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# The gl(1|1) super-current algebra: the rôle of twist and logarithmic fields

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**Abstract**

A free field representation of the $\mathrm{gl}(1|1)_k$ current algebra at arbitrary level $k$ is given in terms of two scalar fields and a symplectic fermion. The primary fields for all representations are explicitly constructed using the twist and logarithmic fields in the symplectic fermion sector. A closed operator algebra is described at integer level $k$. Using a new super spin-charge separation involving $\mathrm{gl}(1|1)_N$ and $\mathrm{su}(N)_0$, we describe how the $\mathrm{gl}(1|1)_N$ current algebra can describe a non-trivial critical point of disordered Dirac fermions. Local gl(1|1) invariant lagrangians are defined which generalize the Liouville and sine-Gordon theories. We apply these new tools to the spin quantum Hall transition and show that it can be described as a logarithmic perturbation of the $\mathrm{osp}(2|2)_k$ current algebra at $k = -2$.

## 1 Introduction

A variety of 2D models with Lie super-group symmetry are now understood to be important in many diverse areas of modern theoretical physics.

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A partial list includes applications to disordered systems [1–6], statistical mechanics [7,8], and string theory [9–11]. Although many results are known, much remains to be understood about these models in comparison with their ordinary bosonic counterparts.

The simplest Lie super-algebra is gl(1|1) and it's current algebra, $\mathrm{gl}(1|1)_k$ at level $k$, is the main subject of this paper. We also obtain some results for the $\mathrm{osp}(2|2)_{k=-2}$ case. (osp(2|2) is also referred to as su(2|1) in the literature.) The WZNW sigma-model based on the GL(1|1) super-group was considered long ago by Rozanski and Saleur [12], and more recently by Schomerus and Saleur [13] using harmonic analysis on the super-group. The latter analysis was extended to other super-groups in [14, 15]. In contrast, in our work the starting point is not the WZNW model field theory, but rather the quantum field theory is constructed algebraically using the current algebra itself, as was done for the su(2) theory by Knizhnik and Zamolodchikov [16]. In this way, new results concerning the spectrum of fields are obtained, and explicit constructions of the vertex operators for all representations are given in terms of twist and logarithmic fields.

For the remainder of this introduction, we summarize our main results and describe the organization of the remainder of the paper. After reviewing the definitions of the super-current algebras in Section 2, we construct a free field representation in Section 3 involving two scalar fields and a symplectic fermion. It is known from the work [4] that such a representation exists at level 1, but it was not evident that this extends to arbitrary level $k$ with the *same field content*. In Section 4, the finite-dimensional representations of gl(1|1) are reviewed. Explicit constructions of the vertex operators are given in Section 6 and require the twist fields of the symplectic fermion sector. This rôle of twist fields was previously recognized for the special case of $\mathrm{osp}(2|2)_{-2}$ by Ludwig [17]. These twist fields were first studied by Kausch [18] and their properties are summarized in Section 5. Some additional properties of the twist fields that were needed are derived in Appendix B. The vertex operator construction indicates that the level can be interpreted as a radius of compactification $R = \sqrt{k}$.

The vertex operators for the so-called atypical indecomposable representations are also explicitly constructed and are logarithmic. We wish to emphasize that this is only possible in the second-order description of symplectic fermions because of the additional zero modes that are not present in the first-order description.

The properties of the twist fields place restrictions on the allowed spectrum of primary fields and this shows how to obtain a closed operator algebra (Section 7). We compare the $k = 2$ case with $c = 0$ minimal Virasoro models and thereby show that it is very closely related, but not identical, to percolation.

We consider $N$-copies in Section 8 and present a super version of the ordinary spin-charge separation. More specifically, the stress tensor of $N$ free Dirac fermions and ghosts can be decomposed as the sum of two commuting pieces which are the stress tensors for $\mathrm{gl}(1|1)_{k=N}$ and $\mathrm{su}(N)_{k=0}$. This generalizes the result found in [5] for $N = 2$ to arbitrary $N$. This fact opens up possibilities for the interpretation of $\mathrm{gl}(1|1)_{k=N}$ as a disordered critical point, and we explain one simple scenario. Our generalization of spin-charge separation to arbitrary $N$ differs from the one in [19] which involves $\mathrm{osp}(2|2)_{-2N} \otimes \mathrm{sp}(2N)_0$, and is more relevant to generalizations of the spin quantum Hall transition (SQHT). The two are equivalent at $N = 2$ since $\mathrm{sp}(2) = \mathrm{su}(2)$ and $\mathrm{osp}(2|2)_{-2} = \mathrm{gl}(1|1)_2$ (see Section 11).

Local (non-chiral) operators that are gl(1|1) invariant are constructed in Section 9. For the logarithmic representations, these operators can be expressed explicitly in terms of the free fields and can be used to define gl(1|1) invariant lagrangians (Section 10). In this way, we obtain gl(1|1) invariant versions of the Liouville and sine-Gordon models.

The $\mathrm{osp}(2|2)_k$ current algebra at $k = -2$ is known to describe the critical point of Dirac fermions subject to a random su(2) gauge potential [5,6]. We extend our results to $\mathrm{osp}(2|2)_{-2}$ in Section 11. In addition to recovering the results in [17] from the gl(1|1) embedding, we construct the local field corresponding to the 8-dimensional logarithmic representation.

The application of the tools developed in this paper to critical points of disordered Dirac fermions is initiated in Section 12 where we revisit the SQHT. Based on the renormalization group (RG) analysis studied in [5, 20], we propose that the additional kinds of disorder in the network model for the SQHT can be accounted for by an additional perturbation of the current algebra $\mathrm{osp}(2|2)_{-2}$ by the logarithmic operator in the 8-dimensional indecomposable representation. We argue this perturbation does not drive the theory to a new fixed point but rather gives logarithmic corrections. This is consistent with the work of Read and Saleur which emphasized that the critical point possesses osp(2|2) symmetry but is not precisely a current algebra.

## 2 The $\mathrm{gl}(1|1)_k$ and $\mathrm{osp}(2|2)_k$ super-current algebras

In this section, we define the super-current algebras and present their stress tensors. There are various conventions in the literature for the level $k$.

Our conventions are natural for applications to disordered Dirac fermions. Consider the 2D free conformal field theory for a single component $U(1)$ charged Dirac fermion $\psi_\pm$ and its ghost partners $\beta_\pm$, with action

$$S = \frac{1}{4\pi} \int d^2x \left( \psi_- \partial_{\overline{z}} \psi_+ + \overline{\psi}_- \partial_z \overline{\psi}_+ + \beta_- \partial_{\overline{z}} \beta_+ + \overline{\beta}_- \partial_z \overline{\beta}_+ \right), \tag{2.1}$$

where $z, \overline{z}$ are euclidean light-cone coordinates, $z = (x + iy)/\sqrt{2}$, $\overline{z} = z^*$. The ghost fields have bosonic statistics and the same conformal dimension as the fermions: $\Delta(\beta_\pm) = \frac{1}{2}$. First order systems of this type were treated in generality in [21], in connection with string world sheet ghosts. In particular, the fermions have Virasoro central charge $c = 1$, whereas the bosons have $c = -1$, and the total central charge is zero. The two-point functions of the left-moving fields are

$$\langle \psi_-(z)\psi_+(w)\rangle = \langle \psi_+(z)\psi_-(w)\rangle = \langle \beta_+(z)\beta_-(w)\rangle = -\langle \beta_-(z)\beta_+(w)\rangle = \frac{1}{z-w} \tag{2.2}$$

and similarly for the right-movers, $\langle \overline{\psi}_-(\overline{z})\overline{\psi}_+(\overline{w})\rangle = 1/(\overline{z} - \overline{w})$, etc. In the sequel, we will not display the right-moving counterparts if they are the obvious duplications of the left.

Define the currents

$$H = \psi_+\psi_-, \quad J = \beta_+\beta_-, \quad S_\pm = \pm \psi_\pm \beta_\mp\,. \tag{2.3}$$

$H$ and $J$ are the $U(1)$ currents underwhich $\psi_\pm$ and $\beta_\pm$ have charge $\pm 1$ and $\mp 1$, respectively. Throughout the sequel, we will mainly present our results using operator product expansions (OPEs). Using equation (2.2), the currents satisfy the $\mathrm{gl}(1|1)_k$ super-current algebra OPEs at $k = 1$:

$$\begin{aligned} H(z)H(0) &\sim \frac{k}{z^2}, \quad J(z)J(0) \sim -\frac{k}{z^2}, \\ H(z)S_\pm(0) &\sim J(z)S_\pm(0) \sim \pm\frac{1}{z}\, S_\pm(0), \\ S_+(z)S_-(0) &\sim \frac{k}{z^2} + \frac{1}{z}\, (H - J)(0). \end{aligned} \tag{2.4}$$

The above $k$ dependence establishes our convention for $\mathrm{gl}(1|1)_k$ at arbitrary $k$.

For a general current $J^a$, define its modes $J^a_n$ as follows: $J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n\, z^{-n-1}$. The modes satisfy the affine Lie super-algebra:

$$\begin{aligned} [H_n, H_m] &= -[J_n, J_m] = k\, n\, \delta_{m+n,0}, \\ [H_n, S^\pm_m] &= [J_n, S^\pm_m] = \pm S^\pm_{n+m}, \\ \{S^+_n, S^-_m\} &= k\, n\, \delta_{n+m,0} + H_{n+m} - J_{n+m}. \end{aligned} \tag{2.5}$$

The zero modes $H_0, J_0, S^\pm_0$ satisfy the finite $\mathrm{gl}(1|1)$ algebra.

The algebra $\mathrm{gl}(1|1)_k$ has an inner automorphism that flips the sign of the level $k$:

$$H \to J, \quad J \to H, \quad S_\pm \to \pm S_\pm, \quad k \to -k. \tag{2.6}$$

This implies that results for negative $k$ can be deduced from the case of positive $k$.

The only additional currents one can define in this theory are:

$$J_\pm = \beta_\mp^2, \quad \widehat{S}_\pm = \psi_\mp \beta_\mp. \tag{2.7}$$

The complete set of currents satisfy the $\mathrm{osp}(2|2)_k$ algebra at level $k$. The complete set of relations are presented in Appendix A. Rescaling $J_\pm \to 2\sqrt{2}J_\pm$, $J \to 2J$, one sees that they together satisfy the su(2) current algebra at level $-k/2$. Also, making the redefinition $H \to -H, J \to J, \widehat{S}_\pm \to \pm\widehat{S}_\pm$, one sees that they also satisfy $\mathrm{gl}(1|1)_k$, so that $\mathrm{osp}(2|2)_k$ contains two non-commuting $\mathrm{gl}(1|1)_k$'s.

We will need the Sugawara stress tensor $T(z)$. The algebra gl(1|1) has two independent quadratic casimirs:

$$C_2 = J^2 - H^2 + S^+S^- - S^-S^+, \quad C_2' = (J-H)^2 \tag{2.8}$$

where it is implicit that the above operators are the zero modes of the currents. The stress tensor is fixed by the condition $T(z)J^a(0) \sim J^a(0)/z^2$, which requires it to be built out of both casimirs [12]:

$$T(z) = -\frac{1}{2k}\left(J^2 - H^2 + S_+S_- - S_-S_+\right) + \frac{1}{2k^2}(J-H)^2. \tag{2.9}$$

The leading term in the OPE $T(z)T(0)$ shows that $c = 0$.

