

An angular momentum bound at null infinity

Piotr T. Chruściel^{1,2} and Paul Tod³

¹LMPT, Fédération Denis Poisson, Tours, France

²Mathematical Institute and Hertford College, Oxford, UK
Chrusciel@maths.ox.ac.uk

³Mathematical Institute and St John's College, Oxford, UK
Tod@maths.ox.ac.uk

Abstract

We prove an inequality relating the trace of the extrinsic curvature, the total angular momentum, the centre of mass, and the Trautman-Bondi mass for a class of gravitational initial data sets with constant mean curvature (CMC) extending to null infinity. As an application we obtain non-existence results for the asymptotic Dirichlet problem for CMC hypersurfaces in stationary space-times.

1 Introduction

Let (\mathcal{S}, g, K) be an n -dimensional, $n \geq 3$, constant mean curvature (CMC) general relativistic initial data set with cosmological constant Λ (possibly

zero), thus

$$R = |K|^2 - (\operatorname{tr}_g K)^2 + 2\Lambda + 16\pi\rho, \quad (1.1)$$

$$D_i K^{ij} = -8\pi\mu^j, \quad D_i(\operatorname{tr}_g K) = 0. \quad (1.2)$$

Here ρ is the matter energy density and μ^j is the matter momentum vector.

There is a transformation which maps such initial data sets with $\operatorname{tr}_g K = \kappa$, where κ is a constant, to new initial data sets $(\mathcal{S}, g, \hat{K})$ with $\operatorname{tr}_g \hat{K} = 0$ and Λ shifted by $-(n-1)\kappa^2/2n$: Indeed, if

$$\hat{K}_{ij} = K_{ij} - \frac{\kappa}{n}g_{ij}, \quad (1.3)$$

then (1.2) still holds with K replaced by \hat{K} , while (1.1) becomes

$$R = |\hat{K}|^2 - \underbrace{(\operatorname{tr}_g \hat{K})^2}_0 + 2 \underbrace{\left(\Lambda - \frac{n-1}{2n}\kappa^2\right)}_{\hat{\Lambda}} + 16\pi\rho. \quad (1.4)$$

Equation (1.3) allows one to go back and forth from CMC hyperboloidal initial data sets in space-times with $\Lambda = 0$ to initial data sets in asymptotically anti-de Sitter (adS) space-times with $\Lambda < 0$.

The object of this note is to point out that this transformation, together with the known bounds on total angular momentum and centre of mass for asymptotically adS space-times [11, 22], implies a striking angular-momentum bound for CMC hyperboloidal initial data which are asymptotically flat at null infinity, see (5.3) below.

Our analysis complements Dain's recent upper bound on angular momentum [13] at spatial infinity for axi-symmetric solutions with two asymptotically flat regions.

As an interesting application, we obtain non-existence results for hypersurfaces as above in stationary space-times, see Section 8 below.

Before presenting our inequality it is useful to review the definitions of global charges both with $\Lambda = 0$ and $\Lambda < 0$; we start with the latter.

2 Global charges for asymptotically anti-de Sitter initial data

For the purposes of this work, an n -dimensional initial data set (\mathcal{S}, g, K) will be called adS if \mathcal{S} contains an asymptotic region, diffeomorphic to

the complement of a ball in \mathbb{R}^n , in which K asymptotes to zero while g asymptotes to a Riemannian background metric

$$b = dr^2 + \sinh^2(r) \check{h}, \tag{2.1}$$

where \check{h} is a unit round metric on S^{n-1} . Note that $(b, 0)$ are initial data for adS space-time. We further assume that there exist constants $k \geq 1$, $\alpha > n/2$ and $C > 0$ such that

$$|g - b|_b + |\mathring{D}g|_b + \dots + |\underbrace{\mathring{D} \dots \mathring{D}}_{k \text{ factors}} g|_b + |K|_b + \dots + |\underbrace{\mathring{D} \dots \mathring{D}}_{k-1 \text{ factors}} K|_b \leq Ce^{\alpha r}. \tag{2.2}$$

Here $|\cdot|_b$ denotes the norm of a tensor field with respect to the metric b , and \mathring{D} is the covariant derivative of b .

In particular, the definition enforces the vanishing of $\text{tr}_g K$ for CMC data. Whether the data are CMC or not, (2.2) implies the vanishing of the trace-free part of the extrinsic curvature of the conformal boundary at infinity.

Let X be a Killing vector in the asymptotic region of the background adS space-time, the Hamiltonian associated with the flow along X can be calculated as follows [7, 12, 10, 19]: Let V be the normal component of X with respect to the background adS metric, and let Y be the tangential component thereof; when defined along a space-like hypersurface, such pairs (V, Y) are called Killing Initial Data (KIDs). Then the Hamiltonian $H(V, Y)$ corresponding to X (which we identify with the couple (V, Y)) takes the form

$$H(V, Y) := \lim_{R \rightarrow \infty} \frac{1}{16\pi} \int_{r=R} (\mathbb{U}^i(V) + \mathbb{V}^i(Y)) dS_i, \tag{2.3}$$

where

$$\mathbb{U}^i(V) := 2\sqrt{\det g} \left(V g^{i[k} g^{j]l} \mathring{D}_j g_{kl} + D^{[i} V g^{j]k} (g_{jk} - b_{jk}) \right), \tag{2.4}$$

$$\mathbb{V}^i(Y) := 2\sqrt{\det g} \left(K^i_j - K^k_k \delta^i_j \right) Y^j. \tag{2.5}$$

Here all indices are space indices, running from 1 to n , and \mathring{D} is the Levi-Civita derivative of the space background metric b .

