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Triangle anomalies from Einstein manifolds

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Abstract

The triangle anomalies in conformal field theory, which can be used to determine the central charge a, correspond to the Chern–Simons (CS) couplings of gauge fields in AdS_5 under the gauge/gravity correspondence. We present a simple geometrical formula for the CS couplings

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in the case of type IIB supergravity compactified on a five-dimensional Einstein manifold $\boldsymbol{X}$. When $\boldsymbol{X}$ is a circle bundle over del Pezzo surfaces or a toric Sasaki–Einstein (SE) manifold, we show that the gravity result is in perfect agreement with the corresponding quiver gauge theory. Our analysis reveals an interesting connection with the condensation of giant gravitons or dibaryon operators which effectively induces a rolling among SE vacua.

1 Introduction

Recent years have seen a tremendous progress in developing the Anti de Sitter/conformal field theory (AdS_5/CFT_4) correspondence [1]. The correspondence arises from considering a large number N of D3-branes placed at a singularity, which is locally the tip of a real cone over a five-dimensional (5d) Einstein manifold X. It predicts the equivalence of the field theory on the stack of the D3-branes and the type IIB theory on $AdS_5 \times X$.

The very first check of the correspondence involves the symmetries on the two sides. The conformal group of CFT_4 is mapped to the isometry group of AdS_5. Other global symmetries in the CFT_4 are mapped to gauge symmetries in AdS_5. More precisely, global symmetry currents J_I on the boundary correspond to massless gauge fields A^I in the 5d bulk with the boundary coupling $\int d^4x \, A^I J_I$.

The global symmetries in the CFT side in general have triangle anomalies among them. They are mapped to the Chern–Simons (CS) couplings $(24\pi^2)^{-1} \int c_{IJK} A^I \wedge F^J \wedge F^K$ for the 5d gauge fields, and the matching between them provides a quantitative check of the AdS/CFT correspondence. It was carried out in [2] for $X = S^5$ using supergravity results of [3, 4], but it has not yet been done for other Einstein manifolds. It is well known that triangle anomalies can be extracted by a simple one-loop computation in the gauge theories and that they are topological objects. We thus expect that it should be possible to develop a generic quantitative understanding also on the gravity side of the duality because they should belong to "protected sectors" of the AdS_5/CFT_4 correspondence.

Other types of "protected sectors" of the AdS/CFT correspondence are given by the Bogomol'ny–Prasad–Sommerfield (BPS) operators, which are protected by supersymmetry. In this case, one can map the scaling dimensions of the BPS operators to the energy of the corresponding BPS states in type IIB string theory on $AdS_5 \times X$. We can expect it to be possible to understand the dual BPS objects on the gravity side in general, without the need of having the explicit metrics. This is indeed the case, for instance, for dimensions of baryonic BPS operators, corresponding to the volumes of supersymmetric (SUSY) cycles, which can be computed with the procedure uncovered in [5]. In the same way, we expect that the CS coefficients can be calculated in the gravity side without the knowledge of the explicit metrics.

The 5d CS coefficients also appear prominently in the analysis of M-theory on Calabi–Yau 3-folds. They are given in terms of the triple intersections of three four-cycles of the Calabi–Yau. Hence, we expect to find a similarly robust formula for the CS coefficients in the case of type IIB supergravity on compact, positively curved, Einstein manifolds X.

Thus, our first objective is to obtain a geometrical formula for the CS coefficients c_{IJK} for type IIB supergravity on $\mathrm{AdS}_5 \times X$. The result we will obtain is so elegant that we would like to give the formula here. It is given by

$$c_{IJK} = \frac{N^2}{2} \int_X \omega_{\{I} \wedge \iota_{k_J} \omega_{K\}}. \tag{1.1}$$

Here, N is the number of the flux of the self-dual five-form F_5 through X, and three-forms ω_I of X appears in the fluctuation of F_5 via

$$\delta F_5 = N(A^I \wedge \omega_I), \tag{1.2}$$

where A^I are gauge fields on AdS_5. Therefore, ω_I determines the distribution of F^I in the internal manifold and thus is usually called wave function.

The Killing vectors k_I measure the non-closedness of ω_I by the relation

$$d\omega_I + \iota_{k_I} \mathrm{vol}^\circ = 0, \tag{1.3}$$

where vol° is the volume form normalized to have $\int_X \mathrm{vol}^\circ = 1$. The index I runs from 1 to $d = \ell + b^3$, where ℓ is the number of isometries of X and b^3 is the third Betti number of X^5.

We will show that this formula gives robust topological quantities in a precise sense. In particular, explicit knowledge of the Einstein metric on X is not necessary to evaluate formula (1.1).

While our formula (1.1) is valid for any Einstein manifold X, the case when X is Sasaki–Einstein (SE) is especially interesting. In this case, by definition, the cone over X is Calabi–Yau. Then, minimal $\mathcal{N} = 1$ supersymmetry is preserved and we get an $\mathcal{N} = 1$ superconformal field theory (SCFT). Other than the round S^5, the only SE space with an explicitly known metric was $T^{1,1}$ for a long time; its SCFT dual was first studied by Klebanov and Witten in [6]. We now have a countably infinite number of explicit SE metrics

[7–9] and the corresponding quiver gauge theories [10–13]. There is a nice interaction between "topological" objects and objects protected by supersymmetry. Thus, we have many examples to test our formula against field theory expectations.

For generic 4d $\mathcal{N}=1$ SCFTs, triangle anomalies encode a lot of physical information and are related to various important correlators of the symmetry currents and the energy–momentum tensor [14]. In particular, the SUSY partner of the energy–momentum tensor is an Abelian global symmetry, which is called the R-symmetry. One important point in AdS/CFT correspondence is that the triangle anomaly of the R-symmetry, c_{RRR}, which is the central charge a of the SCFT [14], is inversely proportional to the volume of the SE manifold [15, 16].

Quantitative analysis on the field theory side can be done thanks to "a-maximization" [17], which determines the R-symmetry. On the gravity side, the R-symmetry is mapped to the so-called "Reeb vector" of the internal manifold X. In the case X is *toric* SE, i.e., the isometry group of X contains a $U(1)^3$, "Z-minimization" [5] determines the Reeb vector; it is thus possible to compare the volume of X and of SUSY three-cycles with gauge theory, as was done in [18]. In the case of the recently found $Y^{p,q}$ and $L^{p,q,r}$, checks of the duality have been given for c_{RRR} in [10,19], for BPS mesonic operators in [11, 20–22] and for various SUSY branes in [23]. There are also works which clarify the relation between a-maximization and Z-minimization through 5d theory in AdS_5 [24, 25]. Note that their results are valid also in the non-toric case.