## 3 Free field representation

In this section, we present a free field representation of $\mathrm{gl}(1|1)_k$ for any level $k$. The free Dirac fermion can be bosonized with a single scalar field. The results in [21] show that the first-order bosonic $\beta_\pm$ system can be represented in terms of a single scalar field for the $U(1)$ current and another first-order fermionic $\eta - \xi$ system. The latter can be formulated as a second-order symplectic fermion (see Appendix B). Thus, it is clear that the $k=1$ representation of the last section constructed out of the fields $\psi_\pm, \beta_\pm$ can be represented with two scalar fields and a symplectic fermion. What is not so evident is that this same field content is sufficient to provide a free field construction of $\mathrm{gl}(1|1)_k$ at any level. This is in contrast to $\mathrm{su}(2)_k$ for example where the higher level case requires additional $Z_k$ parafermions. (For a review of $2D$ conformal field theory see [22, 23]).

Introduce two scalar fields $\boldsymbol{\phi}^a$ and a symplectic fermion $\boldsymbol{\chi}^a$, $a = 1, 2$, with the following free action:

$$S = \frac{1}{8\pi} \int d^2x \sum_{a,b=1}^{2} \left( \eta_{ab} \partial_\mu \boldsymbol{\phi}^a \partial_\mu \boldsymbol{\phi}^b + \epsilon_{ab} \partial_\mu \boldsymbol{\chi}^a \partial_\mu \boldsymbol{\chi}^b \right) \tag{3.1}$$

where

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.2}$$

and $\partial_\mu \partial_\mu = 2\partial_z \partial_{\overline{z}}$. The $\boldsymbol{\chi}$ fields are Grassman: $(\boldsymbol{\chi}^a)^2 = 0$. Note that the metric for the bosonic fields has indefinite signature. The equations of motion imply that the fields can be decomposed into left- and right-moving parts:

$$\begin{aligned} \boldsymbol{\phi}^a(z, \overline{z}) &= \phi^a(z) + \overline{\phi}^a(\overline{z}), \\ \boldsymbol{\chi}^a(z, \overline{z}) &= \chi^a(z) + \overline{\chi}^a(\overline{z}). \end{aligned} \tag{3.3}$$

In the sequel, we will continue to display local fields in bold face. The two-point functions are

$$\langle \phi^a(z) \phi^b(w) \rangle = -\eta^{ab} \log(z - w), \quad \langle \chi^a(z) \chi^b(w) \rangle = -\epsilon^{ab} \log(z - w). \tag{3.4}$$

(Our conventions are $\eta^{ab} = \eta_{ab}, \epsilon^{ab} = \epsilon_{ab}$.)

It is straightforward to verify the following representation of the OPEs in equation (2.4):

$$\begin{aligned} &H = i\sqrt{k}\, \partial_z \phi^1, \quad J = i\sqrt{k}\, \partial_z \phi^2, \\ &S_+ = \sqrt{k}\, \partial_z \chi^1\, \mathrm{e}^{i(\phi^1 - \phi^2)/\sqrt{k}}, \quad S_- = -\sqrt{k}\, \partial_z \chi^2\, \mathrm{e}^{-i(\phi^1 - \phi^2)/\sqrt{k}}. \end{aligned} \tag{3.5}$$

In the sequel, where there is no cause for confusion, we will simply write $\partial \phi$ for $\partial_z \phi(z)$.

## 4 Finite-dimensional representations of gl(1|1)

The complete solution of the current algebra as a quantum field theory requires the determination of the spectrum of fields. The chiral primary fields $V_r(z)$ transform as finite-dimensional representations $r$ of gl(1|1), which is equivalent to the OPE:

$$J^a(z)\, V_r(0) \sim \frac{1}{z}\, t_r^a\, V_r(0), \tag{4.1}$$

where $J^a, a=1,\ldots,4$ are the $\mathrm{gl}(1|1)_k$ currents and $t_r^a$ is the finite-dimensional matrix representation of $r$ of gl(1|1). (In the sequel we will continue to refer to general super-currents as $J^a$).

Before explicitly constructing the primary fields $V_r$, we first describe the relevant finite dimensional representations [13, 24, 25]. The gl(1|1) algebra has the following non-zero (anti) commutation relations:

$$[H, S_\pm] = [J, S_\pm] = \pm S_\pm, \quad \{S_+, S_-\} = H - J. \tag{4.2}$$

The fermionic operators are nilpotent: $S_\pm^2 = 0$. (It is implicit that the above generators are the zero modes of the currents.) First, there are 1-dimensional representations where $S_\pm = 0$, $H = J = h$. We will denote these as $\langle h\rangle_{(1)}$.

The so-called typical representations are 2-dimensional:

$$H = \begin{pmatrix} h & 0 \\ 0 & h-1 \end{pmatrix}, \quad J = \begin{pmatrix} j & 0 \\ 0 & j-1 \end{pmatrix}, \tag{4.3}$$

$$S_+ = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \tag{4.4}$$

where $bc = h - j$. Let us denote these representations as $\langle h, j\rangle$. When $h \neq j$, these representations are irreducible. The tensor product of two typical representations can be deduced by simply considering the $U(1)$'s:

$$\langle h_1, j_1\rangle \otimes \langle h_2, j_2\rangle = \langle h_1 + h_2, j_1 + j_2\rangle \oplus \langle h_1 + h_2 - 1, j_1 + j_2 - 1\rangle. \tag{4.5}$$

When $h = j$, the representations are reducible but indecomposable. There are two different representations depending on whether $b$ or $c$ equals zero. For $b = 0$, the representation will be referred to as $\langle h, h\rangle$ and for $c = 0$ as $\langle h, h\rangle'$. They can be reduced as $\langle h, h\rangle = \langle h\rangle_{(1)} \oplus \langle h-1\rangle_{(1)}$; however they are indecomposable since $S_- : \langle h\rangle_{(1)} \to \langle h-1\rangle_{(1)}$.

Finally there are 4-dimensional indecomposable representations which will be important in the sequel, and we denote as $\langle h\rangle_{(4)}$. They arise in the tensor product of typical representations when $h_1 + h_2 = j_1 + j_2$:

$$\langle h_1, j_1\rangle \otimes \langle h_2, j_2\rangle = \langle h_1 + h_2 - 1\rangle_{(4)}. \tag{4.6}$$

From equation (4.5), one sees that $\langle h\rangle_{(4)}$ can be reduced into $\langle h+1, h+1\rangle \oplus \langle h, h\rangle$; however, $S_\pm$ mixes these two representations. The generators in $\langle h\rangle_{(4)}$ are

$$S_+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$
$$H = J = \begin{pmatrix} h+1 & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & h-1 \end{pmatrix} \tag{4.7}$$

The indecomposability can be represented by the diagram in Figure 1.

If there exists primary fields corresponding to the representation $r$, then the conformal scaling dimension $\Delta_r$ follows from the Sugawara form (2.9) and the values of the casimirs $C_2, C_2'$ in the representation $r$. This way one finds

$$\Delta_{\langle h,j\rangle} = \frac{(h-j)^2}{2k^2} + \frac{(h-j)(h+j-1)}{2k} \tag{4.8}$$

and $\Delta_{\langle h\rangle_{(1)}} = \Delta_{\langle h,h\rangle} = \Delta_{\langle h,h\rangle'} = 0$. Note that $\Delta_{\langle h,j\rangle}(k) = \Delta_{\langle j,h\rangle}(-k)$, in accordance with the automorphism equation (2.6).

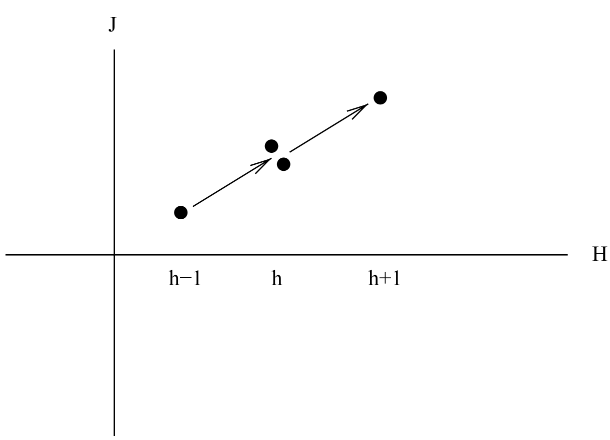


Figure 1: The 4-dimensional indecomposable representation $\langle h\rangle_{(4)}$ of gl(1|1). The arrows indicate the action of $S_+$.

The novel feature of the representation $\langle h\rangle_{(4)}$ is that the casimir $C_2$ is not diagonal:

$$C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.9}$$

As we will see, this leads to logarithmic properties of the fields, as explained in [3] for osp(2|2).

The original fields in the $k = 1$ representation of Section 2 correspond to:

$$(\psi_+, \beta_+) \leftrightarrow \langle 1, 0\rangle, \quad (\beta_-, \psi_-) \leftrightarrow \langle 0, 1\rangle. \tag{4.10}$$

A consistency check is $\Delta_{\langle 1,0\rangle} = \Delta_{\langle 0,1\rangle} = \frac{1}{2}$ when $k = 1$.

## 5 Twist and logarithmic operators in the symplectic fermion theory

In this section, we present the two important features of symplectic fermions we will need in order to construct the primary fields of the current algebra, namely the logarithmic and twist fields. More details are provided in Appendix B.

### 5.1 Logarithmic fields

The $c = -2$ symplectic fermion theory is the simplest and most studied example of a logarithmic conformal field theory [18, 26]; for a review see [27, 28]. The original investigation of these properties was based on the null vector differential equation for the $c = -2$ minimal model four-point functions [26]. In this abstract approach, the explicit construction of the logarithmic operators is not evident. The logarithmic operator is also not contained in the first-order $\eta - \xi$ description [18] (see Appendix B). An important feature of the second-order $\chi$ description is that the logarithmic fields are explicitly contained in the theory because of the additional zero modes.

Generally, let $\ell_0(z), \ell(z)$ denote a logarithmic pair with scaling dimension $\Delta$. By definition they satisfy the following OPE with the stress tensor:

$$\begin{aligned} T(z)\,\ell_0(0) &\sim \frac{\Delta}{z^2}\,\ell_0(0) + \frac{1}{z}\,\partial\ell_0(0), \\ T(z)\,\ell(0) &\sim \frac{\Delta}{z^2}\,\ell(0) + \frac{a}{z^2}\,\ell_0(0) + \frac{1}{z}\partial\ell(0), \end{aligned} \tag{5.1}$$

which implies

$$L_0|\ell_0\rangle = \Delta|\ell_0\rangle, \quad L_0|\ell\rangle = \Delta|\ell\rangle + a|\ell_0\rangle. \tag{5.2}$$

The conformal Ward identities lead to the following two-point functions [3, 29]:

$$\langle \ell_0(z)\ell_0(0)\rangle = 0, \quad \langle \ell(z)\ell_0(0)\rangle = \frac{C}{z^{2\Delta}}, \quad \langle \ell(z)\ell(0)\rangle = \frac{C' - 2aC\log z}{z^{2\Delta}}, \tag{5.3}$$

where $C, C'$ are constants.

