermionic determinant in (2.11), where it is raised to the power $1/4 = 1/2 \times 1/2$ because the fermion ψ satisfies the 10-dimensional Weyl and Majorana conditions (see footnote 14). It is convenient to compute the square (we omit the subscript 0 hereafter)

$$\begin{aligned} (i\Gamma^M D_M)^2 &= -D^2 1_{32} + \frac{i}{2} \Gamma^{MN} F_{MN} \\ &= \begin{pmatrix} -D^2 1_2 + \frac{i}{2} \sigma^{\mu\nu} F_{\mu\nu} & \\ & -D^2 1_2 \end{pmatrix} \otimes 1_8, \end{aligned} \quad (\text{B.5})$$

where we defined $\sigma^{\mu\nu} \equiv (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)/2$ and used the anti-self-duality of

$F_{\mu\nu}$. Thus the fermionic contribution is

$$\begin{aligned} [\det_f (i\Gamma^M D_M)]^{1/4} &= [\det' (i\Gamma^M D_M)]^{1/8} \\ &= \det' \left(-D^2 1_2 + \frac{i}{2} \sigma^{\mu\nu} F_{\mu\nu} \right) \det' (-D^2)^2, \end{aligned} \quad (\text{B.6})$$

where the prime indicates the omission of zero modes.

For the bosons, we have

$$[\det_b (-\delta^{MN} D^2 + 2iF^{MN})]^{-1/2} = \det' (-D^2)^{-3} \det' (-D^2 \delta_{\mu\nu} + 2iF_{\mu\nu})^{-1/2}. \quad (\text{B.7})$$

Observe that

$$-D^2 \delta_{\mu\nu} + 2iF_{\mu\nu} = \frac{1}{2} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \left(-D^2 1_2 + \frac{i}{2} \sigma^{\rho\sigma} F_{\rho\sigma} \right) \alpha^\beta \delta_\alpha^{\dot{\beta}} \sigma_{\nu\beta\dot{\beta}}. \quad (\text{B.8})$$

By treating $(\dot{\alpha}\alpha)$ and $(\beta\dot{\beta})$ as single indices taking four values, we can regard $\bar{\sigma}_\mu^{\dot{\alpha}\alpha}$ and $\sigma_{\nu\beta\dot{\beta}}$ as 4×4 matrices. Then we see that

$$\det' (-D^2 \delta_{\mu\nu} + 2iF_{\mu\nu}) = \det' \left(-D^2 1_2 + \frac{i}{2} \sigma^{\mu\nu} F_{\mu\nu} \right)^2. \quad (\text{B.9})$$

Thus the bosonic contribution can be written as

$$\begin{aligned} &[\det_b (-\delta^{MN} D^2 + 2iF^{MN})]^{-1/2} \\ &= \det' (-D^2)^{-3} \det' \left(-D^2 + \frac{i}{2} \sigma^{\mu\nu} F_{\mu\nu} \right)^{-1}. \end{aligned} \quad (\text{B.10})$$

Finally, the ghost contribution is simply given by

$$\det_g (-D^2) = \det' (-D^2). \quad (\text{B.11})$$

We see that the three contributions (B.6), (B.10) and (B.11) cancel out in (2.11):

$$\frac{[\det_f (i\Gamma^M D_M)]^{1/4} \det_g (-D^2)}{[\det_b (-\delta^{MN} D^2 + 2iF^{MN})]^{1/2}} = 1. \quad (\text{B.12})$$

When the theta angle is turned on, the background fields change to

$$F_{0i} = ig^2 \theta \frac{B}{16\pi^2} \frac{x^i}{r^3}, \quad F_{ij} = \frac{B}{2r^3} \epsilon_{ijk} x^k, \quad F_{4i} = \frac{B}{2r^3} x^i \left(1 + \frac{g^4 \theta^2}{64\pi^4} \right)^{1/2} \quad (\text{B.13})$$

with other components of F_{MN} vanishing. If we rotate the gauge field and Gamma matrices into

$$\begin{pmatrix} A'_0 \\ A'_4 \end{pmatrix} \equiv R(\theta) \begin{pmatrix} A_0 \\ A_4 \end{pmatrix}, \quad \begin{pmatrix} \Gamma'^0 \\ \Gamma'^4 \end{pmatrix} \equiv R(\theta) \begin{pmatrix} \Gamma^0 \\ \Gamma^4 \end{pmatrix} \quad (\text{B.14})$$