We enlarge this impressive list of checks of the correspondence by providing an explicit evaluation of c_{IJK}, through (1.1), for large sets of SE manifolds, namely, circle bundles over del Pezzo surfaces and toric SE manifolds. The evaluation utilizes the flow triggered by the condensation of the giant gravitons. We analyze the field theory side using the same flow by the Higgsing using the dibaryon operators, and we find complete agreement on the gravity side and the field theory side. For toric SE, we obtain

$$c_{IJK} = \frac{N^2}{2} |\det(k_I, k_J, k_K)|, \tag{1.4}$$

where $k_I \in \mathbb{Z}^3$ is the I-th generator of the toric cone. We also call k_I the toric data, as is customary in string theory literature. In other words, c_{IJK} is simply given by the area of a triangle formed by the three toric data. We recover the formula (1.4) from field theory, thus providing a very general check of AdS/CFT.

We will also analyze the BPS operators which are related to giant gravitons, emphasizing the interplay between objects protected by SUSY and topological properties of X. Throughout the analysis, we will see that there is an intricate mixing of the angular momenta and baryonic charges, which reflects the fact that the D3-branes wrapping three-cycles in the SE manifold is partly a giant graviton. This unifies the study of two kind of SUSY states important for the AdS/CFT correspondence. One is the giant gravitons, corresponding to determinant operators in $\mathcal{N} = 4$ super Yang–Mills, and the other is the D3-branes wrapped on SUSY three-cycles, corresponding to *dibaryon* operators in the dual quiver theory.

The organization of this paper is the following: first we sketch in Section 2 the supergravity reduction which gives the formula for the CS terms and gauge coupling constants. Then, we discuss the normalization of gauge fields and the charges in Section 3, where we will see that the formula for the CS terms is topological in a precise sense. We evaluate the formulae for toric SE manifolds and for the circle bundles over del Pezzo surfaces in Section 4. In Section 5, we turn to field theory dual and show, based on explicit examples, that the results obtained in previous sections match with predictions based on AdS_5/CFT_4 correspondence. In Section 6, we explain the simplicity of our results in Section 5 using the flow triggered by the condensation of dibaryons. We conclude with some discussions in Section 7. Appendix A contains the detail of the supergravity reduction, while in Appendix B we obtain the triangle anomaly for quiver theories corresponding to generic toric SE manifolds. Finally, in Appendix C, we elaborate on the mathematics behind the charge lattice associated to the 5d Einstein manifold with isometries.

2 Perturbative supergravity reduction

Consider type IIB theory on $AdS_5 \times X$, where X is an Einstein manifold of dimension five. Let us carry out the Kaluza–Klein reduction and retain only the massless gauge fields. The corresponding 5d action has the form

$$S = \frac{1}{2}\int \tau_{IJ} F^I \wedge *F^J + \frac{1}{24\pi^2}\int c_{IJK} A^I \wedge F^J \wedge F^K + \cdots , \tag{2.1}$$

which yields the equation of motion

$$\tau_{IJ} d * F^I = \frac{1}{8\pi^2} c_{IJK} F^J \wedge F^K. \tag{2.2}$$

We would like to calculate the CS interaction c_{IJK} of the gauge fields. We will eventually choose the indices $I, J, \ldots$ to label the integral basis of the gauge fields in the next section, but in this section we take them arbitrarily. We chose the numerical coefficient $(24\pi^2)^{-1}$ so that $c_{IJK} = \mathrm{tr}\, Q_I Q_J Q_K$ under the AdS/CFT correspondence, where Q_I is the global symmetry corresponding to the gauge field A_I, and the trace is over the label of Weyl fermions.

The arguments which are to be presented in Sections 2.1 and 2.2 only uses the fact that the metric is Einstein, so it is applicable, e.g., to the manifolds $T^{a,b}$ for $(a, b) \neq (1, 1)$.

2.1 The ansatz and its reduction

Since the detail of the reduction is rather tedious, we present only a rough argument in this section. Interested readers can consult Appendix A for the details. We use three kinds of Hodge stars, namely on X, on AdS_5 and on $\mathrm{AdS}_5 \times X$. We denote the last one by $*_{10}$ and the first two by $*$. We hope the context makes clear which one we used.

The equations of motion and the Bianchi identity in type IIB supergravity are

$$R_{\mu\nu} = \frac{c}{24} F_{\mu\alpha\beta\rho\sigma} F_\nu{}^{\alpha\beta\rho\sigma}, \quad F_5 = *_{10} F_5, \quad dF_5 = 0, \tag{2.3}$$

where $R_{\mu\nu}$ is the Ricci curvature of the 10d metric and F_5 is the self-dual five-form field strength. The constant c depends on conventions. We set all other form fields and fermions to zero and the dilaton to constant throughout the analysis.

Let N units of five-form flux penetrate X, where we normalize the five-form F_5 to have $\int F_5 \in 2\pi\mathbb{Z}$. The zero-th order solution is

$$ds^2 = L^2\, ds^2_{\mathrm{AdS}} + L^2\, ds^2_X, \tag{2.4}$$

$$F_5 = \frac{2\pi N}{V}(\mathrm{vol}_X + \mathrm{vol}_{\mathrm{AdS}}), \tag{2.5}$$

where vol is the volume form of X and $V = \int_X \mathrm{vol}$. We take the convention $R_{\mu\nu} = -4g_{\mu\nu}$ for ds^2_{AdS} and $R_{\mu\nu} = 4g_{\mu\nu}$ for ds^2_X as usual. L sets the physical length scale.

Suppose X has ℓ $U(1)$ isometries k^i_a, $(a = 1, \ldots, \ell)$ so that $\exp(2\pi k^i_a \partial_i)$ is the identity. For toric SE manifolds, $\ell = 3$. Let us expand the fluctuation around the zero-th order solution in modes. One can consistently set to zero all the modes which are not invariant under the $U(1)$ isometries. We take the usual Kaluza–Klein ansatz for the metric

$$ds_X^2 = \sum_i (e^i + k_a^i A^a)^2 \tag{2.6}$$

where e^i are the fünfbein forms of the compact manifold X, and A^a are one-forms on AdS_5.