The stress tensor of the symplectic fermion is

$$T(z) = \frac{1}{2}\epsilon_{ab}\partial\chi^a\partial\chi^b. \tag{5.4}$$

Define

$$\ell(z) = -\frac{1}{2}\epsilon_{ab}\chi^a\chi^b = -\chi^1\chi^2. \tag{5.5}$$

Then $\ell(z)$ and $\ell_0 = 1$ satisfy the OPEs (5.1) with $\Delta = 0$. On the states $|\ell\rangle = \ell(0)|0\rangle$ and $|0\rangle = |\ell_0\rangle$, the Virasoro zero mode $L_0$ has the usual form for a logarithmic pair: $L_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

The field can be expanded as follows:

$$\chi^a(z) = \chi_0^a - i\widetilde{\chi}_0^a \log(z) + i\sum_{n\neq 0}\frac{1}{n}\chi_n^a\, z^{-n}, \tag{5.6}$$

where $\{\chi_0^a, \widetilde{\chi}_0^b\} = i\epsilon^{ab}$ and $\{\chi_m^a, \chi_n^b\} = m\epsilon^{ab}\delta_{m+n,0}$. In terms of the zero modes:

$$|\ell\rangle = -\chi_0^1\chi_0^2|0\rangle. \tag{5.7}$$

Functional integrals over $\boldsymbol{\chi}$ are zero unless they contain the zero modes:

$$\langle \ell\rangle = 1, \quad \langle 1\rangle = 0 \tag{5.8}$$

consistent with equation (5.3).

## 5.2 Twist fields

As for the spin fields of the Ising model, the twist fields modify the boundary conditions of the fundamental field $\chi$:

$$\begin{aligned} \chi^1(\mathrm{e}^{2\pi i}z)\mu_\lambda(0) &= \mathrm{e}^{-2\pi i\lambda}\chi^1(z)\mu_\lambda(0), \\ \chi^2(\mathrm{e}^{2\pi i}z)\mu_\lambda(0) &= \mathrm{e}^{2\pi i\lambda}\chi^2(z)\mu_\lambda(0). \end{aligned} \tag{5.9}$$

The properties of these fields were studied in [18]. It is clear from the above equation that $2\pi\lambda$ is a phase and is restricted to $-1 < \lambda < 1$. In the presence of $\mu_\lambda$, the mode expansion is twisted:

$$\begin{aligned} \chi^1(z) &= \chi^1_0 + i \sum_{n\in\mathbb{Z}} \frac{1}{n+\lambda} \chi^1_{n+\lambda}\, z^{-n-\lambda}, \\ \chi^2(z) &= \chi^2_0 + i \sum_{n\in\mathbb{Z}} \frac{1}{n-\lambda} \chi^2_{n-\lambda}\, z^{-n+\lambda}. \end{aligned} \tag{5.10}$$

The expansion equation (5.6) arises as $\lambda \to 0$.

We will need the OPE of the twist fields with $\partial\chi$, which involves new fields $\sigma^a_\lambda$. The results are derived in Appendix B:

$$\begin{aligned} &\partial\chi^1(z)\, \mu_\lambda(0) \sim \frac{\sqrt{1-\lambda}}{z^\lambda}\, \sigma^1_\lambda(0), \quad \partial\chi^2(z)\, \sigma^1_\lambda(0) \sim \frac{\sqrt{1-\lambda}}{z^{2-\lambda}}\, \mu_\lambda(0), \\ &\partial\chi^2(z)\, \mu_\lambda(0) \sim \frac{\sqrt{\lambda}}{z^{1-\lambda}}\, \sigma^2_\lambda(0), \quad \partial\chi^1(z)\, \sigma^2_\lambda(0) \sim -\frac{\sqrt{\lambda}}{z^{1+\lambda}}\, \mu_\lambda(0). \end{aligned} \tag{5.11}$$

The scaling dimensions of these fields is

$$\Delta(\mu_\lambda) = \Delta^{(\chi)}_\lambda, \quad \Delta(\sigma^1_\lambda) = \Delta^{(\chi)}_{\lambda-1}, \quad \Delta(\sigma^2_\lambda) = \Delta^{(\chi)}_{\lambda+1} \tag{5.12}$$

where we have defined:

$$\Delta^{(\chi)}_\lambda \equiv \frac{\lambda(\lambda-1)}{2}. \tag{5.13}$$

The powers of $z$ in equation (5.11) are fixed by these scaling dimensions.

The factors of $\sqrt{\lambda}, \sqrt{1-\lambda}$ are not arbitrary and will be important in the next section. They are fixed once the normalizations $\langle \mu_{1-\lambda} | \mu_\lambda \rangle = 1$ and $\langle \sigma^a_{1-\lambda} | \sigma^b_\lambda \rangle = \epsilon^{ab}$ are fixed (see Appendix B).

## 6 Vertex operators

In this section, we explicitly construct the chiral (left-moving) vertex operators for the gl(1|1) representations in Section 4. We present formulas for $k > 0$; negative $k$ results follow from the $k \to -k$ automorphism (2.6). For a general current $J^a$, the vertex operators for a finite-dimensional representation

$r$ are a vector of fields $V_r^\alpha(z)$, $\alpha = 1, 2, \ldots, \dim(r)$ satisfying the OPE

$$J^a(z)\, V_r^i(0) = \frac{1}{z}\; t^a_{ji} V_r^j\,, \tag{6.1}$$

where $t^a$ is the finite-dimensional matrix representation of $r$.

Introduce the notation for the bosonic sector:

$$\mathcal{V}^\phi_{h,j} \equiv \mathrm{e}^{i(h\phi^1 - j\phi^2)/\sqrt{k}}. \tag{6.2}$$

The above field has $U(1)$ charges $(H, J) = (h, j)$ and conformal dimension

$$\Delta^\phi_{h,j} = \frac{h^2 - j^2}{2k}. \tag{6.3}$$

The vertex operators $V_{\langle h\rangle_{(1)}}$ for the 1-dimensional representation $\langle h\rangle_{(1)}$ are purely bosonic:

$$V_{\langle h\rangle_{(1)}} = \mathcal{V}^\phi_{h,h}. \tag{6.4}$$

Let $V_{\langle h,j\rangle}$ denote the vertex operator for the 2-dimensional typical representation with $h \neq j$. They require the twist fields with $\lambda = \frac{h-j}{k}$. For $h > j$ one has

$$V_{\langle h,j\rangle} = (h-j)^{1/4} \begin{pmatrix} -\mu_\lambda\; \mathcal{V}^\phi_{h,j} \\ \sigma^2_\lambda\; \mathcal{V}^\phi_{h-1,j-1} \end{pmatrix}, \quad \lambda = \frac{h-j}{k}. \tag{6.5}$$

For $h < j$ the proper expression is

$$V_{\langle h,j\rangle} = (j-h)^{1/4} \begin{pmatrix} \sigma^1_{1+\lambda}\; \mathcal{V}^\phi_{h,j} \\ \mu_{\lambda+1}\; \mathcal{V}^\phi_{h-1,j-1} \end{pmatrix}, \quad \lambda = \frac{h-j}{k}. \tag{6.6}$$

To verify that these expressions satisfy equation (6.1), one uses the explicit expressions for the $\mathrm{gl}(1|1)_k$ currents (3.5), the representations $t^a$ given in Section 3, and the OPEs (5.11). In doing so, one finds that the factors of $\sqrt{\lambda}, \sqrt{1-\lambda}$ in the OPEs (5.11) are necessary. The construction is also consistent with the scaling dimension $\Delta_{\langle h,j\rangle}$ in equation (4.8):

$$\Delta_{\langle h,j\rangle} = \Delta^\phi_{h,j} + \Delta^{(\chi)}_\lambda = \Delta^\phi_{h-1,j-1} + \Delta^{(\chi)}_{\lambda+1}, \quad \lambda = \frac{h-j}{k}. \tag{6.7}$$

When $h = j$, the vertex operators for the representations $\langle h, h\rangle$ and $\langle h, h\rangle'$ are

$$V_{\langle h,h\rangle} = \begin{pmatrix} \chi^1\; \mathcal{V}^\phi_{h,h} \\ -\sqrt{k}\; \mathcal{V}^\phi_{h-1,h-1} \end{pmatrix}, \quad V_{\langle h,h\rangle'} = \begin{pmatrix} -\sqrt{k}\; \mathcal{V}^\phi_{h,h} \\ \chi^2\; \mathcal{V}^\phi_{h-1,h-1} \end{pmatrix}. \tag{6.8}$$

The vertex operators for $\langle h\rangle_{(4)}$ are novel because they are logarithmic. The zero modes of the $\chi$ fields span a 4-dimensional vector space $|0\rangle, \chi^1|0\rangle, \chi^2|0\rangle, \chi^1\chi^2|0\rangle$, and the vertex operator is built on this structure:

$$V_{\langle h\rangle_{(4)}} = \begin{pmatrix} \chi^1\ \mathcal{V}^{\phi}_{h+1,h+1} \\ \sqrt{k}\ \mathcal{V}^{\phi}_{h,h} \\ \chi^1\chi^2\ \mathcal{V}^{\phi}_{h,h}/\sqrt{k} \\ \chi^2\ \mathcal{V}^{\phi}_{h-1,h-1} \end{pmatrix}. \tag{6.9}$$

The two middle fields $\ell_0' = \sqrt{k}\mathcal{V}^{\phi}_{h,h}$, $\ell' = \chi^1\chi^2\mathcal{V}^{\phi}_{h,h}/\sqrt{k}$ form a logarithmic pair (5.1) with $\Delta = 0$ and $a = -1/k$ since $\ell_0 = 1$ and $\ell(z)$ in equation (5.5) form such a pair. As explained in [3], this logarithmic property is reflected in the fact that the casimir $C_2$ is not diagonal for $\langle h\rangle_{(4)}$, equation (4.9). Using the Sugawara form (2.9) and equation (4.9), one indeed sees that on $\langle h\rangle_{(4)}$:

$$L_0 = -\frac{1}{k}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.10}$$

where $L_0$ is the zero mode of $T(z) = \sum_n L_n z^{-n-2}$, which is consistent with the form of the vertex operator (6.9).