by a complex orthogonal matrix

$$R(\theta) = \begin{pmatrix} \left(1 + \frac{g^4 \theta^2}{64\pi^4}\right)^{1/2} & -i \frac{g^2 \theta}{8\pi^2} \\ i \frac{g^2 \theta}{8\pi^2} & \left(1 + \frac{g^4 \theta^2}{64\pi^4}\right)^{1/2} \end{pmatrix}, \quad (\text{B.15})$$

with $A'_M = A_M$, $\Gamma'^M = \Gamma^M$ for other M , the corresponding field strength $F'_{\mu\nu}$ remains anti-self-dual, and we have the relation

$$\Gamma^{MN} F_{MN} = \Gamma'^{\mu\nu} F'_{\mu\nu}. \quad (\text{B.16})$$

Thus the proof for the cancellation of non-zero modes still goes through for the straight line. Relation (B.16) also implies that the background remains BPS for $\theta \neq 0$.

These arguments for the line can be mapped to the circular loop formally by a conformal transformation and by using different gauge fixing terms that are generated by the transformation. Since such cancellation does not depend on the gauge fixing procedure, the non-zero modes are also cancelled for the circular loop.

Appendix C Volumes of groups and coset spaces

To relate the integral in (3.5) to $\text{Vol}(H/T)$, let us consider the special case when $w = 0$. The Wilson loop expectation value is then unity. By substituting (3.2) and $w = 0$ in (3.5), we obtain

$$1 = \left(\frac{2}{\pi g^2}\right)^{\dim(G)/2} \frac{\text{Vol}(G/T)}{|\mathcal{W}|} \int_{\mathfrak{t}} [dX] e^{-\frac{2}{g^2} \langle X, X \rangle} \prod_{\alpha > 0} \alpha(X)^2. \quad (\text{C.1})$$

This formula is very general and can be applied to the case when G is

replaced by H , the stability group of the highest weight w , and is given by

$$1 = \left(\frac{2}{\pi g^2} \right)^{\dim(H)/2} \frac{\text{Vol}(H/T)}{|\mathcal{W}(H)|} \int_{\mathfrak{t}} [dX] e^{-\frac{2}{g^2} \langle X, X \rangle} \prod_{\substack{\alpha > 0 \\ \langle \alpha, w \rangle = 0}} \alpha(X)^2. \quad (\text{C.2})$$

We can use (C.2) to eliminate the integral in (3.5) in favor of $\text{Vol}(H/T)$. Though (C.1) and (C.2) are sufficient for our purposes, we note that such integrals have been explicitly evaluated in [34] in terms of the exponents of the Lie algebras.

Let us now derive relation (4.7). First consider setting (g^2, w) to $(2, B)$. By taking the ratio of (C.1) and (C.2), we obtain

$$\text{Vol}(G/H) = \pi^{\dim(G/H)/2} \frac{|\mathcal{W}|}{|\mathcal{W}(H)|} \frac{\int_{\mathfrak{t}} [dX] e^{-\langle X, X \rangle} \prod_{\alpha > 0, \alpha(B)=0} \alpha(X)^2}{\int_{\mathfrak{t}} [dX] e^{-\langle X, X \rangle} \prod_{\alpha > 0} \alpha(X)^2}. \quad (\text{C.3})$$

Next by setting $(g^2, G, H, \mathcal{W}, \mathcal{W}(H), \dots)$ to $(2/n_{\mathfrak{g}}, {}^L G, {}^L H, \mathcal{W}({}^L G), \mathcal{W}({}^L H), \dots)$, we derive by the same procedure

$$\begin{aligned} \text{Vol}({}^L G/{}^L H) &= \left(\frac{\pi}{n_{\mathfrak{g}}} \right)^{\dim({}^L G/{}^L H)/2} \frac{|\mathcal{W}({}^L G)|}{|\mathcal{W}({}^L H)|} \\ &\times \frac{\int_{{}^L \mathfrak{t}} [d{}^L X] e^{-n_{\mathfrak{g}} \langle {}^L X, {}^L X \rangle} \prod_{L\alpha > 0, \langle Lw, L\alpha \rangle = 0} L\alpha({}^L X)^2}{\int_{{}^L \mathfrak{t}} [d{}^L X] e^{-n_{\mathfrak{g}} \langle {}^L X, {}^L X \rangle} \prod_{L\alpha > 0} L\alpha({}^L X)^2}. \end{aligned} \quad (\text{C.4})$$