The ansatz for F_5 is rather intricate already at first order. We write F_5 as the sum of components $F_{p,q}$ which has p legs in AdS_5 and q legs in X so that

$$F_5 = F_{0,5} + F_{1,4} + F_{2,3} + F_{3,2} + F_{4,1} + F_{5,0}. \tag{2.7}$$

Then, we take the ansatz to be

$$F_{0,5} = \frac{2\pi N}{V}\mathrm{vol}_X, \quad F_{5,0} = \frac{2\pi N}{V}\mathrm{vol}_{\mathrm{AdS}}, \tag{2.8}$$

$$F_{1,4} = \frac{2\pi N}{V} A^a \wedge \iota_{k_a}\,\mathrm{vol}_X + *F_{4,1}, \tag{2.9}$$

$$F_{2,3} = NF^I \wedge \omega_I, \quad F_{3,2} = N(*F^I) \wedge *\omega_I. \tag{2.10}$$

Here, ω_I are three-forms on X to be determined later, and F^I are two-forms on AdS_5, respectively. The range in which I can take values is also determined later. The first term in (2.9) is necessary because equation (2.6) modifies the Hodge star.

The exterior derivative is decomposed to $d = d_X + d_{\mathrm{AdS}}$, where $d_{X,\mathrm{AdS}}$ is the exterior derivative on the respective spaces. Then, $dF_5 = 0$ imposes

$$d_{\mathrm{AdS}} F_{p,q+1} + d_X F_{p+1,q} = 0. \tag{2.11}$$

$F_{4,1}$ can be shown to yield massive degrees of freedom, so we set $F_{4,1} = 0$. Moreover, in order to have massless equation of motion $dF^I = 0$ and $d * F^I = 0$, there must be constants c_I^a such that

$$d * \omega_I = 0, \quad d\omega_I = \frac{2\pi}{V} c_I^a \iota_{k_a}\,\mathrm{vol}_X, \tag{2.12}$$

for ω_I and

$$dA^a = c_I^a F^I \tag{2.13}$$

for F^I. One important property is the non-closedness of ω_I, which was already pointed out in [25]. If $d\omega_I = 0$ in (2.12), the allowed number of F^I would be precisely $b^3 = \dim H^3(X)$. The presence of $\iota_{k_a}\,\mathrm{vol}_X$ enlarges the dimension of the space of wavefunctions ω_I for massless gauge fields by the number of isometries, ℓ. Thus, the index I runs from 1 to d where

$$d = \ell + b^3. \tag{2.14}$$

Let us introduce $\mathrm{vol}^\circ \equiv \mathrm{vol}/V$ and $k_I \equiv 2\pi c_I^a k_a$. Equation (2.12) becomes

$$d\omega_I + \iota_{k_I} \mathrm{vol}_X^\circ = 0. \tag{2.15}$$

We now consider the CS couplings. One contribution to the CS interaction arises as follows. The Hodge star $*$ for the metric ansatz (2.6) forces F_5 to have a second-order contribution of the form

$$\delta^{(2)} F \propto A^a \wedge F^I \wedge \iota_{k_a} \omega_I, \tag{2.16}$$

just as we had $A^a \wedge \iota_{k_a} \mathrm{vol}_X$ term in (2.9). Then, $d_{\mathrm{AdS}} F_{3,2} + d_X F_{4,1} = 0$ requires the presence of $F^a \wedge F^I$ terms in the right hand side of the equation of motion. After combining with the other contribution, the resulting equation of motion for F^I turns out to be

$$d * F^I \int_X \left(\omega_K \wedge *\omega_I + \frac{1}{16V^2} (k_K \cdot k_I) \mathrm{vol} \right) = \frac{1}{8\pi} F^I \wedge F^J \int_X \omega_{\{I} \wedge \iota_{k_J} \omega_{K\}} \tag{2.17}$$

where $(a \cdot b)$ for two one-forms $a = a_i\, dx^i$, $b = b_i\, dx^i$ is defined by $(a \cdot b) = a_i b_j g^{ij}$, and $\{IJK\} = IJK + IKJ + \cdots$ is the total symmetrization without 1/6. Again, consult Appendix A for details.

2.2 Comparison to the 5d Lagrangian

Let us write down the formula for c_{IJK} and τ_{IJ}. In order to determine the combination of τ_{IJ} and c_{IJK} entering the 5d action, we need the normalization of the kinetic term of F_5 entering the 10d action. One can resort to string worldsheet perturbation theory, but there is a quicker way out. We are normalizing F_5 to have $\int F_5 \in 2\pi\mathbb{Z}$. Then a D3-brane sources the field $F_5 = dC_4$ by the coupling $S = \int_{D3} C_4$. D3-branes are their own electromagnetic dual, thus one D3-brane should create five-form flux which satisfies the same quantization condition $\int F_5 \in 2\pi\mathbb{Z}$. Thus the supergravity action for F_5 is fixed to be

$$S_{F_5} = \frac{1}{4\pi} \int_{\mathrm{AdS} \times X} \mathcal{F}_5 \wedge *\mathcal{F}_5, \tag{2.18}$$

where $\mathcal{F}_5 = F_{0,5} + F_{1,4} + F_{2,3}$.

Plugging (2.6) and (2.10) into the 10d action, we obtain

$$\tau_{IJ} = \frac{N^2}{2\pi}\int_X \left(\omega_J \wedge *\omega_I + \frac{1}{16V^2}(k_J \cdot k_I)\mathrm{vol}\right), \tag{2.19}$$

where the first and second terms come from the kinetic terms for the five-form and the metric, respectively. This expression for τ_{IJ} agrees with the one presented in [25]. Then, from (2.17), we finally obtain

$$c_{IJK} = \frac{N^2}{2}\int_X \omega_{\{I} \wedge \iota_{k_J}\omega_{K\}}. \tag{2.20}$$

2.3 a and the volume

Before moving to the explicit evaluation of c_{IJK} for various SE manifolds, let us determine the central charge a from our formula (2.20) and check that it is inversely proportional to the volume. In this subsection, we assume X is not just an Einstein manifold but also SE.

Let J be the Kähler form of the cone $C(X)$ over X and $e_r = r\partial_r$ the dilation on the cone direction. Let e be the one-form $\iota_{e_r}J$. It endows X with the structure of a contact manifold so that $\mathrm{vol}_X = e \wedge J \wedge J/2$ and $de = 2J$. The Reeb vector is ie_r.