A preferred set of background Killing vector fields is provided by those which are b -normal to the initial data surface. The resulting Hamiltonians are usually interpreted as energies. In contradistinction with the asymptotically flat case, where only one normal background Killing vector field exists, if one assumes that conformal infinity has spherical space-like sections, then there are several normal background Killing vector fields. This

implies that there is not a *single* energy, but rather an *energy functional* M . This functional M is uniquely characterized by $n + 1$ numbers M_μ , $\mu = 0, 1, \dots, n$, which transform as a Lorentz vector under asymptotic isometries of g , see [12]. (The component M_0 coincides with the Abbott–Deser mass under appropriate restrictions [12].) It follows that the Lorentzian length of M_μ is a geometric invariant of (\mathcal{S}, g) . The asymptotically adS-positive-energy theorem implies that M_μ is causal, future pointing [16, 17, 22] (compare [8, 20, 26, 27]), unless (\mathcal{S}, g, K) are initial data for adS space–time. Let us assume that we are not in this last situation.

It is convenient to view the hyperbolic space as a unit space-like hyperboloid in \mathbb{R}^{n+1} , the latter equipped with the Minkowski metric. Assuming that M_μ is timelike,¹ after applying an asymptotic isometry to obtain

$$M_\mu = (m, 0, \dots, 0),$$

the background Killing vector fields tangent to \mathcal{S} can now be split into rotations and “boosts.” It is customary to define the rest-frame angular momentum as

$$j_i := H(0, \beta_{(i)}),$$

where the $\beta_{(i)}$ ’s are the generators of rotations of S^{n-1} , when embedded in \mathbb{R}^n ; for example, in space dimension $n = 3$ a natural choice is

$$\beta_{(i)} = \epsilon_{ijk} x^j \partial_k.$$

The numerical values of the remaining n Hamiltonians generating boost transformations will be denoted by c_i . For initial data which are asymptotically flat in space-like directions, the c_i ’s have the interpretation of the centre of mass, and we will retain the name of centre of mass for the vector $\vec{c} = (c_i)$.

For reasons that are discussed in Section 9 below, from now on we restrict our attention to $n = 3$. Assuming that (\mathcal{S}, g) is complete, that the dominant energy condition holds,

$$|\mu|_g \leq \rho, \tag{2.6}$$

where μ and ρ are as in (1.1) and (1.2), and that the total matter energy as defined by²

$$\int_{\mathcal{S}} (1 + e^r) \rho \, d\mu_g \tag{2.7}$$

¹One expects that M_μ cannot be null, see [11] for some partial results.

²We take this opportunity to correct [11], where the weight factor e^r in (2.7) has been inadvertently omitted from the hypotheses of the positive charges theorem.

(with r as in (2.1)) is finite, it is shown in [22] (compare [11]) that the positive energy theorem implies the following inequality:

$$m \geq \sqrt{-\Lambda/3} \sqrt{|\vec{c}|^2 + |\vec{j}|^2 + 2|\vec{c} \times \vec{j}|}, \quad (2.8)$$

where $\vec{c} \times \vec{j}$ is the vector product, while $|\vec{j}| = \sqrt{j_1^2 + j_2^2 + j_3^2}$, etc.

The inequality also holds if \mathcal{S} is complete with boundary, as long as the boundary satisfies one of the “trapping” conditions: the boundary is either *weakly future trapped*, which means that

$$\mathrm{tr}_h \lambda + h^{ab} K_{ab} \leq 0, \quad (2.9)$$

or *weakly past trapped*, which corresponds to changing the sign in front of the K term in (2.9). Yet another such condition is obtained [8, 22] by setting $k(\nu) = K_{ia} \nu^i dx^a$, where the x^a 's are coordinates on $\partial\mathcal{S}$, then the positivity of the global charges will hold if

$$\mathrm{tr}_h \lambda + |k(\nu)|_h \leq \sqrt{\frac{-2(n-1)\Lambda}{n}} \quad (2.10)$$

(see [8, Remark 4.8] for a discussion of (2.10) when $k(\nu) = 0$).

It has been proved in [11] that equality in (2.8) holds only for initial data in adS space–time *provided* the associated space–time has a Scri with a sufficiently large time extent. Our application of (2.8) in Section 5 makes it clear that it would be of interest to obtain a proof without such a condition.