## 7 Closed operator algebras and the spectrum of fields

As for ordinary current algebras, not all representations of gl(1|1) correspond to primary fields. For example, for $su(2)_k$, only the primary fields with spin $j \leq k/2$ are present in the spectrum [30]. For $gl(1|1)_k$ there are similar restrictions depending on the level $k$. Since the twist fields $\mu_\lambda$ are defined for $-1 \leq \lambda \leq 1$ and the vertex operators $V_{\langle h,j\rangle}$ involve $\lambda = \frac{h-j}{k}$, it is clear that:

$$-k \leq h - j \leq k. \tag{7.1}$$

The above restriction can also be understood directly in the affine superalgebra. Let $|h,j\rangle_{hw}$ denote a highest weight state satisfying

$$S^{\pm}_n|h,j\rangle_{hw} = S^{+}_0|h,j\rangle_{hw} = 0, \quad n > 0. \tag{7.2}$$

Consider the modes $S^{\pm}_{1}, S^{\pm}_{-1}$ which satisfy two gl(1|1)'s:

$$\{S^+_1, S^-_{-1}\} = k + H_0 - J_0, \quad \{S^+_{-1}, S^-_1\} = -k + H_0 - J_0. \tag{7.3}$$

Then one has

$$\{S^+_{-1}, S^-_1\}|h, j\rangle_{hw} = S^-_1 S^+_{-1}|h, j\rangle_{hw} = (h - j - k)|h, j\rangle_{hw} = 0. \tag{7.4}$$

This means there is a null state $S^+_{-1}|h, j\rangle_{hw} = 0$ if $h - j = k$. Using this null state inside a three-point function one deduces that the primary fields must satisfy equation (7.1). The fusion rules are as in equation (4.5) where only fields satisfying $h - j \leq k$ are kept on the right hand side.

Thus far there is no restriction on the level $k$. The manner in which $k$ enters the vertex operator construction shows that in the bosonic sector $\mathcal{V}^{\phi}_{h,j}$, $k$ can be interpreted as a radius of compactification $R = \sqrt{k}$.

For generic irrational $k$, one does not have a closed operator algebra. A closed operator algebra is obtained when $k$ is an integer and $(h, j)$ are integers. These are the "minimal models" based on gl$(1|1)_k$. This situation arises naturally in the application to disordered systems since the fundamental fields $\psi_\pm, \beta_\pm$ at $k = 1$ correspond to the doublets $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$, equation (4.10). It will be shown in the next section how one can obtain higher integer $k$ in the multi-copy theory via a super spin charge separation. Thus it appears the locality considerations in [1], which led to the restriction $k = 1/m$ with $m$ is an integer, is too restrictive.

The closed operator algebra at higher integer level $k$ is generated by repeated OPE of the two vertex operators $V_{\langle 1,0\rangle}$ and $V_{\langle 0,1\rangle}$, and are subject to the restriction (7.1). Note that the scaling dimensions follow the pattern

$$\Delta_{\langle h+n, j+n\rangle} = \Delta_{\langle h,j\rangle} + \frac{n(h - j)}{k}. \tag{7.5}$$

### 7.1 The case of $k = 2$

Let us illustrate these features in the next simplest case of $k = 2$. The twist fields $\mu_{1/2}$ and $\sigma^a_{1/2}$ have $\Delta$ equal to $-1/8$ and $3/8$, respectively. These fields have the following OPE [18]

$$\begin{aligned}
\mu_{1/2}(z)\, \mu_{1/2}(0) &= z^{1/4} \left(\ell(0) + \log(z) + \cdots\right), \\
\sigma^a_{1/2}(z)\, \sigma^b_{1/2}(0) &= \frac{1}{z^{3/4}} \epsilon^{ab} \left(\ell(0) + \log(z) + \cdots\right), \\
\mu_{1/2}(z)\, \sigma^a_{1/2}(0) &= -\frac{1}{\sqrt{2}} \frac{1}{z^{1/4}} \left(\chi^a(0) + \cdots\right).
\end{aligned} \tag{7.6}$$

The vertex operators $V_{\langle 1,0\rangle}$ and $V_{\langle 0,1\rangle}$ both have conformal dimension 1/8 and take the form

$$V_{\langle 1,0\rangle} = \begin{pmatrix} -\mu_{1/2}\,\mathcal{V}^{\phi}_{1,0} \\ \sigma^2_{1/2}\,\mathcal{V}^{\phi}_{0,-1} \end{pmatrix}, \quad V_{\langle 0,1\rangle} = \begin{pmatrix} \sigma^1_{1/2}\,\mathcal{V}^{\phi}_{0,1} \\ \mu_{1/2}\,\mathcal{V}^{\phi}_{-1,0} \end{pmatrix}. \tag{7.7}$$

Using the OPEs (7.6), one finds

$$V_{\langle 1,0\rangle}(z)\, V_{\langle 0,1\rangle}(0) \sim \frac{1}{z^{1/4}} V_{\langle 0\rangle_{(4)}}, \tag{7.8}$$

where $V_{\langle 0\rangle_{(4)}}$ is the 4-dimensional logarithmic field (6.9).

To find the other OPEs, we need the $\lambda = 0, 1$ limit of the twist fields in the expressions (6.5) and (6.6) for the vertex operators $V_{\langle 2,0\rangle}, V_{\langle 1,-1\rangle}, V_{\langle 0,2\rangle}, V_{\langle -1,1\rangle}$. The following linear combinations are consistent with the $\lambda = 0, 1$ limit of the OPEs in equation (5.11):

$$\begin{aligned} \mu_1 &= a + b\,\chi^2, \quad \sigma^2_1 = a\,\partial\chi^2 + b\,\partial\chi^2\,\chi^2, \\ \mu_0 &= c + d\,\chi^1, \quad \sigma^1_0 = c\,\partial\chi^1 + d\,\partial\chi^1\,\chi^1, \end{aligned} \tag{7.9}$$

where $a, b, c, d$ are constants. Which linear combinations appear in the vertex operators follows from equation (7.6). One finds

$$\begin{aligned} V_{\langle 1,0\rangle}(z)\, V_{\langle 1,0\rangle}(0) &\sim z^{-1/4}\; V_{\langle 1,-1\rangle} + z^{3/4}\; V_{\langle 2,0\rangle}, \\ V_{\langle 0,1\rangle}(z)\, V_{\langle 0,1\rangle}(0) &\sim z^{-1/4}\; V_{\langle 0,2\rangle} + z^{3/4}\; V_{\langle -1,1\rangle}, \end{aligned} \tag{7.10}$$

where

$$\begin{aligned} V_{\langle 2,0\rangle} &= \sqrt{2}\begin{pmatrix} -\mathcal{V}^{\phi}_{2,0} \\ \partial\chi^2\mathcal{V}^{\phi}_{1,-1} \end{pmatrix}, \quad V_{\langle 1,-1\rangle} = \sqrt{2}\begin{pmatrix} -\chi^2\mathcal{V}^{\phi}_{1,-1} \\ \partial\chi^2\chi^2\mathcal{V}^{\phi}_{0,-2} \end{pmatrix} \\ V_{\langle 0,2\rangle} &= \sqrt{2}\begin{pmatrix} \partial\chi^1\chi^1\mathcal{V}^{\phi}_{0,2} \\ \chi^1\mathcal{V}^{\phi}_{-1,1} \end{pmatrix}, \; V_{\langle -1,1\rangle} = \sqrt{2}\begin{pmatrix} \partial\chi^1\mathcal{V}^{\phi}_{-1,1} \\ \mathcal{V}^{\phi}_{-2,0} \end{pmatrix} \end{aligned} \tag{7.11}$$

The remaining low dimension fields are $V_{\langle 2,1\rangle}$ and $V_{\langle 1,2\rangle}$ with $\Delta = 5/8$, $-3/8$, respectively. By virtue of equation (7.5), the other fields have dimension which differs by an integer from the fields considered thus far.

Since $\mathrm{gl}(1|1)_2$ has $c = 0$, it is interesting to compare it with the $c = 0$ minimal Virasoro model. Let us refer to the minimal model fields at $c = 0$ as $\Phi_{m,n}$ with conformal dimension $\Delta^{(\mathrm{min})}_{m,n}$ (see [22, 23]). The two models share the dimensions 1/8 and 5/8 since $\Delta^{(\mathrm{min})}_{2,2} = 1/8$ and $\Delta^{(\mathrm{min})}_{2,1} = 5/8$. The latter determines the correlation length exponent for percolation, $\nu_{\mathrm{perc.}} = (2(1 - \frac{5}{8}))^{-1} = 4/3$. Note that the field $\Phi_{1,3}$ with $\Delta = 1/3$ is not present in the gl(1|1)$_2$ theory. The field $\Phi_{1,3}$ is known to determine the correlation length exponent for self-avoiding walks, $\nu_{\rm SAW} = 3/4$. Thus, as a possible disordered critical point, gl(1|1)$_2$ is more closely related to percolation. However it is not entirely equivalent to it since it does not contain for example all the hull exponents considered in [31]. We will return to this point in Section 12 where we discuss applications to the SQHT.

## 8 Super spin-charge separation and disordered critical points.

Spin-charge separation for ordinary su(2) Dirac fermions has many important applications, for example to Luttinger liquids in $1d$. In this section, we present the extension of this construction to super-current algebras. We also explain how the higher level gl(1|1)$_k$ theory can arise as a disordered critical point.

Consider the action (2.1) for Dirac fermions only, extended to $N$-copies:

$$S^{\rm N-copy} = \frac{1}{4\pi} \int d^2x \sum_{\alpha=1}^{N} \left( \psi_-^\alpha \partial_{\overline{z}} \psi_+^\alpha + \overline{\psi}_-^\alpha \partial_z \overline{\psi}_+^\alpha \right). \tag{8.1}$$

The model now has an su$(N)_{k=1}$ symmetry with currents

$$L_\psi^a = \psi_-^\alpha t_{\alpha\beta}^a \psi_+^\beta, \tag{8.2}$$

where here $t^a$ are a matrix representation of the vector of su$(N)$. The model also has a $u(1)$ symmetry which commutes with su$(N)$. Spin-charge separation is the statement that the full stress tensor for the free theory can be decomposed into commuting parts:

$$T_{\rm free}^{\rm N\text{-}copy} = -\frac{1}{2} \sum_\alpha \psi_-^\alpha \partial_z \psi_+^\alpha = T_{u(1)} + T_{{\rm su}(N)_1}, \tag{8.3}$$

where $T_{{\rm su}(N)_1}$ is the Sugawara stress tensor and $T_{u(1)}$ is the stress tensor for a single scalar field (see for instance [23]). A check of the above decomposition is the central charge. The su$(N)_k$ theory has $c_{{\rm su}(N)_k} = \frac{k(N^2-1)}{(k+N)}$, whereas the $u(1)$ has $c = 1$. When $k = 1$, the total $c$ equals $N$, as appropriate for $N$ Dirac fermions. The other check involves the scaling dimension. The $N$-dimensional vector representation at level $k$ has

$$\Delta_{{\rm su}(N)_k} = \frac{N^2 - 1}{2N(k+N)}. \tag{8.4}$$

The $u(1)$ is at radius $R = \sqrt{N}$, with $\Delta_{u(1)} = \frac{1}{2N}$, and one verifies $\Delta(\psi_\pm) = \Delta_{u(1)} + \Delta_{{\rm su}(N)_1} = \frac{1}{2}$.