Since X is now SE, the corresponding CFT is $\mathcal{N} = 1$ SUSY. Let the R-symmetry in the superconformal algebra be the linear combination $R^I Q_I$. Then, the central charge a is given by

$$a = \frac{9}{32}c_{IJK}R^I R^K R^K = \frac{N^2}{2}\frac{27}{16}\int \omega_R \wedge \iota_{k_R}\omega_R, \tag{2.21}$$

where $\omega_R = R^I\omega_I$ and $k_R = R^I k_I$. It is known through the work [26] that ω_R is a multiple of $e \wedge J$. We should normalize it so that k_R is proportional to the Reeb vector, and the holomorphic three-form Ω on $C(X)$ has charge 2 under k_R. Thus, we obtain

$$k_R = 2\pi\frac{2}{3}ie_r \tag{2.22}$$

because Ω scales as r^3 and the natural holomorphic one-form is re. The extra factor of 2π comes from our convention $k_I = 2\pi c_I^a k_a$ relating k_I and the k_a in the metric ansatz.

Thus, we have

$$\omega_R = -\frac{\pi e \wedge J}{3V} \tag{2.23}$$

from (2.15). Then equation (2.21) becomes

$$a = \frac{N^2}{2}\frac{27}{16}\frac{4\pi^3}{27}\frac{\int e \wedge J \wedge J}{V^2} = \frac{N^2}{4}\frac{\pi^3}{V}, \tag{2.24}$$

which is precisely the relation established in [15, 16].

3 Properties of the supergravity formula

3.1 Giant gravitons and the normalization of ω_I

We have found so far the formula (2.20) for the CS coefficient c_{IJK} given in terms of three-forms ω_I on the Einstein manifold X. The gauge field in the AdS space has these forms as wavefunctions. In order to compare the result to the field theory in four dimensions, first we need to find the basis of the gauge fields so that charged objects have integral charges with respect to these gauge fields.

Let us recall the situation in the compactification of the M-theory on a Calabi–Yau Y. In that case, a massless gauge field arises from the M-theory three-form, with a harmonic two-form ω on Y as the wavefunction, and harmonic two-form naturally corresponds to $H^2(Y, \mathbb{R})$. M2-branes wrapped on a two-cycle C in the Calabi–Yau give rise to the charged particles in the non-compact dimensions, and the charge is given by $\int_C \omega$. Thus, $H^2(Y, \mathbb{Z}) \subset H^2(Y, \mathbb{R})$ gives the integral basis we wanted.

Similarly, in our case, D3-branes wrapped on three-cycles in the Einstein manifold X give rise to charged objects in the AdS side.[1] There are $b^3(X)$ homologically independent three-cycles. We also have ℓ Kaluza–Klein angular momenta associated to the ℓ isometries. For example, gravitons moving inside X will give charged objects from the AdS point of view. In all, there are $d = b^3(X) + \ell$ types of charged objects which match the number of the massless gauge fields.

[1] The R-charge of the wrapped D3-branes was studied in [26]. The analysis of the R-charge and the baryonic charges in the regular SE manifolds was carried out in detail in [27].

Let us give a simple argument showing that ordinary homology of three-cycles is not the correct mathematical object to classify the charges of the SUSY wrapped D3-branes. For S^5, the homology is trivial but there are giant gravitons. A less simple example comes from the $Y^{p,q}$ geometries (where the topology is simply $S^2 \times S^3$): there are various SUSY three-cycles which are homologically equivalent but have different volumes. D3-branes wrapped on different cycles correspond to different operators in the dual quiver gauge theory. These SUSY three-cycles are invariant under the $U(1)^l = U(1)^3$ isometries. The point is that we cannot deform one such SUSY three-cycle to another keeping it invariant under the isometries. It is thus clear that we need some kind of homology that keeps track also of the isometries, which show up in AdS_5 as Kaluza–Klein momenta.

Alert readers might be puzzled by now by the fact that the wavefunctions ω_I are not closed in general. Then the charge of a wrapped D3-brane depends not only on its homology class, but also on extra data, as expected also from the discussion in the previous paragraph. The Kaluza–Klein gauge fields coming from the metric also enter the expansion of F_5 because in expansion (2.10)

$$\delta F_5 = d(A^I \wedge N\omega_I), \tag{3.1}$$

A_I includes the gauge fields from the metric through (2.13). The non-closedness of ω_I allows a D3-brane wrapping a topologically trivial cycle C to have non-zero coupling to A^I given by

$$N \int_C \omega_I. \tag{3.2}$$

For instance, if we consider type IIB theory on S^5 with N units of five-form flux and we wrap a D3-brane on S^3 at the equator, it will give rise to a soliton with N unit of Kaluza–Klein momenta. This is precisely the maximal giant gravitons treated in [28, 29].

For simplicity, let us restrict our attention to branes which are not moving in the SE. In order for them to be charge eigenstates, their worldvolume should be invariant under the isometry. Let us introduce an equivalence relation such that $C \sim C'$ if $C - C' = \partial B$, where B is an invariant four-chain. Then, the coupling of the branes to the gauge fields A^I depends only on the equivalence class because

$$\int_C \omega_I - \int_{C'} \omega_I = \int_{\partial B} \omega_I = \int_B d\omega_I = \int_B \iota_{k_I} \operatorname{vol}^\circ, \tag{3.3}$$

and the integral of ι_k acting on anything vanishes if the integration region B is invariant under k. It is because the integrand is zero when k is degenerating on B and the interior product kills the legs along B when k does not degenerate on B.

Suppose X has $U(1)^\ell$ isometry and the third Betti number is b^3. In the explicit examples we will treat in the following sections, there are always $d = \ell + b^3$ of independent invariant three-cycles, although we could not find a general proof in the mathematical literature.[2] Assuming this, D3-branes wrapping on invariant three-cycles comprise a good basis of charged objects with respect to the gauge fields A^I. Let us denote the basis by C^I, $(I = 1, \ldots, d)$. Then,

$$\int_{C^I} \omega_J = \delta^I_J \tag{3.4}$$

determines the dual basis for the wavefunctions of the gauge fields A_I. Then a D3-brane wrapping the cycle C^I has charge N under A_I and charge 0 for other gauge fields.

3.2 Metric independence of c_{IJK}

First we recall the situation for the M-theory on Calabi–Yau 3-fold case. There, after the Kaluza–Klein reduction, the 5d CS interaction c_{IJK} of the massless gauge fields A^I is given by

$$c_{IJK} \propto \int \omega_I \wedge \omega_J \wedge \omega_K, \tag{3.5}$$

where ω_I is the two-form on the Calabi–Yau which appears in the Kaluza–Klein ansatz for the M-theory three-form C,

$$\delta C = A^I \wedge \omega_I. \tag{3.6}$$

The masslessness of A_I requires ω_I to be harmonic, and explicitly finding the harmonic form is quite difficult. Fortunately, the formula above (3.5) is independent of the shift of ω_I by exact forms. It implies that c_{IJK} becomes independent of the metric.