3 Hamiltonian global charges in space–times asymptotically flat at \mathcal{I}^+

In this section, we briefly review the space–time version of the approach in [9]. Let $(\mathcal{M}, {}^4g)$ be a four-dimensional space–time with a smooth, or polyhomogeneous, conformal boundary completion at null infinity $\tilde{\mathcal{M}} = \mathcal{M} \cup \mathcal{I}^+$ à la Penrose. Let \mathcal{S} be a smooth space-like hypersurface in $\tilde{\mathcal{M}}$ which intersects \mathcal{I}^+ transversally at a smooth section $S = \partial\mathcal{S} = \mathcal{S} \cap \mathcal{I}^+$. Such a section singles out a six parameter family of Bondi coordinate systems, by the requirement that in the chosen Bondi coordinates we have $S = \{u = 0\}$. Now, every such coordinate system defines a flat background

metric b in a neighbourhood of S :

$$b = b_{\mu\nu} dx^\mu dx^\nu \equiv -du^2 - 2 du dr + r^2 \check{h}_{AB} dx^A dx^B. \tag{3.1}$$

The resulting metrics are independent of the Bondi coordinate system chosen, within the six parameter freedom available, as those coordinate systems differ from each other by a Lorentz transformation. We can thus define a unique six parameter family of BMS generators which are singled out by the requirement that they are tangent to S , and that they are Killing vector fields of the background metric b .

Consider, near $S \subset \mathcal{I}^+$, a Bondi-type coordinate system (u, x, v^A) as above with $u \in (-\epsilon, \epsilon)$, $x \in [0, \epsilon)$, for some $\epsilon > 0$, while the v^A 's are coordinates on S^2 . Here the usual Bondi coordinate r is replaced by $1/x$ so that the space-time metric 4g , when conformally rescaled by r^{-2} , takes the form

$$x^2 {}^4g_{\mu\nu} dx^\mu dx^\nu = -Vx^3 e^{2\beta} du^2 + 2e^{2\beta} du dx + h_{AC}^B(dx^A - U^A du)(dx^C - U^C du), \tag{3.2}$$

$$\frac{\partial(\det h_{AC}^B)}{\partial x} = 0. \tag{3.3}$$

If the matter fields decay sufficiently fast then, for smooth conformally rescaled metrics, one has the following asymptotics:

$$h_{AB}^B = \check{h}_{AB} + \frac{\chi_{AB}(v)}{r} + O(r^{-2}), \tag{3.4}$$

$$\beta = -\frac{\check{h}^{AB}\check{h}^{CD}\chi_{AC}\chi_{BD}}{32r^2} + O(r^{-3}), \tag{3.5}$$

$$U^A = -\frac{\check{D}_B\chi^{AB}}{2r^2} + \frac{2N^A(v)}{r^3} + \frac{\check{D}^A(\chi^{CD}\chi_{CD})}{16r^3} + o(r^{-3}),$$

$$V = r - 2\mu_{TB} + O(r^{-1}), \tag{3.6}$$

where \check{h} is the unit round metric on S^2 , \check{D} is the corresponding derivative operator, and μ_{TB} is the Bondi mass aspect function.

In terms of these variables, the Hamiltonian associated to rotations and boosts reads [9, equation (6.117)]

$$H_L(X, \mathcal{S}) = -\frac{1}{64\pi} \int_{S^2} \left(24N_A + 2\chi_{AB}\chi^{BC} \right)_{||C} + \frac{1}{2}(\chi_{BC}\chi^{BC})_{||A} X^A|_{x=0} \sin\theta d\theta d\varphi, \tag{3.7}$$

where the vector fields X in (3.7) belong to the six-dimensional vector space of b -Killing vectors uniquely singled out by $S = \partial\mathcal{S}$.

The above definition has several good properties, discussed in [9], some of which are used in Section 8 below. For a discussion of alternative definitions of angular momentum at \mathcal{S} , see [25].

4 The global charges of hyperboloidal initial data sets with $\Lambda = 0$

We continue with a review of the initial data version of the analysis in [9]. Consider an asymptotically CMC hyperboloidal initial data set (\mathcal{S}, g, K) . In [10, Appendix C.3] a construction is given of an embedding $\iota : \mathcal{S} \rightarrow \mathcal{M}^B$ of such an initial data set into a space-time (\mathcal{M}^B, g^B) coordinatized as in (3.2), with the property that the conformal boundary of \mathcal{S} is mapped to $u = 0$. Both the embedding ι and (\mathcal{M}^B, g^B) are constructed so that ι^*g^B is asymptotic to infinite order to g at the conformal boundary of \mathcal{S} ; similarly, the pull-back to \mathcal{S} of the extrinsic curvature of $\iota(\mathcal{S})$ is asymptotic to infinite order to K . The angular momentum and the centre of mass of (\mathcal{S}, g, K) are then defined using (3.7).