Consider now $N$ copies of the theory (2.1) with ghosts $\beta_\pm^\alpha$. This theory has the maximal $\mathrm{osp}(2N|2N)_1$ symmetry. In the ghost sector the currents

$$L_\beta^a = \beta_- t^a \beta_+ \tag{8.5}$$

satisfy $\mathrm{su}(N)_{k=-1}$. We will need the following basic result. Given two copies of the same current algebra with currents $J_1^a$ at level $k_1$ and $J_2^a$ at level $k_2$ which furthermore commute, $[J_1^a(z), J_2^b(w)] = 0$. Then $J^a = J_1^a + J_2^a$ satisfies the current algebra at level $k_1 + k_2$. The complete $\mathrm{su}(N)$ currents $L^a = L_\psi^a + L_\beta^a$ thus have level $k = 0$.

The model also has a $\mathrm{gl}(1|1)$ symmetry generated by the currents:

$$H = \sum_\alpha \psi_+^\alpha \psi_-^\alpha, \quad J = \sum_\alpha \beta_+^\alpha \beta_-^\alpha, \quad S_\pm = \pm \sum_\alpha \psi_\pm^\alpha \beta_\mp^\alpha. \tag{8.6}$$

Since the above currents are sums of the currents in each copy with level $k = 1$, they satisfy $\mathrm{gl}(1|1)_k$ with $k = N$. It is also important that these $\mathrm{gl}(1|1)_N$ currents commute with the $\mathrm{su}(N)_{k=0}$. The super spin charge separation is the non-trivial statement:

$$T_{\text{free}}^{\text{N-copy}} = -\frac{1}{2}\sum_\alpha (\psi_-^\alpha \partial_z \psi_+^\alpha + \beta_-^\alpha \partial_z \beta_+^\alpha) = T_{\mathrm{gl}(1|1)_N} + T_{\mathrm{su}(N)_0}. \tag{8.7}$$

As we will show in section X, $T_{\mathrm{gl}(1|1)_2} = T_{\mathrm{osp}(2|2)_{-2}}$, and this form of the relation (8.7) was proved for $k = 2$ in [5], see also [32]. The more general relation above for any $N$ can be proved similarly. Note that since both current algebras have $c = 0$, this is consistent with $c_{\text{free}} = 0$. A more non-trivial check at arbitrary $N$ is based on the conformal dimensions. The fields $(\psi_+, \beta_+), (\beta_-, \psi_-)$ transform in the $\langle 1, 0\rangle, \langle 0, 1\rangle$ representations of $\mathrm{gl}(1|1)_N$ with $\Delta_{\langle 1,0\rangle} = \Delta_{\langle 0,1\rangle} = \frac{1}{2N^2}$. The vector representation of $\mathrm{su}(N)_0$ has $\Delta_{\mathrm{su}(N)_0} = \frac{N^2-1}{2N^2}$ so that

$$\Delta(\psi_\pm, \beta_\pm) = \Delta_{\langle 1,0\rangle} + \Delta_{\mathrm{su}(N)_0} = \frac{1}{2}. \tag{8.8}$$

The $\mathrm{gl}(1|1)_N$ theory can arise as a disordered critical point as follows. More generally, consider two commuting current algebras $\mathcal{G}_A$ and $\mathcal{G}_B$ with currents $J_A, J_B$. Furthermore, let us suppose that the stress tensor for a given conformal theory separates as in equation (8.7). Consider the perturbation of the conformal field theory by left–right current–current perturbations:

$$S = S_{\text{cft}} + \int \frac{d^2x}{2\pi} \left( g_A\, J_A \cdot \overline{J}_A + g_B\, J_B \cdot \overline{J}_B \right), \tag{8.9}$$

where $J \cdot \overline{J}$ is the invariant built on the quadratic casimir. Since the currents commute, the RG beta-functions decouple; to 1-loop the result is

$$\frac{dg_A}{d\ell} = C_A^{\text{adj}}\, g_A^2, \quad \frac{dg_B}{d\ell} = C_B^{\text{adj}}\, g_B^2, \tag{8.10}$$

where $\ell$ is the logarithm of the length scale and $C_A^{\text{adj}}$ is the casimir for the adjoint representation of $\mathcal{G}_A$. Let us suppose that the physical regime corresponds to positive $g_{A,B}$. If $C_B^{\text{adj}}$ is positive, then the coupling $g_B$ is marginally relevant and the flow is to infinity. This is a massive sector as in the Gross–Neveu model. These massive $\mathcal{G}_B$ degrees of freedom are decoupled at low energies. We will refer to the $\mathcal{G}_B$ degrees of freedom as being "gapped-out" in the RG flow to low energies. If $C_A^{\text{adj}}$ is negative, then the coupling $g_A$ is marginally irrelevant. This results in the fixed point defined by the theory with current algebra symmetry $\mathcal{G}_A$. If the original conformal field theory corresponds to the current algebra $\mathcal{G}_{\max}$, then the fixed point may be viewed as the coset $\mathcal{G}_{\max}/\mathcal{G}_B$. For $N$-copies of Dirac fermions and ghosts, $\mathcal{G}_{\max} = \text{osp}(2N \mid 2N)_1$. This scenario was proposed for generic fixed points of marginal current–current perturbations in [33], however, here it is a somewhat trivial example of the GKO construction [34] because of the decomposition of the stress tensor. In fact, what was missing in the arguments in [33] was precisely the spin-charge separation.

Returning to disordered Dirac fermions, the N-copy version is relevant for the computation of averages of multiple moments (multi-fractality) or can be part of the definition of the 1-copy theory, as in the SQHT which has an su(2) symmetry from the very beginning and thus the 1-copy theory corresponds to $N = 2$. Disorder averaging generally leads to left–right current–current perturbations. For certain kinds of disorder, where perhaps some of the disorder is set to zero, the disorder averaged effective action takes the form (8.9) with $\mathcal{G}_A = \text{gl}(1|1)_N$ and $\mathcal{G}_B = \text{su}(N)_0$. The $\text{su}(N)_0$ current interactions can arise from a disordered su($N$) gauge field, but not necessarily so; other scenarios will be described in [35].

For the $\text{su}(N)_0$ currents $L^a$, $C^{\text{adj}} > 0$. For super-current algebras like $\text{osp}(2N|2N)$, $C^{\text{adj}} < 0$. For gl(1|1) the situation is somewhat more subtle because there are two quadratic casimirs [4]. Consider

$$S = S_{\text{gl}(1|1)_k} + \int \frac{d^2x}{2\pi} \left( g\, \left( J\overline{J} - \overline{H}H + S_+\overline{S}_- - S_-\overline{S}_+ \right) + g'(J-H)(\overline{J}-\overline{H}) \right), \tag{8.11}$$

where $S_{\text{gl}(1|1)_k}$ formally represents the conformal theory with $\text{gl}(1|1)_k$ symmetry. The latter can be taken to have the free field form (3.1). Then the 1-loop beta-function for $g$ is zero, whereas $dg'/d\ell = -g^2$. Therefore the gl(1|1) current interactions are marginally irrelevant. This is to be contrasted with the situation for the model in [4] since there $g'$ corresponded to the variance of disordered *imaginary* $u(1)$ gauge field and this changes the sign of the coupling. The higher loop corrections computed in [33] do not alter this picture.

For $N = 2$ the analogue of this flow to $\mathrm{osp}(2|2)_{-2}$ for pure su(2) gauge disorder was proposed in [5]. In [6] the $\mathrm{osp}(2|2)_{-2}$ description of strongly disordered gauge fields was shown to be consistent with other approaches such as [1, 2]. More interesting models, such as the SQHT, have additional kinds of disorder besides pure gauge field disorder and thus should correspond to relevant perturbations of the current algebra. We will return to this issue in Section 12, where we show that the perturbation is by a logarithmic operator.

## 9 Local gl(1|1) invariant operators

The left and right sectors must be put together in a consistent manner in order to obtain local operators $\mathbf{\Phi}(z, \overline{z})$ with single-valued correlation functions. In this section we describe how to construct such operators that are also gl(1|1) invariant.

We first need to fix our conventions for the right-moving sector. Given the decomposition (3.3), we define the right-moving currents as

$$\overline{H} = -i\sqrt{k}\,\partial_{\overline{z}}\overline{\phi}^1, \quad \overline{J} = -i\sqrt{k}\,\partial_{\overline{z}}\overline{\phi}^2. \tag{9.1}$$

Right-moving vertex operators of charge $(h, j)$ are

$$\overline{\mathcal{V}}^{\phi}_{h,j} = \mathrm{e}^{-i(h\overline{\phi}^1 - j\overline{\phi}^2)/\sqrt{k}}. \tag{9.2}$$

Local $u(1)$ invariant bosonic vertex operators are then

$$\mathbf{V}^{\phi}_{h,j} = \mathcal{V}^{\phi}_{h,j}\,\overline{\mathcal{V}}^{\phi}_{-h,-j} = \mathrm{e}^{i(h\boldsymbol{\phi}^1 - j\boldsymbol{\phi}^2)/\sqrt{k}}. \tag{9.3}$$

Imposing locality for the symplectic fermion $\boldsymbol{\chi}(\mathrm{e}^{i\alpha}z, \mathrm{e}^{-i\alpha}\overline{z}) = \boldsymbol{\chi}(z, \overline{z})$ identifies the zero modes $\widetilde{\chi}_0 = \overline{\widetilde{\chi}}_0$ so that the field has the expansion

$$\boldsymbol{\chi}(z, \overline{z}) = \boldsymbol{\chi}_0 - i\widetilde{\boldsymbol{\chi}}_0 \log(z\overline{z}) + i\sum_{n\neq 0}\left(\frac{1}{n}\chi_n\, z^{-n} + \frac{1}{n}\overline{\chi}_n\, \overline{z}^{-n}\right) \tag{9.4}$$

with $\{\boldsymbol{\chi}_0^a, \widetilde{\boldsymbol{\chi}}_0^b\} = i\epsilon^{ab}$. This implies for example that the local version of the logarithmic field $\ell(z)$ in (5.5) is simply:

$$\boldsymbol{\ell}(z,\overline{z}) = -\frac{1}{2}\epsilon_{ab}\boldsymbol{\chi}^a\boldsymbol{\chi}^b. \tag{9.5}$$

It satisfies

$$T(z)\boldsymbol{\ell}(0) \sim \frac{1}{z^2} + \frac{1}{z}\partial_z\boldsymbol{\ell}(0), \quad \overline{T}(\overline{z})\boldsymbol{\ell}(0) \sim \frac{1}{\overline{z}^2} + \frac{1}{\overline{z}}\partial_{\overline{z}}\boldsymbol{\ell}(0). \tag{9.6}$$

When we encounter $\chi^1\overline{\chi}^2$ this is thus equated to $\boldsymbol{\chi}^1\boldsymbol{\chi}^2$. For the remainder of this section and the next, we will not display the local fields $\boldsymbol{\phi}, \boldsymbol{\chi}, \boldsymbol{\ell}$ in bold face but simply as $\phi, \chi, \ell$.