Similarly, we found in Section 2 the form ω_I is co-closed and "closed up to isometry" (2.15). We show in this section that c_{IJK} and the normalization

[2] In [30,31], one can find interesting discussions on the construction of the SUSY three-cycles using the complex algebraic geometry of the cone over the SE manifolds.

condition do not change under the shift

$$\omega_I \longrightarrow \omega_I + d\alpha_I + \iota_{k_I}\beta \tag{3.7}$$

where α_I are two-forms, β is a four-form, both of which are assumed to be invariant under $U(1)^\ell$ action.

First we discuss the shift $\omega_I \to \omega_I + d\alpha_I$. The normalization condition (3.4) is not affected. The change in c_{IJK} is zero because

$$\begin{aligned}\delta c_{IJK} \propto \int d\alpha_{\{I} \wedge \iota_{k_J}\omega_{K\}} &= -\int \alpha_{\{I} \wedge \iota_{k_J} d\omega_{K\}} \\ &= -\int \alpha_{\{I} \wedge \iota_{k_J}\iota_{k_{K\}}}\mathrm{vol}^\circ_X = 0.\end{aligned} \tag{3.8}$$

Secondly, we turn to the shift $\omega_I \to \iota_{k_I}\beta$. Here, we need to shift all of the forms ω_I simultaneously using the same β. It induces the change in c_{IJK} by

$$\delta c_{IJK} = \int \iota_{k_{\{I}}\beta \wedge \iota_{k_J}\omega_{K\}} = 0. \tag{3.9}$$

Hence, it does not change the CS coefficient. As for the normalization (3.4), the cycles C^I are assumed to be invariant under the isometry. Then we have $\int_{C^I} \iota_{k_J}\beta = 0$, using the same argument as before.

From relation (2.15), the shift $\omega_I \to \iota_{k_I}\beta$ is accompanied by the shift $\mathrm{vol}^\circ \to \mathrm{vol}^\circ - d\beta$. It means that we are free to take any five-form which integrates to one as vol° in determining ω_I through (2.15). Equation (2.15) fixes ω_I only up to the addition of exact forms, which was shown not to affect c_{IJK} above.

Let us recapitulate the method to calculate c_{IJK}.

- We first take any invariant five-form vol° which satisfies $\int \mathrm{vol}^\circ = 1$.
- Then find ω_I with the normalization $\int_{C^J} \omega_I = \delta_I^J$ (3.4).
- Next, we define k_I as the linear combination of ℓ isometries such that the condition $d\omega_I + \iota_{k_I}\mathrm{vol}^\circ = 0$, (2.15) is satisfied.
- Finally we plug these quantities to formula (2.20) and evaluate.

The procedure does not require knowledge of the Einstein metric on X. We would like to emphasize that the Sasaki structure on X is not necessary in the calculation of c_{IJK} either. The only ingredient is the action of $U(1)^\ell$ on X. In this sense, we claim that c_{IJK} is a topological invariant of the manifold with $U(1)^\ell$ action.

4 Explicit evaluation of the supergravity formula

4.1 SE manifolds with one $U(1)$ isometry

We first treat the case where there is only one isometry k on the SE manifold X. We take the period of k to be 2π. Then, the isometry determines on X an S^1 fibration

$$\begin{array}{ccc} S^1 & \longrightarrow & X \\ & & \downarrow \\ & & B \end{array} \tag{4.1}$$

over a Kähler–Einstein base B. Let the one-form e be $e = g_{ij}k^i\, dx^j$. Then, the SE condition implies that the curvature of the circle bundle de is equal to twice the Kähler class J of the base B, that is,

$$de = 2J. \tag{4.2}$$

We have $\mathrm{vol}^\circ \propto e \wedge J \wedge J$. Then, an elementary calculation shows that elements of $H^3(X)$ correspond to elements of $H^2(B)$ annihilated by $J\wedge$. Thus, $b^3(X) = b^2(X) - 1$. Since we assumed $\ell = 1$, the number of the gauge field d is

$$d = \ell + b^3(X) = b^2(B). \tag{4.3}$$

Thus, we need to find $b^2(B)$ of three-cycles C^I and three-forms ω_I in X which satisfy constraints (2.15) and (3.4). To this end, take a basis of two-cycles $D^1, \ldots, D^d$ in B and the dual basis of two-forms $\gamma_1, \ldots, \gamma_d$ on B such that $\int_{D^I} \gamma_J = \delta^I_J$. Let us take C^I to be the three-cycle above D^I in the fibration and $\omega_I = (2\pi)^{-1} e \wedge \gamma_I$. Then, the normalization (3.4) is automatic, and from $d\omega_I + \iota_{k_I} \mathrm{vol}^\circ = 0$ (2.15), we have

$$k_I = -2 \left(\int_B J \wedge \gamma_I \right) k. \tag{4.4}$$

Thus, we obtain

$$c_{IJK} = \frac{N^2}{2} \int_B \frac{J}{\pi} \wedge \gamma \left\{ I \int_B \gamma_J \wedge \gamma_K \right\}. \tag{4.5}$$

4.2 Higher del Pezzo surfaces

Circle bundles over del Pezzo surfaces are prime examples of 5d SE manifolds, where the n-th del Pezzo surface dP_n for $n < 9$ is $\mathbb{CP}^2$ blown up at generic n points. For $n = 1, 2, 3$, they are toric, which will be treated in the next subsection. In this subsection, we evaluate (4.5) for del Pezzo surfaces with $n \geq 4$, which have only one isometry which rotates the circle fiber. We compare the result with the field theory result in Section 5.2.

Let us take γ_0 as the two-form dual to the base $\mathbb{CP}^2$, and γ_i, $i = 1, \ldots, n$, be the two-forms dual to the i-th exceptional cycle. The intersection pairing is Lorentzian, i.e.,

$$\int_{dP_n} \gamma_I \wedge \gamma_J = \mathrm{diag}(+1, -1, \ldots, -1), \tag{4.6}$$

where $I, J = 0, 1, \ldots, n$. The Kähler form J is chosen to be equal to negative of the Chern class of the anti-canonical bundle,

$$J = \frac{\pi}{3}\left(3\gamma_0 - \sum_{i=1}^{n} \gamma_i\right). \tag{4.7}$$

The area of the dP_n is $\int_{dP_n} J \wedge J/2 = \pi^2(9-n)/18$. Formula (4.5) can be conveniently packed in the cubic polynomial

$$P_n(a_0, a_1, \ldots, a_n) \equiv c_{IJK} a^I a^J a^K = 3N^2 \int_{dP_n} \frac{J}{\pi} \wedge \gamma \int_{dP_n} \gamma \wedge \gamma \tag{4.8}$$

by introducing indeterminate variables a^I, $I = 0, \ldots, n$ and $\gamma \equiv \gamma_I a^I$. It can be easily evaluated to be

$$P_n(a^I) = N^2 \left(3a^0 + \sum_i a^i\right)\left((a^0)^2 - \sum_i (a^i)^2\right). \tag{4.9}$$

An obvious consequence is that we have

$$P_n(a^0, a^1, \ldots, a^n) = P_{n+1}(a^0, a^1, \ldots, a^n, a^{n+1} = 0). \tag{4.10}$$

We will clarify the physical mechanism behind this result in later sections.