The coordinates (x, v^A) on \mathcal{M}^B , when composed with ι , induce coordinates near the conformal boundary of \mathcal{S} which will be denoted by the same symbols. One can then write $\iota(\mathcal{S})$ as a graph:

$$u = \alpha(x, v^A), \quad \alpha(0, v^A) = 0,$$

and we have (see [10])

$$\alpha_{,x} \Big|_{x=0} = \frac{9}{2(\text{tr}_g K)^2}, \tag{4.1}$$

$$\alpha_{xx} \Big|_{x=0} = -\frac{1}{2} \left(\frac{3}{\text{tr}_g K} \right)^3 (\text{tr}_g K)_{,x}, \tag{4.2}$$

$$x^2 g = \left(2 \frac{\partial \alpha}{\partial x} + O(x) \right) dx^2 + O(x) dx dx^A + (\check{h}_{AB} + x \chi_{AB} + O(x^2)) dx^A dx^B. \tag{4.3}$$

Thus the extrinsic curvature of the conformal boundary at infinity, say $\tilde{\lambda}_{AB}$, is proportional to χ_{AB} :

$$\tilde{\lambda}_{AB} = -\frac{6}{\text{tr}_g K} \chi_{AB}. \tag{4.4}$$

Hence $\tilde{\lambda}$ vanishes if and only if χ does; this will be relevant shortly.

5 The angular momentum inequality

With these preliminaries, we may now state the inequality. Let κ be a constant, and consider a CMC hyperboloidal initial data set (\mathcal{S}, g, K) with $\dim \mathcal{S} = 3$, $\text{tr}_g K = \kappa$ and $\Lambda = 0$. Suppose that (\mathcal{S}, g) is complete and that the dominant energy condition (2.6) holds. In this section we will assume that

$$\begin{aligned} & \text{the trace-free part of the extrinsic curvature} \\ & \text{of the conformal boundary at infinity vanishes;} \end{aligned} \quad (5.1)$$

an argument indicating that (5.1) can be removed will be presented in Section 6 below. (Note, however, that (5.1) has been invoked in the literature in the context of CMC hyperboloidal surfaces [4, 14, 18].) It follows from (4.4) that this is equivalent to the hypothesis that, in Newman-Penrose terminology, the associated Bondi cone is asymptotically shear free. Performing the transformation (1.3), the initial data set $(\mathcal{S}, g, \hat{K})$ satisfies the constraint equations with

$$\Lambda = -\frac{\kappa^2}{3}.$$

We need to analyse what happens with the global charges under (1.3). First, using the formulae in [10, Appendix F] one checks that, both for translations and rotations, any trace terms in (2.3) integrate out to zero, so that the extrinsic curvature contributions to (2.3) from K_{ij} and \hat{K}_{ij} coincide. The same is true for boost generators if (5.1) is assumed. Next, it follows from [9, Appendix C.3] that [10, equation (3.13)] holds, which implies that the functional [10, equation (3.11)] coincides with (2.3) (see [10, equation (3.14)]). Letting m be the Hamiltonian mass of $(\mathcal{S}, g, \hat{K})$, and m_{TB} the Trautman-Bondi mass of (\mathcal{S}, g, K) , the equality

$$m = m_{\text{TB}}. \quad (5.2)$$

follows now from Theorem 5.3 of [10].

For the remaining charges, observe that *under* (5.1) integrals (2.3) are equal to their linearizations. Now, it has been shown in [10, Appendix B] that,³ again under (5.1), the linearization of the functional [10, equation (3.14)] equals the linearization of the functional H_{boundary} of [9]. The

³Note that the terms quadratic in χ in the last equation of [10, Appendix B] might seem to be incompatible with the fact that a linearized expression is considered. This apparent contradiction is resolved by observing that some coefficients of the metric, which enter linearly in the integral, are themselves quadratic in the free Bondi functions χ and their derivatives.

calculations in [9, Sections 6.4 and 6.5] then show that the angular momenta of K and \hat{K} coincide. Now, the centre of mass for $(\mathcal{S}, g, \hat{K})$ is calculated using only the first term at the right-hand side of [9, equation (6.57)], whereas the calculation for (\mathcal{S}, g, K) uses the whole right-hand side of that equation. Nevertheless, both quantities are equal under (5.1).

If we furthermore assume that ρ decays fast enough so that the total energy as defined by (2.7) is finite, then all the conditions needed for (2.8) are met, and we conclude that

$$m_{\text{TB}} \geq \frac{|\text{tr}_g K|}{3} \sqrt{|\vec{c}|^2 + |\vec{j}|^2 + 2|\vec{c} \times \vec{j}|}. \tag{5.3}$$

Here m_{TB} is the Trautman–Bondi mass, \vec{j} is the angular momentum vector (the Hamiltonian associated with rotations) in the rest frame (i.e., a conformal frame in which space-momentum vanishes), and \vec{c} is the centre of mass (the Hamiltonian associated with boosts) in that frame. In particular, we have the striking bounds

$$m_{\text{TB}} \geq \frac{|\text{tr}_g K|}{3} |\vec{j}|, \quad m_{\text{TB}} \geq \frac{|\text{tr}_g K|}{3} |\vec{c}|. \tag{5.4}$$

In the light of the earlier discussion of (2.8), it is expected that equality in (5.3) can occur only for initial data in Minkowski space–time; it would be of interest to prove this.