Under the isomorphisms between $\mathfrak{t}, \mathfrak{t}^*, {}^L \mathfrak{t}$ and ${}^L \mathfrak{t}^*$, we can identify $({}^L X, L\alpha)$ with (X, α) . This involves rescaling of the metric (see footnote 22) $\langle {}^L X, {}^L X \rangle = \langle X, X \rangle / n_{\mathfrak{g}}$ and the relation

$$L\alpha({}^L X) = \frac{2\alpha(X)}{n_{\mathfrak{g}} \langle \alpha, \alpha \rangle}. \quad (\text{C.5})$$

Noting that $\dim {}^L G = \dim G, \dim {}^L H = \dim H, \mathcal{W} \simeq \mathcal{W}({}^L G), \mathcal{W}({}^L H) \simeq \mathcal{W}(H)$, we can put (C.4) into the form

$$\begin{aligned} \text{Vol}({}^L G/{}^L H) &= (\pi n_{\mathfrak{g}})^{\dim(G/H)/2} \frac{|\mathcal{W}|}{|\mathcal{W}(H)|} \left(\prod_{\substack{\alpha > 0 \\ \alpha(B) \neq 0}} \frac{\langle \alpha, \alpha \rangle}{2} \right)^2 \\ &\times \frac{\int_{\mathfrak{t}} [dX] e^{-\langle X, X \rangle} \prod_{\alpha > 0, \alpha(B)=0} \alpha(X)^2}{\int_{\mathfrak{t}} [dX] e^{-\langle X, X \rangle} \prod_{\alpha > 0} \alpha(X)^2}. \end{aligned} \quad (\text{C.6})$$

By cancelling the ratios of integrals in (C.3) and (C.6), and using the relation $\langle L\alpha, Lw \rangle = 2\alpha(B) / \langle \alpha, \alpha \rangle n_{\mathfrak{g}}$, we finally obtain (4.7).

Appendix D Examples of Wilson loop expectation values

To illustrate the formula (3.6) for the Wilson loop and compare it with the corresponding formula (2.15) for the 't Hooft loop, let us give some examples.

- $G = \text{SU}(2)$ and $\text{SO}(3)$. Irreducible representations are labeled by the spin j . According to the formula (3.6) the expectation value at large g^2 is

$$\langle W(j) \rangle_{G,\tau} = \exp\left(\frac{j^2}{4}g^2\right) j^2 g^2. \tag{D.1}$$

- $G = U(N)$. A highest weight $w = [m_1, \dots, m_N]$ satisfies $m_1 \geq \dots \geq m_N$.
 - $w = [k, 0, \dots, 0]$. For the rank- k symmetric representation

$$\langle W([k, 0, \dots, 0]) \rangle_{G,\tau} = \exp\left(\frac{k^2}{8}g^2\right) \left(\frac{k^2 g^2}{4}\right)^{N-1} \frac{1}{(N-1)!}. \tag{D.2}$$

- $w = [\underbrace{1, \dots, 1}_{k \text{ times}}, 0, \dots, 0]$. The rank- k antisymmetric representation gives

$$\begin{aligned} &\langle W([1, \dots, 1, 0, \dots, 0]) \rangle_{G,\tau} \\ &= \exp\left(\frac{k}{8}g^2\right) \left(\frac{g^2}{4}\right)^{k(N-k)} \frac{\prod_{n=1}^{k-1} n!}{\prod_{n=1}^k (N-n)!}. \end{aligned} \tag{D.3}$$

- $w = [m_1, m_2, \dots, m_N]$ with $m_1 > m_2 > \dots > m_N$. For such a representation

$$\begin{aligned} &\langle W([m_1, m_2, \dots, m_N]) \rangle_{G,\tau} \\ &= \exp\left(\frac{\sum_i m_i^2}{8}g^2\right) \left(\frac{g^2}{4}\right)^{\frac{N(N-1)}{2}} \frac{\prod_{i < j} (m_i - m_j)^2}{\prod_{n=1}^{N-1} n!}. \end{aligned} \tag{D.4}$$

Results (D.1) to (D.4) agree with (2.16) to (2.19) via the S -duality map $g^2 \rightarrow Lg^2 = g^2|\tau|^2$.

D.1 The method of orthogonal polynomials

In Section 3 we have derived an expression for the expectation value of a Wilson loop $W(R)$ at strong coupling (3.6) that is valid for an arbitrary

gauge group G . For illustration purposes, we present here an alternative derivation of $\langle W(R) \rangle$ for the case $G = U(N)$ obtained with the method of orthogonal polynomials.