4.3 Toric SE manifolds

We would like to move on to the case where there are three isometries in the SE manifold X, i.e., $\ell = 3$. In that case, the Calabi–Yau cone over X is toric, thus X is called a toric SE manifold. Let us describe X as a T^3 fibration over a two-dimensional d-gon B, where the coordinates of T^3 are $\theta_{1,2,3}$ and those of the base are $y^{1,2}$. We take the periodicity of θ_i to be 1. Denote the edges by E^I, $I = 1, \ldots, d$, the three-cycles above them by C^I. It is known that $H^3(X) = d - 3$ so that the number of the edges is precisely the number of gauge fields which we obtain by compactifying type IIB string on X. Let $k_I = k_{iI}\partial/\partial\theta_i$ be the degenerating Killing vector at C^I, see figure 1.

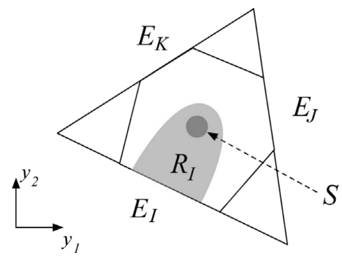

Figure 1: Construction of ω_I. The polygon designates the image of the moment map. The dark grey blob S is the support of $\mathcal{F}$ and the pale grey region R_I is the support of $\mathcal{A}_I$.

We will see shortly that the calculation of c_{IJK} only depends on $k_{I,J,K}$ and not on the other $k_{L\neq I,J,K}$ or the number of the edges. From now on, all the forms are assumed to depend only on $y^{1,2}$.

First, take a two-form $\mathcal{F}$ on the base B supported on a region S with $\int \mathcal{F} = 1$. S is marked with red in the figure 1. Choose

$$\mathrm{vol}^\circ = \mathcal{F} \wedge d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \tag{4.11}$$

as the normalized volume form.

Secondly, for each edge E_I, draw a region R_I which contains S and touches only with E_J with $J = I$ (cf. figure 1). Choose the one-form $\mathcal{A}_I$ on the base B which is non-zero only in R_I such that $d\mathcal{A}_I = \mathcal{F}$. Notice that $\int_{E^I} \mathcal{A}_J = \delta^I_J$, since $\mathcal{A}_J$ is only non-zero on R_J and $\sum_J \int_{E_J} \mathcal{A}_I = \int_B \mathcal{F} = 1$.

We need to ensure furthermore[3] that $\mathcal{A}_I$ has only components parallel to the edge E_I. Then,

$$\omega_I \equiv -\mathcal{A}_I \wedge \iota_{k_I} d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \qquad \text{(no summation on } I\text{)} \tag{4.12}$$

is a well-behaved form on X, since the existence of ι_{k_I} guarantees that ω_I is regular near E_I, and the fact A_I vanishes outside the blue region guarantees ω_I is regular near $E_{J\neq I}$. It also satisfies constraints (2.15) and (3.4) almost by construction.

Now we can clearly see that the forms $\omega_{I,J,K}$ can be taken to be the same irrespectively of, for example, whether we are calculating c_{IJK} for the hexagon inside or the triangle outside in the figure. Thus, c_{IJK} depends only

[3]The construction of the forms $\mathcal{A}_I$ can be done as follows: Let the x-axis be along the edge E_I, the y-axis be perpendicular to it, and the region R_I be given by $0 \leq y \leq a(x)$. Denote $\mathcal{F} = F(x,y)dx \wedge dy$. Then, $\mathcal{A}_I = dx \int_y^{a(x)} F(x,y)dy$ satisfies the required properties. It can be done similarly for other more complicated shape of R_I.

on $k_{I,J,K}$ and not at all on $k_{L\neq I,J,K}$. It is even independent of the number of the edges, i.e.,

$$c_{IJK} = f(k_I, k_J, k_K). \tag{4.13}$$

First of all if two of $k_{I,J,K}$ are equal, then f is obviously zero because the integrand is zero. Next, let us consider the case when they are all different. We can assume the base B is a triangle without loss of generality. We will show that X is an orbifold of S^5, which allows us to obtain c_{IJK}.

Take the universal cover U of X, that is, remove the periodicity $\theta_i \sim \theta_i + 1$. X can be obtained by dividing S^5 with the lattice N generated by $(\theta_1, \theta_2, \theta_3) = (1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Instead, consider a manifold Y by dividing U by the lattice L generated by k_I, k_J, and k_K. Along the edges of B, precisely the direction $k_{I,J,K}$ degenerates. Thus we have shown that Y is topologically an S^5 and $X = S^5/\Gamma$, where Γ is the finite group L/N. The order of Γ is

$$\#\Gamma = \left|\det(k_I, k_J, k_K)\right|. \tag{4.14}$$

Let us denote the corresponding quantities on S^5 by adding tildes and the projection map by $i : S^5 \to S^5/\Gamma = X$, we find

$$i^*\omega_I = (\#\Gamma)\tilde{\omega}_I, \quad i^* \mathrm{vol}^\circ = (\#\Gamma)\,\widetilde{\mathrm{vol}}^\circ, \quad \text{and} \quad i^* k_I = \tilde{k}_I. \tag{4.15}$$

Then

$$\int_{S^5/\Gamma} \omega_{\{I} \wedge \iota_{k_J}\omega_{K\}} = (\#\Gamma)^{-1} \int_{S^5} i^*\omega_{\{I} \wedge \iota_{k_J} i^*\omega_{K\}} = \#\Gamma \int_{S^5} \tilde{\omega}_{\{I} \wedge \iota_{\tilde{k}_J}\tilde{\omega}_{K\}}, \tag{4.16}$$