6 A possible direct proof

In this section, we indicate an argument that could remove the restrictive condition (5.1). We start with some notation. In space–time dimension n , we view the hyperbolic space as the open unit ball $B^n(1) \subset \mathbb{R}^n$ equipped with the metric $b = {}^n b = \omega^{-2} \delta$, where δ is the standard flat metric on \mathbb{R}^n , and

$$\omega = \frac{1 - |x|^2}{2}.$$

If we write the Minkowski metric η as $-dt^2 + \delta_{ij} dy^i dy^j$, and set

$$\tau = t - \sqrt{1 + |y|^2}, \quad y^i = \omega^{-1} x^i, \quad r = |x|, \tag{6.1}$$

we obtain

$$\eta = -d\tau^2 + \omega^{-2} (-2rd\tau dr + \delta_{ij} dx^i dx^j).$$

The KID-decompositions of the Minkowskian Killing vectors at $\mathcal{S} := \{\tau = 0\}$ read

$$\begin{aligned} \partial_t &= V_{(0)}n + Y_{(0)} = \frac{1 + |x|^2}{1 - |x|^2}n - x^i\partial_{x^i}, \\ \partial_{y^i} &= V_{(i)}n + Y_{(i)} = -\omega^{-1}x^i n + \omega\partial_{x^i} + x^i x^j \partial_{x^j}, \\ t\partial_{y^i} + y^i\partial_t &= 0 \cdot n + C_{(i)} = \frac{1 + |x|^2}{2}\partial_{x^i} - x^i x^j \partial_{x^j}, \\ y^i\partial_{y^j} - y^j\partial_{y^i} &= 0 \cdot n + \Omega_{(i)(j)} = x^i\partial_{x^j} - x^j\partial_{x^i}, \end{aligned}$$

where n is the unit normal to \mathcal{S} .

The standard proof of positivity of Trautman–Bondi mass proceeds by solving the Witten equation:

$$\gamma^i \nabla_i \psi = 0, \quad \text{where } \nabla_i := D_i + \frac{1}{2}K_{ij}\gamma^j\gamma^0. \tag{6.2}$$

One further requires ψ to asymptote to spinors $\mathring{\psi}$ which are restrictions to a hyperboloid of covariantly constant spinors in Minkowski space–time. For hyperboloids with $\mathring{K}_{ij} = -b_{ij}$ the spinors solve⁴

$$\mathring{D}_i \mathring{\psi} = \frac{1}{2}\gamma_i \gamma^0 \mathring{\psi}. \tag{6.3}$$

In the obvious spin frame associated with the above conformal representation,⁵ the solutions of (6.3) read

$$\psi_u = \omega^{-1/2}(1 + x^k \gamma^k \gamma^0)u \tag{6.4}$$

(summation over k), where u is a spinor with constant entries, while the anti-Hermitian matrices γ^k with constant entries satisfy the flat space Clifford relations

$$\gamma^i \gamma^j + \gamma^j \gamma^i = -2\delta^{ij}.$$

(The ψ_u 's exhaust the space of solutions because the map that assigns u to, e.g., $\psi_u(0)$ is a bijection.) Further, γ^0 is a Hermitian matrix, with constant

⁴We use the conventions of [10], in which the standard unit future hyperboloid in Minkowski space–time $\mathbb{R}^{1,n}$ satisfies $\text{tr}_g K = -n$.

⁵More precisely, we take a spin frame that projects to the frame $\theta^i = \omega^{-1}dx^i$, with e_i dual to θ^i , and a local basis of the spinor bundle in which the γ^μ 's are constant matrices.

entries, satisfying

$$(\gamma^0)^2 = 1, \quad \gamma^0 \gamma^j + \gamma^j \gamma^0 = 0.$$

(The spinor bundle can always be chosen so that such a matrix exists.) The KID (V_u, Y_u^i) associated to ψ_u takes the form

$$V_u := \langle \psi_u, \psi_u \rangle = 2 \left(|u|^2 \underbrace{\frac{1 + |x|^2}{1 - |x|^2}}_{V_{(0)}} - \langle u, \gamma^k \gamma^0 u \rangle \underbrace{\frac{(-2)x^k}{1 - |x|^2}}_{V_{(k)}} \right), \tag{6.5}$$

$$\begin{aligned} Y_u^i \partial_i &:= \langle \psi_u, \gamma^0 \gamma^i \psi_u \rangle e_i \\ &= -2 \left(|u|^2 \underbrace{x^i \partial_i}_{Y_{(0)}} + \langle u, \gamma^k \gamma^0 u \rangle \underbrace{\left(\frac{1 - |x|^2}{2} \delta_k^i + x^i x^k \right) \partial_i}_{Y_{(k)}} \right). \end{aligned} \tag{6.6}$$

This, together with the usual Witten argument, implies that the boundary term in the Witten equation will only carry information about the global charges associated with space-time translations of $\mathbb{R}^{1,n}$.