Let $Q^a = \frac{1}{2\pi i}\oint J^a(z)$ denote the left-moving charge for the current $J^a$ and similarly for $\overline{Q}^a$. The vertex operators in the representations $r, \overline{r}$ satisfy

$$[Q^a, V_r^i] = t^a_{ji}\, V_r^j, \quad [\overline{Q}^a, \overline{V}_{\overline{r}}^j] = \overline{t}^a_{ji}\, \overline{V}_{\overline{r}}^j. \tag{9.7}$$

Introduce the notation

$$V_r \cdot \overline{V}_{\overline{r}} = d_{ij} V_r^i \overline{V}_{\overline{r}}^j. \tag{9.8}$$

The operator $V_r \cdot \overline{V}_{\overline{r}}$ is invariant under the diagonal gl(1|1) symmetry $Q^a + \overline{Q}^a$ if the following relation holds:

$$t^a d + d\,\overline{t}^a = 0, \quad \forall\ a. \tag{9.9}$$

Using the explicit matrix representations $t^a$ in Section 4, one finds the following local gl(1|1) invariant operators:

(i) Typical representations with $h \neq j$. An invariant is

$$\boldsymbol{\Phi}_{\langle h,j\rangle} = V_{\langle h,j\rangle} \cdot \overline{V}_{\langle 1-h,1-j\rangle}, \quad d = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{9.10}$$

The structure of the $h, j$ charges is dictated by the $U(1)$ symmetries $H, J$. It will also prove useful to define

$$\widetilde{\boldsymbol{\Phi}}_{\langle h,j\rangle} = \overline{V}_{\langle 1-h,1-j\rangle} \cdot V_{\langle h,j\rangle}, \tag{9.11}$$

which can differ from $\boldsymbol{\Phi}_{\langle h,j\rangle}$ by fermionic signs which arise when left and right are interchanged.

(ii) Two-dimensional representations with $h = j$. There are four types of such operators:

$$\mathbf{\Phi}_{\langle h,h\rangle} = V_{\langle h,h\rangle} \cdot \overline{V}_{\langle 1-h,1-h\rangle}, \quad d = \begin{pmatrix} 0 & 1 \\ -1 & a\,\delta_{h,1/2} \end{pmatrix}, \tag{9.12}$$

$$'\mathbf{\Phi}'_{\langle h,h\rangle} = V_{\langle h,h\rangle'} \cdot \overline{V}_{\langle 1-h,1-h\rangle'}, \quad d = \begin{pmatrix} a\,\delta_{h,1/2} & 1 \\ -1 & 0 \end{pmatrix}, \tag{9.13}$$

$$\mathbf{\Phi}'_{\langle h,h\rangle} = V_{\langle h,h\rangle} \cdot \overline{V}_{\langle 1-h,1-h\rangle'}, \quad d = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{9.14}$$

$$'\mathbf{\Phi}_{\langle h,h\rangle} = V_{\langle h,h\rangle'} \cdot \overline{V}_{\langle 1-h,1-h\rangle}, \quad d = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{9.15}$$

Above $a$ is a free parameter that is only allowed if $h = \frac{1}{2}$ by $u(1)$ invariance.

(iii) 4-dimensional indecomposable representations. Finally there is a local field based on the representation $\langle h\rangle_{(4)}$:

$$\mathbf{\Phi}_{\langle h\rangle_{(4)}} = V_{\langle h\rangle_{(4)}} \cdot \overline{V}_{\langle -h\rangle_{(4)}}, \quad d = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & a & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{9.16}$$

As before, $a$ is a free parameter.

The local fields based on the atypical representations are of interest since they are expressed in terms of the original local fields $\phi, \chi$. The fields $\mathbf{\Phi}'_{\langle h,h\rangle} = k\mathbf{V}^{\phi}_{h-1,h-1}$ and $'\mathbf{\Phi}_{\langle h,h\rangle} = k\mathbf{V}^{\phi}_{h,h}$ are purely bosonic singlets. $\mathbf{\Phi}_{\langle h,h\rangle}$ and $'\mathbf{\Phi}'_{\langle h,h\rangle}$ are fermionic when $a = 0$. Thus the most interesting field is the logarithmic one

$$\mathbf{\Phi}_{\langle h\rangle_{(4)}} = \chi^1\chi^2 \left(\mathrm{e}^{i(h+1)(\phi^1-\phi^2)/\sqrt{k}} + \mathrm{e}^{i(h-1)(\phi^1-\phi^2)/\sqrt{k}}\right) + a\,k\,\mathrm{e}^{ih(\phi^1-\phi^2)/\sqrt{k}}. \tag{9.17}$$

The case of the $h = 0$, which arises in the OPE of $\mathbf{\Phi}_{\langle 1,0\rangle}$ with $\mathbf{\Phi}_{\langle 0,1\rangle}$, is real:

$$\mathbf{\Phi}_{\langle 0\rangle_{(4)}} = 2\chi^1\chi^2 \cos\left(\frac{\phi^1-\phi^2}{\sqrt{k}}\right). \tag{9.18}$$

In the sequel, we will need the explicit forms of some additional local operators in the case of $k = 2$. The fundamental field involves the twist fields:

$$\mathbf{\Phi}_{\langle 1,0\rangle} = \mathrm{e}^{i\phi^1/\sqrt{2}}\,\boldsymbol{\mu}_{1/2} + \mathrm{e}^{i\phi^2/\sqrt{2}}\,\boldsymbol{\sigma}_{1/2}, \tag{9.19}$$

where $\boldsymbol{\mu}_{1/2} = \mu_{1/2}\overline{\mu}_{1/2}$ and $\boldsymbol{\sigma}_{1/2} = \sigma^2_{1/2}\overline{\sigma}^1_{1/2}$. In addition to the field $\mathbf{\Phi}_{\langle 0\rangle_{(4)}}$, at $k=2$, the fields $\mathbf{\Phi}_{\langle 2,0\rangle}, \mathbf{\Phi}_{\langle -1,1\rangle}, \mathbf{\Phi}_{\langle 1,-1\rangle}$ and $\mathbf{\Phi}_{\langle 0,2\rangle}$ are also expressed in terms of the original fields:

$$\begin{aligned}
\mathbf{\Phi}_{\langle 2,0\rangle} - \widetilde{\mathbf{\Phi}}_{\langle -1,1\rangle} &= 4\partial_\mu\chi^1\partial_\mu\chi^2 \cos\Big((\phi^1+\phi^2)/\sqrt{2}\Big) - 4\cos(\sqrt{2}\phi^1),\\
\mathbf{\Phi}_{\langle 1,-1\rangle} - \widetilde{\mathbf{\Phi}}_{\langle 0,2\rangle} &= 4\,\chi^1\chi^2 \cos\Big((\phi^1+\phi^2)/\sqrt{2}\Big)\\
&\quad + 4(\partial_\mu\chi^1\partial_\mu\chi^2)(\chi^1\chi^2)\cos(\sqrt{2}\phi^2).
\end{aligned} \tag{9.20}$$

## 10 Logarithmic perturbations and local Lagrangians

Using the constructions of the last section, we can consider a variety of local perturbations of the free action that preserve gl(1|1). The simplest and most interesting are based on the 4-dimensional indecomposable representation $\langle h\rangle_{(4)}$. Consider first a perturbation by $\mathbf{\Phi}_{\langle 0\rangle_{(4)}}$:

$$\begin{aligned}
S &= S_{\mathrm{gl}(1|1)_k} + \int \frac{d^2x}{8\pi}\,\mathbf{\Phi}_{\langle 0\rangle_{(4)}}\\
&= \int \frac{d^2x}{8\pi}\left(\sum_{a,b=1}^{2} \eta_{ab}\,\partial_\mu\phi^a\partial_\mu\phi^b + \epsilon_{ab}\,\partial_\mu\chi^a\partial_\mu\chi^b + g\,\chi^1\chi^2\cos\Big((\phi^1-\phi^2)/\sqrt{k}\Big)\right).
\end{aligned} \tag{10.1}$$

The above action may be viewed as a gl(1|1) invariant generalization of the sine-Gordon theory. The interaction is a $\Delta = 0$ logarithmic operator.

Next consider a perturbation by $\mathbf{\Phi}_{\langle 1\rangle_{(4)}}$:

$$S = \int \frac{d^2x}{8\pi}\left(\sum_{a,b=1}^{2} \eta_{ab}\,\partial_\mu\phi^a\partial_\mu\phi^b + \epsilon_{ab}\,\partial_\mu\chi^a\partial_\mu\chi^b + g\,\chi^1\chi^2\, e^{2i(\phi^1-\phi^2)/\sqrt{k}}\right). \tag{10.2}$$

(We set the free parameter $a=0$.) This may be viewed as a gl(1|1) invariant Liouville theory. As for the usual Liouville, background charges $\vec{q}_0$ can be introduced to give the perturbation $\Delta = 1$:

$$T_{\vec{\phi}} = -\frac{1}{2}\partial\vec{\phi}\cdot\partial\vec{\phi} + \frac{i}{2}\vec{q}_0\cdot\partial^2\vec{\phi}, \tag{10.3}$$

where $\vec{\phi}\cdot\vec{\phi} \equiv \eta_{ab}\phi^a\phi^b$. The dimensions are

$$\Delta(e^{i\vec{q}\cdot\vec{\phi}}) = \frac{1}{2}\vec{q}\cdot(\vec{q}-\vec{q}_0). \tag{10.4}$$

The new central charge in the bosonic sector is $c_{\text{bosonic}} = 2 - 3\vec{q}_0 \cdot \vec{q}_0$. For the perturbation in (10.2), $\vec{q} = 2(1,1)/\sqrt{k}$ with $\vec{q} \cdot \vec{q} = 0$. Choosing $\vec{q}_0 = \sqrt{k}(-1,1)/2$ endows it with dimension 1. Note that since $\vec{q}_0 \cdot \vec{q}_0 = 0$, the total central charge remains zero. Note also that $\vec{q}_0 = \sqrt{k}(-1,1)$ renders the full cosine term of the sine-Gordon version with $\Delta = 1$. We will not pursue adding background charges further in this paper.

An important feature of logarithmic perturbations such as in equation (10.1) is the following. Because of the indefinite metric for the bosons, OPEs of the dimension zero operators in the bosonic sector are regular:

$$e^{ih(\phi^1-\phi^2)(z)}\, e^{ih'(\phi^1-\phi^2)(w)} \sim \text{regular}. \tag{10.5}$$

Therefore in perturbation theory, the perturbation by $\mathbf{\Phi}_{\langle 0 \rangle_{(4)}}$ behaves like a mass term $\chi^1\chi^2$. As for a mass term, it simply leads to logarithmic corrections to correlation functions without changing the exponents.

More generally consider a conformal field theory perturbed by a logarithmic operator $\Phi_\ell$ with action

$$S = S_{\text{cft}} + \int \frac{d^2x}{2\pi}\, g\, \Phi_\ell(x). \tag{10.6}$$

The RG beta-function for $g$ is determined by OPE of $\Phi_\ell$ with itself. If $\Phi_\ell$ has $\Delta = 0$, then the singular term in the OPE is at worse a logarithm:

$$\Phi_\ell(x)\, \Phi_\ell(0) = \gamma \log(x^2)\, \Phi_\ell(0) + \cdots \tag{10.7}$$

Introducing a short distance cut-off $a$, $\int_a d^2x \log(x^2) = \pi a^2(1 - \log(a)) +$ const., then the cut-off dependent coupling is $g(a) = g + \gamma g^2 a^2 (1 - 2\log(a))/4$. This implies that as $a \to 0$, the beta-function $dg(a)/d\log(a) \to 0$. Thus, in general one does not expect logarithmic perturbations to drive the theory to a new fixed point. This feature was also discussed in [37].