As we saw in Section 3, the Wilson loop $W(R)$ can be written as a sum over weights of the representation R . At large g , the terms that dominate are the ones corresponding to the weights obtained from the highest weight w by the action of the Weyl group. For $U(N)$, the highest weight can be labeled as $w = (m_1, \dots, m_N)$, where the integers m_i are ordered: $m_1 \geq m_2 \geq \dots \geq m_N$. We introduce integers N_I ($I = 1, \dots, M$) such that $w = (m_1, \dots, m_N)$ contains M distinct integers, with the I th one appearing N_I times. One can rotate the matrix M in (3.1) to a diagonal configuration with eigenvalues $\{x_i\}$ at the cost of introducing a Vandermonde determinant $\Delta^2 = \prod_{i < j} (x_i - x_j)^2$ in the integration measure. This determinant can be rewritten in terms of polynomials that are orthogonal with respect to the Gaussian measure, so that the integral to compute becomes

$$\langle W(R) \rangle \simeq \frac{1}{\prod_{I=1}^M N_I!} \int \left(\prod_{i=1}^N dx_i \right) \det(\{P_{j-1}(x_i)\})^2 e^{-\sum_i x_i^2} e^{\frac{g}{\sqrt{2}} \sum_i m_i x_i}.$$

These polynomials are normalized Hermite polynomials: $P_n(x) \equiv H_n(x) / \sqrt{2^n n! \sqrt{\pi}}$ [23]. Completing the squares in the exponentials one readily finds

$$\begin{aligned} \langle W(R) \rangle &\simeq \frac{1}{\prod_{I=1}^M N_I!} e^{\frac{g^2}{8} \sum_i m_i^2} \\ &\times \int \left(\prod_{i=1}^N dx_i \right) \det(\{P_{j-1}(x_i)\})^2 e^{-\sum_i (x_i - m_i \frac{g}{2\sqrt{2}})^2}. \end{aligned} \quad (D.5)$$

To obtain the polynomial corrections to the exponential behavior, we need

$$\mathcal{I}_{ij}^{(k)} \equiv \int_{-\infty}^{\infty} dx e^{-\left(x - k \frac{g}{2\sqrt{2}}\right)^2} P_i(x) P_j(x).$$

For this, it is useful to use the contour integral representation of the Hermite polynomials

$$H_n(x) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} dt \frac{e^{-t^2 + 2tx}}{t^{n+1}},$$

where \mathcal{C} encircles the origin counterclockwise. We find

$$\mathcal{I}_{ij}^{(k)} = \frac{1}{\sqrt{i!j!}} \left(\frac{kg}{2}\right)^{i+j} \sum_{\ell=0}^{\min(i,j)} \ell! \binom{i}{\ell} \binom{j}{\ell} \left(\frac{2}{kg}\right)^{2\ell},$$

which can be expressed in terms of a confluent hypergeometric function of the second kind $U(a, b, z)$ as

$$\mathcal{I}_{ij}^{(k)} = \frac{(-1)^i}{\sqrt{i!j!}} \left(\frac{kg}{2}\right)^{j-i} U\left(-i, 1-i+j, -\frac{k^2g^2}{4}\right). \tag{D.6}$$

By applying this to (D.5), we arrive to

$$\begin{aligned} \langle W(R) \rangle &\simeq \frac{(-1)^{\frac{N(N-1)}{2}} e^{\frac{g^2}{8} \sum_i m_i^2}}{\prod_{l=1}^M N_l! \prod_{n=0}^{N-1} n!} \sum_{\sigma, \sigma' \in S_N} \text{sign}(\sigma) \text{sign}(\sigma') \\ &\times \prod_{i=0}^{N-1} m_{i+1}^{\sigma'(i)-\sigma(i)} U\left(-\sigma(i), 1-\sigma(i)+\sigma'(i), -\frac{m_{i+1}^2g^2}{4}\right), \end{aligned} \tag{D.7}$$

where σ and σ' permute $\{0, 1, \dots, N-1\}$.

We can now specialize (D.7) to the representations of $G = U(N)$ that we have considered above, see equations (D.2) to (D.4).