Now, our argument so far leading to the angular momentum bound can be rephrased as follows: instead of (6.2) one considers

$$\gamma^i \hat{\nabla}_i \psi = 0, \quad \text{where } \hat{\nabla}_i := D_i + \frac{1}{2} \left(K_{ij} - \frac{\text{tr}_g K}{n} g_{ij} \right) \gamma^j \gamma^0 - \frac{i \text{tr}_g K}{2n} \gamma_i, \tag{6.7}$$

where the ψ 's asymptote now to imaginary Killing spinors $\hat{\psi}$ of b which, for $\text{tr}_g K = -n$, solve

$$\overset{\circ}{D}_i \hat{\psi} = -\frac{i}{2} \gamma_i \hat{\psi}. \tag{6.8}$$

Those take the form

$$\hat{\psi}_u = \omega^{-1/2} (1 - ix^k \gamma^k) u \tag{6.9}$$

(summation over k), where u is again a spinor with constant entries. Instead of (6.5) and (6.6), the KID (\hat{V}_u, \hat{Y}_u^i) associated to $\hat{\psi}_u$ takes the form

$$\hat{V}_u := \langle \hat{\psi}_u, \hat{\psi}_u \rangle = 2 \left(|u|^2 \underbrace{\frac{1 + |x|^2}{1 - |x|^2}}_{V_{(0)}} + \langle u, i \gamma^k u \rangle \underbrace{\frac{(-2)x^k}{1 - |x|^2}}_{V_{(k)}} \right), \tag{6.10}$$

$$\begin{aligned}
 \hat{Y}_u^i \partial_i &:= \langle \hat{\psi}_u, \gamma^0 \gamma^i \hat{\psi}_u \rangle e_i \\
 &= 2 \langle u, \gamma^0 \gamma^k u \rangle \underbrace{\left(\frac{1 + |x|^2}{2} \delta_k^i - x^i x^k \right)}_{C_{(k)}} \partial_i \\
 &\quad + \frac{1}{2} \langle u, i \gamma^0 (\gamma^k \gamma^i - \gamma^i \gamma^k) u \rangle \underbrace{(x_k \partial_i - x_i \partial_k)}_{\Omega_{(k)(i)}}, \tag{6.11}
 \end{aligned}$$

so that the boundary term in the Witten identity will carry now information about all global charges.

We are ready to prove that the existence of solutions of (6.7) with the above boundary condition, without assuming the vanishing of χ . Indeed, from inspection of the positivity proof of [10, Section 5.4] one infers that one needs to justify

$$\gamma^i \hat{\nabla}_i \hat{\psi} \in L^2, \tag{6.12}$$

compare the proof of Lemma 5.9 in [10]. In what follows, notations and conventions of [10] are used unless explicitly indicated otherwise.⁶ Now, after a constant rescaling so that $\text{tr}_g K = -n$, from (6.7) we obtain

$$\gamma^i \hat{\nabla}_i \hat{\psi} = \gamma^i D_i \hat{\psi} + \underbrace{\left(K_{ij} - \frac{\text{tr}_g K}{n} g_{ij} \right)}_0 \gamma^i \gamma^j \gamma^0 \hat{\psi} - \frac{ni}{2} \hat{\psi}.$$

By (6.8) we have

$$X(\hat{\psi}) = \frac{1}{4} \hat{\omega}_{ij}(X) \gamma^i \gamma^j \hat{\psi} - \frac{i}{2} \sum_{\ell} \hat{X}^{\ell} \gamma^{\ell} \hat{\psi}, \tag{6.13}$$

where \hat{X}^{ℓ} are the components of X in the b -orthonormal frame \hat{e}_i as in [10, Appendix C]: $X = \hat{X}^i \hat{e}_i$. In (6.13), we have indicated explicitly the summation over ℓ since both ℓ 's are superscripts there. Letting $\hat{f}_i = \hat{M}_i^j \hat{e}_j$ be the g -orthonormal frame as in [10, Appendix C], it follows that

$$\begin{aligned}
 \gamma^{\ell} \hat{\nabla}_{\ell} \hat{\psi} &= \gamma^{\ell} \hat{f}_{\ell}(\hat{\psi}) - \frac{1}{4} \omega_{ij}(\hat{f}_{\ell}) \gamma^{\ell} \gamma^i \gamma^j \hat{\psi} - \frac{ni}{2} \hat{\psi} \\
 &= \frac{1}{4} \left(\hat{\omega}_{ij}(\hat{f}_{\ell}) - \omega_{ij}(\hat{f}_{\ell}) \right) \gamma^{\ell} \gamma^i \gamma^j \hat{\psi} - \frac{i}{2} \sum_j (\hat{M}_i^j - \delta_i^j) \gamma^i \gamma^j \hat{\psi}. \tag{6.14}
 \end{aligned}$$

⁶We take the opportunity to point out the following misprints there: first, γ^0 is assumed to be hermitian and γ^i – anti-hermitian, in spite of what is said at the beginning of page 122 of [10]. Next, $\sqrt{\det g}$ should be estimated as $O(x^{-3})$ in the penultimate displayed equation of Appendix D of [10]. In [10, equation (5.14)] the factor $1/4\pi$ should be 4π .