## 11 Aspects of osp(2|2)$_{-2}$

Normally, larger dimensional algebras such as osp(2|2)$_k$ require more fields to be represented. However it was shown by Ludwig [17] that the special case of $k = -2$ has a free field representation with the same field content as above. In this section, we explain how this follows from our results on gl(1|1)$_k$ and use this connection to present additional results.

First note that osp(2|2)$_{-2}$ has two gl(1|1)$_{-2}$ subalgebras which do not commute. Let us try to represent $H, J$ and both $S_\pm$ and $\widehat{S}_\pm$ with the same field content as in equation (3.5). The problem with generic $k$ is that the

OPE $S_+(z)\widehat{S}_+(0) \propto 1/(z^{2+2/k})$ and thus does not close on integer powers, except for $k=-2$. The resulting free field representation at $k=-2$ then follows from previous expressions (3.5) with $\sqrt{k}=i\sqrt{2}$:

$$\begin{aligned} H &= -\sqrt{2}\,\partial\phi^1, \quad J = -\sqrt{2}\,\partial\phi^2, \quad J_\pm = \pm 2\,\mathrm{e}^{\mp\sqrt{2}\phi^2} \\ S_\pm &= \pm i\sqrt{2}\;\partial\chi^\pm\;\mathrm{e}^{\pm(\phi^1-\phi^2)/\sqrt{2}}, \quad \widehat{S}_\pm = \pm i\sqrt{2}\;\partial\chi^\mp\;\mathrm{e}^{\mp(\phi^1+\phi^2)/\sqrt{2}}, \end{aligned} \tag{11.1}$$

where here for notational simplicity we have defined $\chi^{1,2}=\chi^{+,-}$. The OPEs of the above currents is the same as in Appendix A up to some inconsequential minus signs. Note that $J, J_\pm$ have the standard $\mathrm{su}(2)_1$ representation in terms of a single boson. For the remainder of this section gl(1|1) refers to the $\mathrm{gl}(1|1)_{-2}$ algebra generated by $H, J, S_\pm$.

The finite dimensional representations of osp(2|2) can be labelled by the su(2) with generators $J, J_\pm$ and by the $u(1)$ charge $H$. The typical, irreducible representations will be denoted as $[b,s]^{\mathrm{osp}}$ where $s\in\{0,\frac{1}{2},1,\frac{3}{2},\ldots\}$ is an su(2) spin and $b=H/2$. These representations are $8s$ dimensional. In order to describe their $\mathrm{su}(2)\otimes u(1)$ decomposition, let $[b,s]^{\mathrm{su}}$ denote the $2s+1$ dimensional representation with $J/2=s_3=-s,-s+1,...,s$ and $H=2b$. The generic decomposition is [3, 36]

$$[b,s]^{\mathrm{osp}} = [b,s]^{\mathrm{su}} \oplus [b+\tfrac{1}{2}, s-\tfrac{1}{2}]^{\mathrm{su}} \oplus [b-\tfrac{1}{2}, s-\tfrac{1}{2}]^{\mathrm{su}} \oplus [b, s-1]^{\mathrm{su}}. \tag{11.2}$$

The stress tensor is built from the single quadratic casimir [3]:

$$\begin{aligned} T_{\mathrm{osp}(2|2)} = \frac{1}{2(2-k)}\Big[ & J^2 - H^2 - \frac{1}{2}(J_+J_- + J_-J_+) + (S_+S_- - S_-S_+) \\ & + (\widehat{S}_-\widehat{S}_+ - \widehat{S}_+\widehat{S}_-)\Big]. \end{aligned} \tag{11.3}$$

Since the $k=-2$ case has the same free field construction as $\mathrm{gl}(1|1)_{-2}$, and there exists the $k\to -k$ automorphism (2.6), one must have $T_{\mathrm{osp}(2|2)_{-2}} = T_{\mathrm{gl}(1|1)_{-2}} = T_{\mathrm{gl}(1|1)_2}$. The typical representations with $b^2\neq s^2$ have conformal dimension

$$\Delta^{\mathrm{osp}}_{[b,s]} = \frac{2(s^2-b^2)}{2-k}. \tag{11.4}$$

The vertex operators follow from the results of Section 6 and the decomposition of the osp(2|2) representations in terms of gl(1|1). The later can be deduced from (11.2) with the identification $H = 2b, J = 2s_3$. For example

$$[b, \tfrac{1}{2}]^{\text{osp}} = \langle 2b, 1\rangle \oplus \langle 2b+1, 0\rangle, \tag{11.5}$$

where $\langle h, j\rangle$ are the 2-dimensional gl(1|1) representations of Section 4. A check of the above is the scaling dimension:

$$\Delta^{\text{osp}}_{[b,\frac{1}{2}]} = \Delta^{\text{gl}}_{\langle 2b,1\rangle} = \Delta^{\text{gl}}_{\langle 2b+1,0\rangle}, \tag{11.6}$$

where $\Delta^{\text{gl}}_{\langle h,j\rangle}$ are the gl(1|1) scaling dimensions in equation (4.8) at $k = -2$.

The vertex operator for the 4-dimensional representation $[b, \frac{1}{2}]^{\text{osp}}$ is then

$$V^{\text{osp}}_{[b,\frac{1}{2}]} = \begin{pmatrix} V_{\langle 2b,1\rangle} \\ V_{\langle 2b+1,0\rangle} \end{pmatrix}. \tag{11.7}$$

Throughout this section, $V_{\langle h,j\rangle}$ refers to the $k = -2$ vertex operators, which are simply related to the $k = 2$ expressions in Section 6 by the automorphism (2.6).

As for gl(1|1), there are atypical, indecomposable but reducible representations at $b^2 = s^2$. The simplest is 8-dimensional and arises in the following tensor product

$$[0, \tfrac{1}{2}]^{\text{osp}} \otimes [0, \tfrac{1}{2}]^{\text{osp}} = [0, 1]^{\text{osp}} \oplus [8]^{\text{osp}}. \tag{11.8}$$

The gl(1|1) decomposition is

$$[8]^{\text{osp}} = \langle 0\rangle_{(4)} + \langle 2, 0\rangle + \langle -1, 1\rangle, \tag{11.9}$$

and the vertex operator is

$$V^{\text{osp}}_{[8]} = \begin{pmatrix} V_{\langle 2,0\rangle} \\ V_{\langle 0\rangle_{(4)}} \\ V_{\langle -1,1\rangle} \end{pmatrix}. \tag{11.10}$$

This is a $\Delta = 0$ logarithmic operator due to the presence of $V_{\langle 0\rangle_{(4)}}$, and reflects the fact that the quadratic casimir is not diagonal on the $[8]^{\text{osp}}$-representation [3]. The structure of the $[8]^{\text{osp}}$ is shown in Figure 2. The explicit form of the $V_{\langle 2,0\rangle}$, $V_{\langle -1,1\rangle}$ operators in (11.10) can found by acting on $V_{\langle 0\rangle_{(4)}}$ with the generators according to Figure 2, where $V_{\langle 0\rangle_{(4)}}$ is given in equation (6.9) with $k = -2$; the result is consistent with equation (7.11).

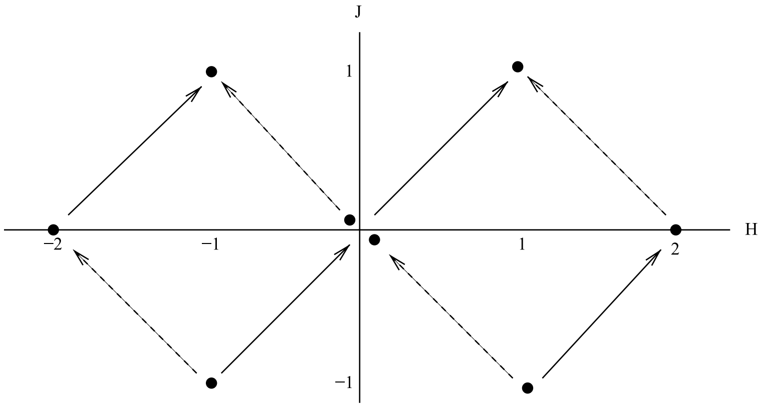


Figure 2: The structure of the osp(2|2) representation $[8]^{\text{osp}}$. The NE arrows (to the right) indicate the action of $S_+$ and the dashed NW arrows indicate the action of $\widehat{S}^+$.

In the next section, we will need the local operator $\mathbf{\Phi}^{\text{osp}}_{[8]} = V^{\text{osp}}_{[8]} \cdot \overline{V}^{\text{osp}}_{[8]}$, which can be constructed as in Section 4. Setting $k = -2$ and recalling the automorphism (2.6), one finds

$$\mathbf{\Phi}^{\text{osp}}_{[8]} = \mathbf{\Phi}_{\langle 2,0\rangle} + \mathbf{\Phi}_{\langle -1,1\rangle} + 2\mathbf{\Phi}_{\langle 0\rangle_{(4)}}. \tag{11.11}$$

The explicit form is

$$\mathbf{\Phi}^{\text{osp}}_{[8]} = 4\,\chi^1\chi^2 \left( \cosh\left(\frac{\phi^1 - \phi^2}{\sqrt{2}}\right) + \cosh\left(\frac{\phi^1 + \phi^2}{\sqrt{2}}\right)\right) + 4(\partial_\mu \chi^1 \partial_\mu \chi^2)(\chi^1\chi^2)\cosh(\sqrt{2}\phi^1). \tag{11.12}$$

(As in Section 9, above the fields $\phi, \chi$ refer to the local fields $\boldsymbol{\phi}(z, \overline{z})$ and $\boldsymbol{\chi}(z, \overline{z})$.)

## 12 Application to the spin quantum Hall transition

In this section, we apply some of the tools developed so far to the SQHT. Since this is tangential to the original scope of this article, details and additional results will be presented elsewhere [35]. Let us begin with a short summary of the relevant background. Like the usual quantum Hall transition, the SQHT has a network model description [39] and can be mapped onto a spin chain [38]. Gruzberg *et al.* mapped the spin chain onto percolation [40]. Critical percolation explains the two main exponents that were studied numerically [38], namely the correlation length $\nu_{\text{perc.}} = 4/3$ and the density of states exponent $\rho(E) \sim E^{1/7}$.

The SQHT can also be formulated in the continuum as a model of disordered Dirac fermions. It has an su(2) gauge disorder with coupling $g_s$, and two additional kinds of mass/potential disorder with couplings $g_c$ and $g_8$. (See [5] for precise definitions). Whereas $g_s$ corresponds to current–current interactions for the $\text{su}(2)_{k=0}$ currents, the coupling $g_c$ corresponds to the $\text{osp}(2|2)_{-2}$ currents. The RG flow for the couplings was studied in [5, 20]. Though a perturbative fixed point was not found in the latter work, one feature related to the super spin-charge separation described in Section 8 emerged as follows. If the $g_8$ disorder is initially set to zero, which amounts to an initial fine-tuning of the model, then the RG flow of the remaining couplings decouples due to the spin-charge separation. At 1-loop, $dg_s/d\ell = g_s^2$ and $dg_c/d\ell = -2g_c^2$, and this decoupling persists to higher orders. As described more generally in Section 8, whereas $g_s$ is marginally relevant, $g_c$ is marginally irrelevant, so that the fixed point of the model at $g_8 = 0$ was argued to be $\text{osp}(2|2)_{-2}$ [5]. For another approach based on replicas, see [41].