- $w = [k, 0, \dots, 0]$. For the rank- k symmetric representation, we need to use the following limits of the confluent hypergeometric function

$$\lim_{x \rightarrow 0} x^{j-i} U(-i, 1-i+j, -x^2) = (-1)^i i! \delta_{ij}, \tag{D.8}$$

i.e., $\mathcal{I}_{ij}^{(0)} = \delta_{ij}$, and

$$\lim_{x \rightarrow \infty} U(-i, 1-i+j, -x) = (-1)^i x^i. \tag{D.9}$$

We get

$$\begin{aligned} \langle W(R) \rangle &\simeq \frac{e^{\frac{g^2}{8} k^2}}{(N-1)! \prod_{n=0}^{N-1} n!} \sum_{\sigma, \sigma' \in S_N} \text{sign}(\sigma) \text{sign}(\sigma') \left(\frac{g}{2}\right)^{2\sigma(0)} \\ &\times k^{\sigma'(0)+\sigma(0)} \prod_{i=1}^{N-1} \sigma(i)! \delta_{\sigma(i)\sigma'(i)}. \end{aligned}$$

Since we are interested in the highest power of g we must select $\sigma(0) = N - 1$. The product over the Kronecker deltas imposes that $\sigma = \sigma'$, whereas the product over the factorials gives $\prod_{n=0}^{N-2} n!$. By summing over $(N - 1)!$ permutations, we have

$$\langle W(R) \rangle \simeq \frac{1}{(N - 1)!} e^{\frac{g^2}{8} k^2} \left(\frac{g}{2}\right)^{2(N-1)} k^{2(N-1)}.$$

This is what we found in (D.2).

- $w = \underbrace{[1, \dots, 1, 0, \dots, 0]}_{k \text{ times}}$. To compute the rank- k antisymmetric representation we use again the limit (D.8) but, because of the degeneracy of the non-zero m_i 's, we need in this case the full expression (D.6) for the confluent hypergeometric function, since some of the leading polynomial corrections cancel for combinatorial reasons. We find

$$\begin{aligned} \langle W(R) \rangle &\simeq \frac{e^{\frac{g^2}{8} k}}{k!(N - k)! \prod_{n=0}^{N-1} n!} \\ &\times \sum_{\sigma, \sigma' \in S_N} \text{sign}(\sigma) \text{sign}(\sigma') \left(\prod_{j=k}^{N-1} \sigma(j)! \delta_{\sigma(j)\sigma'(j)} \right) \\ &\times \left(\prod_{i=0}^{k-1} \sigma(i)! \sigma'(i)! \sum_{\ell=0}^{\min(\sigma(i), \sigma'(i))} \frac{1}{\ell!(\sigma(i) - \ell)! (\sigma'(i) - \ell)!} \left(\frac{g^2}{4}\right)^{\sigma(i) - \ell} \right). \end{aligned}$$

To obtain the leading order in the coupling constant we require $\{\sigma(0), \dots, \sigma(k - 1)\} = \{N - k, \dots, N - 1\}$, from which it also follows $\{\sigma(k), \dots, \sigma(N - 1)\} = \{0, \dots, N - k - 1\}$ and, because of the Kronecker deltas, $\{\sigma'(0), \dots, \sigma'(k - 1)\} = \{N - k, \dots, N - 1\}$ and $\{\sigma'(k), \dots, \sigma'(N - 1)\} = \{0, \dots, N - k - 1\}$. The power of the coupling constant $g^2/4$, without considering the subleading contributions coming from the sums over ℓ 's, is then given by $\sum_{n=N-k}^{N-1} = k(2N - k - 1)/2$. Because of the degeneracy of the m_i 's there are cancellations though and this power is reduced of $\sum_{n=0}^{k-1} n = k(k - 1)/2$ to give $(g^2/4)^{k(N-k)}$. Selecting only the terms in the sums over ℓ 's that produce this power and performing the sums over $k!$ and $(N - k)!$ permutations, one finally finds the same result as (D.3).

- $w = [m_1, m_2, \dots, m_N]$ with $m_1 > m_2 > \dots > m_N$. In this case we simply expand the hypergeometric functions for large g using (D.9). It is easy

to see that (D.7) becomes

$$\begin{aligned} \langle W(R) \rangle &\simeq \frac{e^{\frac{g^2}{8} \sum_i m_i^2}}{\prod_{n=0}^{N-1} n!} \left(\frac{g}{2}\right)^{N(N-1)} \\ &\times \sum_{\sigma, \sigma' \in S_N} \text{sign}(\sigma) \text{sign}(\sigma') \prod_{i=0}^{N-1} m_{i+1}^{\sigma(i)+\sigma'(i)} \\ &= \frac{e^{\frac{g^2}{8} \sum_i m_i^2}}{\prod_{n=0}^{N-1} n!} \left(\frac{g}{2}\right)^{N(N-1)} \prod_{i<j=1}^N (m_i - m_j)^2, \end{aligned}$$

in agreement with (D.4).

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