It has been shown in [10, Appendix D] that the first term in (6.14) can be estimated by $Cx^2|\hat{\psi}|$, which in turn implies that it is in L^2 . Next, by [10, equations (C.21), (C.22), (C.40) and (C.47)], both the anti-symmetric part and the trace of $\hat{M}_i^j - \delta_i^j$ are $O(x^2)$, and (6.12) follows. This, together with the arguments in [8, 10] proves existence of the relevant solutions of the Witten equation. In retrospect, the calculation here is shorter than the one for the original positivity proof, albeit applying to CMC initial data only.

To complete the proof of (5.3) without the restrictive condition (5.1) one needs to analyse the boundary term that appears in the Witten identity associated to the operator (6.7). We are planning to return to this in the near future.

7 The conformal method

Given a spacetime (\mathcal{M}, g) , it is far from clear that whether or not \mathcal{M} contains *any* complete CMC surfaces (see, however, [4]). Furthermore, it is not clear whether or not those surfaces will be sufficiently differentiable at \mathcal{I}^+ as needed above. Therefore, it is reasonable to raise the question of the range of applicability of our bounds. Recall, now, that the conformal method provides a construction of all, say vacuum, CMC general relativistic initial data sets. In the hyperboloidal context, one prescribes a non-zero value of $\text{tr}_g K$, as well as an arbitrary conformally compactifiable Riemannian manifold (\mathcal{S}, \hat{g}) equipped with a seed symmetric trace-free tensor, say A , and constructs (\mathcal{S}, g, K) by solving a set of elliptic equations, see [3] and references therein. In such a construction the resulting initial data set will satisfy condition (5.1) if and only if the trace-free part of the extrinsic curvature of the conformal boundary at infinity of \hat{g} vanishes. Since \hat{g} and A can be chosen arbitrarily, subject to a finite number of compatibility conditions at the conformal boundary [2], we conclude that there exists an infinite-dimensional family of vacuum initial data sets for which (5.3) provides a non-trivial upper bound for \vec{j} and \vec{c} in terms of the total mass. The associated globally hyperbolic vacuum developments [15] provide, in turn, examples of space-times containing hypersurfaces satisfying the hypotheses of our inequality.

8 Obstructions to existence of CMC surfaces

Note that (4.3) shows that $\tilde{\lambda}_{AB}$ is the same for all CMC surfaces asymptotic to a given cut of \mathcal{I} . This leads to the following unexpected consequence of

our analysis: whenever $|\vec{j}| + |\vec{c}| \neq 0$ there exists an upper bound on $|\text{tr}_g K|$ for complete hyperboloidal CMC surfaces satisfying⁷ (5.1) (without boundary, or with boundaries on or beyond horizons) which asymptote to smooth cuts S of \mathcal{I} , namely

$$|\text{tr}_g K| \leq \frac{3m_{\text{TB}}}{\sqrt{|\vec{c}|^2 + |\vec{j}|^2 + 2|\vec{c} \times \vec{j}|}}. \quad (8.1)$$

8.1 CMC surfaces in Schwarzschild

Equation (8.1) does not lead to any restrictions on $\text{tr}_g K$ for CMC hypersurfaces in Schwarzschild space–time which asymptote to spherically symmetric cuts of \mathcal{I}^+ , and indeed there are none [23]. Consider, however, cuts S_α of the Schwarzschildian \mathcal{I}^+ which are obtained by applying a translation $u \rightarrow u + \alpha$ to $S_0 = \{u = 0\}$, where α is a linear combination of $\ell = 0$ and 1 spherical harmonics. As shown in [10, Section 6.6], all such cuts have vanishing angular momentum. More generally, it is shown in [10, Section 6.7] that for all stationary space–times the Hamiltonian angular momentum is independent of the cut of \mathcal{I}^+ chosen, so the discussion that follows applies to any stationary space–time with matter satisfying the dominant energy condition. It is also shown in [10, Sections 6.6 and 6.7] that the change of centre of mass of S_α can be calculated using the standard special-relativistic rule: under a translation by a vector \vec{a} orthogonal to the momentum the centre of mass is shifted by $m\vec{a}$. Since (5.1) is preserved under translations, from (8.1) we conclude that for any translation \vec{a} , *the associated cut S_α in the Schwarzschild space–time cannot span a complete CMC surface meeting \mathcal{I}^+ smoothly (or C^2 and polyhomogeneously) with*

$$|\text{tr}_g K| > \frac{3}{|\vec{a}|}. \quad (8.2)$$

An identical conclusion is reached in space–times that are stationary near \mathcal{I}^+ and have zero angular momentum, and a similar conclusion without assuming that $\vec{j} = 0$.

8.2 CMC hypersurfaces in Kerr space–time?

Both the tensor field χ and the centre of mass vanish for the family of $\{u = \text{const}\}$ cuts of \mathcal{I}^+ in Kerr space–time, where u is an outgoing

⁷In view of the analysis of Section 6, it is rather likely that (5.1) is not needed for the discussion of this section.

Eddington–Finkelstein coordinate, and for these we obtain

$$|\operatorname{tr}_g K| \leq \frac{3}{|a|}, \tag{8.3}$$

where a is the usual angular momentum parameter in the Kerr metric, for any complete CMC surface spanned by those cuts. As above, it follows immediately that no such surfaces exceeding this bound exist.