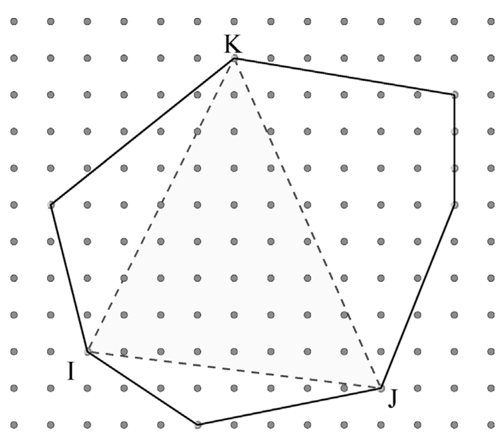

Figure 2: Pictorial representation of the toric formula $c_{IJK} = \frac{N^2}{2}\left|\det(k_I, k_J, k_K)\right|$.

that is, c_{IJK} is $\#\Gamma$ times that of S^5. Finally, for S^5, one can do the explicit calculation to find $c_{IJK} = N^2/2$. Thus, we obtain the formula

$$c_{IJK} = \frac{N^2}{2} \left|\det(k_I, k_J, k_K)\right| , \tag{4.17}$$

which is proportional to the area of the triangle inside the toric diagram, see figure 2.

5 Field theory analysis

From AdS/CFT duality, there are global symmetries Q_I and their currents J_I on the boundary corresponding to the gauge fields A^I in the bulk with the boundary coupling

$$\int d^4 x A^I J_I. \tag{5.1}$$

Thus, the CS interaction c_{IJK} in the 5d action induces the triangle anomaly on the CFT side [2]. The numerical coefficient in (2.1) is chosen such that

$$c_{IJK} = \mathrm{tr}(Q_I Q_J Q_K) \tag{5.2}$$

is satisfied. We obtained a concrete supergravity formulae for c_{IJK} in the previous sections. We also know the corresponding quiver theories which flow to CFTs in the IR through recent developments. We will see that the triangle anomaly calculated in the quiver side completely agrees with the supergravity calculation.

5.1 Cubic anomalies from field theories in the toric case

In this section, we compute all the cubic 't Hooft anomalies in the case of gauge theories dual to toric SE manifolds. In order to perform this computation, we first have to know the structure of the quiver theory. Although we will summarize below only the facts that we need later, the method of obtaining the quiver gauge theory from the toric data and vice versa is a beautiful subject in itself. It has been known for some time that it can be done in principle algorithmically, but the method was unwieldy and required extensive calculation. Now various works [11–13, 18, 32–36] give a technique to obtain the quiver theory in a much more streamlined way by the so-called dimer methods. They accomplished the most difficult parts at the same time, namely the determination of the superpotential of the quiver theory. We would like the reader to refer to the works cited above for these developments.

We will use the following properties of the quiver gauge theories dual to a toric diagram:

1) The gauge group is $SU(N)^{\mathcal{A}}$, where $\mathcal{A}$ is twice the area of the toric diagram.
2) The bifundamental chiral superfields can be grouped in $d(d-1)/2$ sets, which we can call $\mathcal{B}_{ij}$, where i and j label two external (p,q)-legs. In each set $\mathcal{B}_{ij}$ there are
$$|\mathcal{B}_{ij}| = p_i\, q_j - p_j\, q_i \tag{5.3}$$
of bifundamental fields, where (p_i, q_i) is the i-th external (p,q)-leg.
3) All the fields belonging to the same set $\mathcal{B}_{ij}$ have, under the global symmetry $U(1)^d$, the same charges Q_I^{ij}.

The full group of global symmetries, as we saw, is
$$U(1)^d = U(1)_F^3 \times U(1)_B^{d-3} \tag{5.4}$$
if the toric diagram has d points on the boundary.

Before proceeding, let us comment on what is known about the validity of the various properties. Property 1 is a well-established fact. The total number of gauge groups is equal to the total number of compact cycles (zero-, two- and four-cycles) in the completely resolved Calabi–Yau. Since there is no odd-homology, this number is the Euler number of the resolved non-compact Calabi–Yau, which is, in turn, given by twice the area of the toric diagram. Properties 2 and 3 were proposed in [11], under the name of "folded quiver." Property 2[4] was shown for toric del Pezzo surfaces in [37], and there is by now a lot of evidence for it, for instance, the exact quiver gauge theories are known for $Y^{p,q}/L^{p,q|r}$ and they satisfy property 2. We expect it to be possible to give a general proof studying intersection numbers of compact three-cycles in the mirror Calabi–Yau, as was conjectured in [11] on the base of [37]. For recent work, see [34–36]. In particular, using the procedure devised in [34], it is possible to derive formula (5.3) from the counting of the intersection of (p,q)-legs when drawn in the planar torus (again, consult [34] for details). Let us stress that the properties 1 and 2 are inherently topological in the sense that the former depends only on the topology of the Calabi–Yau and the latter that of its mirror. Property 3 instead goes slightly beyond purely topological properties, for instance, the existence of three $U(1)$ flavor symmetries is related to isometries of the

[4]We expect there is always at least one toric phase where the number of the fields is precisely given by the determinant (5.3). This is known to be the case for the set of theories $Y^{p,q}$ and $L^{p,q|r}$. For the $Y^{p,q}$'s, all toric phases have been classified [26], and in some phases, with so-called double impurities, property 2 does not hold as stated. In these cases, there are additional pairs of fields with opposite charges.

Calabi–Yau metric. Let us notice also that in [37] a different interpretation of (5.3) was given, and we now know that the correct interpretation is in terms of property 3.

Very strong evidence for the validity of all the three properties listed above was given in the work of Butti and Zaffaroni [18, 36], where it was shown that the field theory computation of the cubic 't Hooft anomaly c_{RRR} matches precisely the geometric results for the volumes of the SE, as expected from AdS/CFT correspondence. The volumes on the gravity side can be computed using the results of Martelli *et al.* [5], which enables us to compute the volumes just in terms of toric data. We will show that *all cubic 't Hooft anomalies* c_{IJK} *match with the CS coefficients as computed from gravity.*

As an aside, let us note that, beyond 't Hooft anomalies, using the "folded quiver" picture, one can readily compute the scaling dimension of dibaryon operators and succesfully match with string theory. This gives additional evidence for the validity of properties 1, 2, and 3. Also, the topology of some SUSY three-cycle can be matched with this picture [12].