The fixed point $\text{osp}(2|2)_{-2}$ reproduces the main exponents of the SQHT. This is most transparent using the $\text{gl}(1|1)_2$ embedding, since, as explained in Section 7, it has precisely the percolation exponents that are relevant for the SQHT. In the $\text{osp}(2|2)_{-2}$ description, the density operator $\rho$ is identified with the representation $[0, \frac{1}{2}]^{\text{osp}}$ with $\Delta = \frac{1}{8}$, and determines the density of states exponent, $\frac{1}{7} = \frac{\Delta}{1-\Delta}$. The $\Delta = \frac{5}{8}$ field which determines $\nu_{\text{perc.}} = 4/3$ is a descendant of the $\Delta = -\frac{3}{8}$ field $[\pm 1, \frac{1}{2}]^{\text{osp}}$. Note also that 1-hull operator with $\Delta = \frac{1}{3}$ in the theory of percolation is not contained in the $\text{gl}(1|1)_2$ theory (see Section 7). This appears to be consistent with the fact this operator does not play any known rôle in the SQHT. Related comments were made in [7], where there also the $\Delta = \frac{1}{3}$ field was not in the spectrum.

The potential problem with the $\text{osp}(2|2)_{-2}$ fixed point is that the residual $g_8$ perturbations potentially modify it. This is consistent with the study of the spectrum of the spin chain in [7] which suggested that the critical point is a new kind of theory with osp(2|2) symmetry that is not simply a current algebra. The higher-order corrections to the beta functions computed in [20], which are correct up to at least 4-loops [42], do not help to resolve the problem since the flow is to a singular point.

To resolve these difficulties, we propose to carry out the RG flow in two stages. First one sets $g_8 = 0$ and flows to $\text{osp}(2|2)_{-2}$. In the second stage, we restore the $g_8$ coupling as a perturbation of the current algebra. The currents in the $g_8$ coupling transform under the $[8]^{\text{osp}}$ of osp(2|2) and the spin 1 of the su(2). In the RG flow the su(2) is gapped out which leaves a field transforming under the $[8]^{\rm osp}$. The resulting action is

$$S = S_{\rm free} + g_8 \int \frac{d^2x}{2\pi} \, \mathbf{\Phi}^{\rm osp}_{[8]}(x), \tag{12.1}$$

where $S_{\rm free}$ is just the free action for the scalars and symplectic fermion (3.1) and $\mathbf{\Phi}^{\rm osp}_{[8]}$ is the logarithmic operator in equation (11.12).

The proposal equation (12.1) overcomes previous difficulties in a number of ways. First, as argued in Section 10, the dimension zero logarithmic perturbation by $\mathbf{\Phi}^{\rm osp}_{[8]}$ does not modify the scaling dimensions but only leads to logarithmic corrections to the correlation functions. Second, the model has an osp(2|2) symmetry, as expected from the spin-chain description. However, because of the logarithmic perturbation, the critical point is not strictly speaking a conformal current algebra, even though it has the same exponents as the current algebra. This is consistent with observations made in [7].

Further checks of this proposal will be described in [35], where we explain how to obtain the multi-fractal exponents.

## 13 Conclusions

To summarize, using the detailed properties of the twist and logarithmic fields in the symplectic fermion sector, we have explicitly constructed all the primary fields of the $\mathrm{gl}(1|1)_k$ current algebra at arbitrary level $k$. We have also identified a closed operator algebra at integer level. For the indecomposable representations, the explicit construction of the logarithmic operators led to gl(1|1) invariant models as perturbations by these operators, and the simplest have local lagrangians that generalize the Liouville and sine-Gordon models. We also argued that these logarithmic perturbations have a trivial beta functions and just give logarithmic corrections to the correlation functions without changing the anomalous dimensions. We derived a new form of super spin-charge separation and gave general arguments indicating how the $\mathrm{gl}(1|1)_N$ theory can arise as a critical point of disordered Dirac fermions in $2+1$ dimensions. By studying the gl(1|1) embeddings, we also constructed explicitly the local logarithmic field corresponding to the indecomposable representation $[8]^{\rm osp}$ of $\mathrm{osp}(2|2)_{-2}$. Since other super-current algebras typically have $\mathrm{gl}(1|1)_k$ subalgebras, it should be possible to obtain other new results as well.

We initiated the application of these new tools to the investigation of critical points of disordered Dirac fermions by re-examining the SQHT. It was shown that the 1-copy theory is a perturbation of the $\mathrm{osp}(2|2)_{-2}$ current algebra by the logarithmic field corresponding to the $[8]^{\mathrm{osp}}$ indecomposable representation. In [35], we will extend this analysis to $N$-copies and thereby compute the multi-fractal exponents. We will also apply these methods to the original Chalker–Coddington network model for the ordinary quantum Hall transition, where we essentially obtain the gl(1|1) invariant sine-Gordon model (10.1). These results are not presented here since they require more specific details about the disordered Dirac fermion theories.

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## Appendix A. Complete $\mathrm{osp}(2|2)_k$ relations

Our conventions for the $\mathrm{osp}(2|2)_k$ current algebra are based on the level 1 representation in terms of $\psi_\pm, \beta_\pm$ given in Section 2:

$$\begin{aligned}
J(z)J(0) &\sim -\frac{k}{z^2}, \quad H(z)H(0) \sim \frac{k}{z^2}, \\
J(z)J_\pm(0) &\sim \pm\frac{2}{z}\, J_\pm, \quad J_+(z)J_-(0) \sim \frac{2k}{z^2} - \frac{4}{z}J, \\
J(z)S_\pm(0) &\sim \pm\frac{1}{z}S_\pm, \quad J(z)\widehat{S}_\pm(0) \sim \pm\frac{1}{z}\widehat{S}_\pm, \\
H(z)S_\pm(0) &\sim \pm\frac{1}{z}S_\pm, \quad H(z)\widehat{S}_\pm(0) \sim \mp\frac{1}{z}\widehat{S}_\pm, \\
J_\pm(z)S_\mp(0) &\sim \frac{2}{z}\widehat{S}_\pm, \quad J_\pm(z)\widehat{S}_\mp(0) \sim -\frac{2}{z}S_\pm, \\
S_\pm(z)\widehat{S}_\pm(0) &\sim \pm\frac{1}{z}J_\pm, \\
S_+(z)S_-(0) &\sim \frac{k}{z^2} + \frac{1}{z}(H-J), \\
\widehat{S}_+(z)\widehat{S}_-(0) &\sim -\frac{k}{z^2} + \frac{1}{z}(H+J).
\end{aligned} \tag{A.1}$$

## Appendix B: more on symplectic fermions

In this appendix, we provide some derivations of the results used in this paper. Most are already contained in [18, 21].

First consider the first-order system for the bosonic $\beta_\pm$ ghosts (2.1) with $c = -1$. It was shown in [21] that these can be bosonized in terms of a single scalar field $\phi$ for the $U(1)$ current and an additional auxiliary fermionic $\eta - \xi$ system with $c = -2$:

$$\beta_+ = \mathrm{e}^{i\phi}\,\eta, \quad \beta_- = \mathrm{e}^{-i\phi}\partial\xi, \tag{B.1}$$

where $\langle\phi(z)\phi(0)\rangle = \log(z)$. The $\eta - \xi$ system is also first order, with action

$$S_{\eta,\xi} = \frac{1}{4\pi}\int d^2x\,\left(\eta\partial_{\bar z}\xi + \overline{\eta}\partial_z\overline{\xi}\right) \tag{B.2}$$

but now with $\Delta(\eta, \xi) = (1, 0)$.

Before using the equations of motion, we can relate the model to symplectic fermions by identifying $\eta = i\partial_z\chi^1$, $\overline{\eta} = i\partial_{\bar z}\chi^1$, $\partial_{\bar z}\xi = i\partial_{\bar z}\chi^2, \partial_z\overline{\xi} = i\partial_z\chi^2$. In this way, one obtains the second-order action (3.1). After using the equations of motion, the chiral components are identified as follows:

$$\eta(z) = i\partial_z\chi^1, \quad \xi(z) = i\chi^2(z), \tag{B.3}$$

consistent with the conformal dimensions. It is important to note that the $\eta - \xi$ system does not contain the zero mode of $\chi^1$, and thus does not explicitly contain the logarithmic operator $\ell$ in equation (3.5).

The $\eta - \xi$ system can in turn be bosonized with a single scalar $f(z)$

$$\eta = \mathrm{e}^{-if}, \quad \xi = \mathrm{e}^{if} \tag{B.4}$$

with $\langle f(z)f(0)\rangle = -\log(z)$. However, in order to obtain $\Delta(\eta, \xi) = (1, 0)$, $f$ has a background charge:

$$T_f = -\frac{1}{2}(\partial f)^2 + \frac{i}{2}\partial^2 f. \tag{B.5}$$

With this background charge,

$$\Delta(e^{i\alpha f}) = \frac{\alpha(\alpha-1)}{2}, \tag{B.6}$$

and the correlation functions have the charge asymmetry:

$$\langle \mathrm{e}^{i\lambda f}\,\mathrm{e}^{i\lambda' f}\rangle \neq 0 \iff \lambda + \lambda' = 1. \tag{B.7}$$

All the correlation functions can be computed with Coulomb gas techniques.

Let $[\lambda]$ denote the sector $e^{i\lambda f}|0\rangle$ and its decendents. The twist field $\mu_\lambda \in [\lambda]$. Furthermore since $\chi^1 \in [-1], \chi^2 \in [1]$, then $\sigma^1_\lambda \in [\lambda - 1]$ and $\sigma^2_\lambda \in [\lambda + 1]$. In this way, one obtains the conformal dimensions equations (5.12) and (5.13).

The OPEs (5.11) are derived as follows. Define $|\mu_\lambda\rangle = \mu_\lambda(0)|0\rangle$ and $\langle\mu_{1-\lambda}| = \lim_{z\to\infty} z^{2\Delta_\lambda} \langle 0|\mu_{1-\lambda}(z)$. Using the mode expansions (5.10) and $\chi^a_{n-\lambda}|\mu_\lambda\rangle = 0$ for $n > 0$,

$$\langle\mu_{1-\lambda}|\, \partial_z \chi^1(z)\, \chi^2(w)\, |\mu_\lambda\rangle = -\left(\frac{w}{z}\right)^\lambda \frac{1}{z-w}. \tag{B.8}$$

Taking the derivative $\partial_w$ of the above equation, letting $w \to 0$ and $z \to \infty$, and using $\langle\mu_{1-\lambda}|\mu_\lambda\rangle = 1$, one obtains the $\sqrt{\lambda}$ factor in equation (5.11). The $\sqrt{1-\lambda}$ factors are obtained similarly.

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