We wish to present an argument which suggests strongly that no such hypersurfaces exist in Kerr at all. Suppose, for contradiction, that there exists a complete space-like hypersurface \mathcal{S}_{κ_0} in Kerr space–time, satisfying (5.1), with $\operatorname{tr}_g K = \kappa_0$, for some $\kappa_0 < 0$, with two spherical boundaries lying on two different components of \mathcal{I}^+ . We further assume that \mathcal{S}_{κ_0} is contained within four diamond-shaped blocks of the usual maximal analytic extension of Kerr, on two of which $r_- < r < r_+$, while $r > r_+$ on the remaining, asymptotically flat, ones. Choose any κ more negative than $-3/|a|$ (and thus smaller than κ_0) and let \mathcal{S}_n be a sequence of CMC surfaces with $\operatorname{tr}_g K = \kappa$ such that the boundary of \mathcal{S}_n consists of two spherical components lying on \mathcal{S}_{κ_0} , with $\partial\mathcal{S}_n$ approaching \mathcal{I} as n tends to infinity. Such \mathcal{S}_n exist by the results in [5, 6], because \mathcal{S}_{κ_0} provides an upper barrier, whereas a lower barrier is provided by the boundary of the past domain of dependence, say \mathcal{D}_n^- , of that subregion of \mathcal{S}_{κ_0} which is bounded by $\partial\mathcal{S}_n$. To see that \mathcal{D}_n^- is conditionally compact, note that it must be included in the region which is delimited to the future by \mathcal{S}_{κ_0} , and which is delimited to the past⁸ by the hypersurfaces $u = u_0$, and $\hat{u} = u_0$, where u is an Eddington–Finkelstein retarded coordinate the level sets of which provide cuts of \mathcal{I}^+ in the first asymptotic region, while \hat{u} is the analogous Eddington–Finkelstein coordinate associated to the second asymptotically flat region, with $u_0 = \min(\inf_{\partial\mathcal{S}_{\kappa_0}} u, \inf_{\partial\mathcal{S}_{\kappa_0}} \hat{u})$. (Note that one of u and \hat{u} is actually an advanced Eddington–Finkelstein coordinate v in the relevant region $r_- < r < r_+$.) This proves that the compactness condition needed for Bartnik’s theorem [6] of existence of smooth solutions of the Dirichlet problem is satisfied. (An alternative height bound to the past is obtained by the level sets of r near r_- , which are crushing [14] as $r \rightarrow r_-$.) By Bartnik’s interior estimates the sequence \mathcal{S}_n converges, in the compact-open topology, to some smooth hypersurface \mathcal{S}_κ . If one could show — which is not clear (compare [4] where the conditions for the construction of barriers near the boundary preclude a non-vanishing J) — that \mathcal{S}_n is uniformly space like in the conformally rescaled space–time, with a bound independent of n , one would obtain a smooth space-like CMC surface \mathcal{S}_κ spanned on $\partial\mathcal{S}_{\kappa_0} \subset \mathcal{I}^+$.

⁸This follows from the fact that in Eddington–Finkelstein coordinates one has $g_{rr} = \Gamma_{rr}^\mu = 0$, so that the curves $u = u_0$, $\theta = \theta_0$, $\varphi = \varphi_0$ are null geodesics.

If one could further show — which is likely, using the results in [3] — that \mathcal{S}_κ is smooth at \mathcal{S} (polyhomogeneous and C^2 would suffice [10]; compare [24]), one would obtain a contradiction with (8.3) for κ large negative. It would then follow that no CMC hypersurfaces \mathcal{S}_{κ_0} as assumed above exist in Kerr.

9 Higher dimensions

It is interesting to enquire what happens in higher dimensions. Indeed, the positive charges theorem has been proved for hyperboloidal initial data with $\Lambda < 0$ with a spherical conformal infinity under the assumption that \mathcal{S} is spin (compare, however, [1]), together with the asymptotic conditions (2.2) [11]; note that those require the vanishing, up to an overall conformal factor, of $\lfloor n/2 \rfloor$ derivatives of the conformally rescaled metric at the conformal boundary at infinity. Assuming the latter condition, we expect the transformation (1.3) to map all the global charges at null infinity to the adS ones, but no such analysis has been carried out so far. Now, an easy way out is to *define* the charges at null infinity as the values of the adS ones after the transformation (1.3) has been performed. Under suitable global hypotheses, this gives immediately the global charges inequalities of [11]⁹ in any dimension $n \geq 3$. It is then unfortunate that no explicit sharp inequalities are known in space–time dimensions higher than seven. In any case, it would be preferable to express the inequalities in terms of global charges directly definable at \mathcal{S}^+ , compare [21]. Furthermore, similarly to $n = 3$, we expect the asymptotic conditions (2.2) to be overly restrictive for a proper understanding of null infinity.

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⁹See Theorem 2 of the published version, which is Theorem 3.1 of the arxiv version.

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