In order to compute the full set of cubic 't Hooft anomalies, we need to identify the d $U(1)$ global symmetries. We will take all the d symmetries to be R-symmetries (taking linear combinations, it is obvious how to obtain $d-1$ ordinary $U(1)$ symmetries). There is a natural way to associate a $U(1)$ symmetry to every external node in the toric diagram: the charge of a field under the i-th symmetry is one if the i-th node on the right of the arrow corresponding to the field in the folded quiver diagram, zero otherwise. For instance external fields in the folded quiver diagram are charged only under one $U(1)$ symmetry. In this way, all chiral superfields have charges 0 or 1 under the $U(1)^d$ global symmetry. The superpotential corresponds to closed loops of the folded quiver. Thus, its charge under the i-th $U(1)$ symmetry is 1. It implies that the commutation relation between I-th $U(1)$ charge Q_I and the supercharge Q_α is $[Q_I, Q_\alpha] = -Q_\alpha/2$. This in turn means that the gauginos have thus charge $1/2$ and their contribution to cubic anomalies is always $\mathcal{A}N^2/8$. Then, the fermionic component of the bifundamental superfields has thus charge $-1/2$ or $1/2$. We thus see that in this way all the charges are half integral, and every bifundamental field contributes $\pm N^2/8$ to the cubic anomalies. The point is that this basis is precisely the field theory dual of the basis considered in the previous subsection. Indeed, the dibaryon constructed from the field in $\mathcal{B}_{I,I+1}$ has the charge $\delta_{IJ}N$ under the symmetry Q_J, which precisely matches the charge of the D3-brane which wraps the cycle C_I, see (3.4).

Let us report in detail the results for the case of toric diagram with four corners. The charges are given in table 1.

Table 1: Charge assignments for the basic superfields in the case of toric diagrams with four corners.

Field	Number	Q_1	Q_2	Q_3	Q_4	Q_1^F	Q_2^F	Q_3^F	Q_4^F
$\mathcal{B}_{12}$	p	1	0	0	0	1/2	−1/2	−1/2	−1/2
$\mathcal{B}_{23}$	r	0	1	0	0	−1/2	1/2	−1/2	−1/2
$\mathcal{B}_{34}$	q	0	0	1	0	−1/2	−1/2	1/2	−1/2
$\mathcal{B}_{41}$	$p+q-r$	0	0	0	1	−1/2	−1/2	−1/2	1/2
$\mathcal{B}_{13}$	$q-r$	1	1	0	0	1/2	1/2	−1/2	−1/2
$\mathcal{B}_{42}$	$r-p$	1	0	0	1	1/2	−1/2	−1/2	1/2
Gauge	$p+q$	0	0	0	0	1/2	1/2	1/2	1/2

It is straightforward to check that the linear 't Hooft anomalies vanish, i.e., $\mathrm{tr}(Q_j) = 0$. This has to be the case for any superconformal quiver [38, 39]. A general proof of the vanishing of linear anomalies using the folded quiver picture was given in [18]. Since $(Q_i^F)^2 = 1$, $\mathrm{tr}(Q_j) = 0$ also implies that

$$\mathrm{tr}(Q_i^2 Q_j) = \mathrm{tr}(Q_j) = 0. \tag{5.5}$$

The remaining cubic 't Hooft anomalies (recall they are completely symmetric) are easily computed to be

$$\mathrm{tr}(Q_1Q_2Q_3) = \frac{N^2\, r}{2} \tag{5.6}$$

$$\mathrm{tr}(Q_2Q_3Q_4) = \frac{N^2 q}{2} \tag{5.7}$$

$$\mathrm{tr}(Q_3Q_4Q_1) = \frac{N^2(p+q-r)}{2} \tag{5.8}$$

$$\mathrm{tr}(Q_4Q_1Q_2) = \frac{N^2 p}{2}. \tag{5.9}$$

It is now straightforward to check that these are proportional to the area of the triangles

$$|\det(k_I, k_J, k_K)| \tag{5.10}$$

spanned by the corners of the toric diagram of figure 3 or 4. Thus we have shown that, for a toric diagram with four edges, the cubic anomaly c_{IJK} is given by

$$c_{IJK} = \frac{N^2}{2} |\det(k_I, k_J, k_K)|, \tag{5.11}$$

which agrees with the supergravity result (4.17).

This nice result can be proven for a generic toric diagram with arbitrary number of edges, by an easy mathematical induction. We leave the details in the Appendix B.

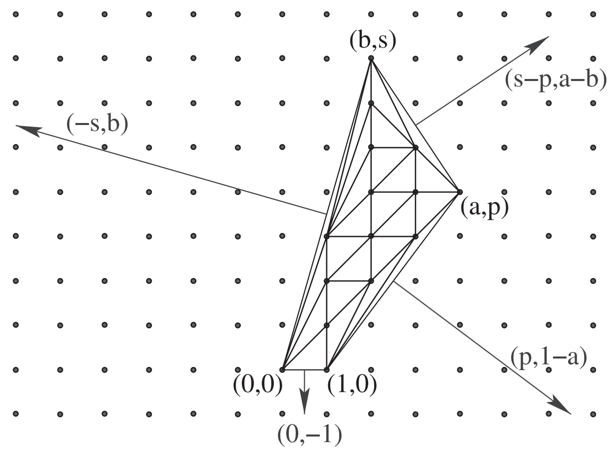

Figure 3: A generic toric diagram with four corners, i.e., a generic $L^{p,q|r}$ and the associated (p,q)-web. We have $s = p + q - r$. The integers a and b are such that $as - bp = q$.

5.2 del Pezzo surfaces

Now we want to discuss the gauge theories corresponding to the complex cones over smooth Kähler-Einstein surfaces, i.e., del Pezzo surfaces dP_n for $3 \leq n \leq 8$. The quivers were constructed in [40] for toric del Pezzo surfaces (dP_1, dP_2, and dP_3) and in [37, 41] for the non-toric ones, i.e., dP_n with

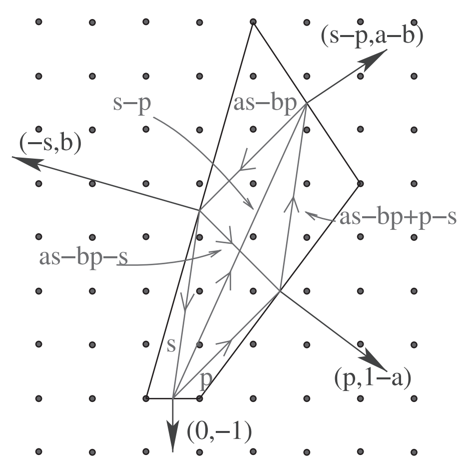

Figure 4: An example of "folded quiver." From a generic toric diagram with four nodes, we can immediately compute the multiplicities of six sets of bifundamental fields.

$4 \le n \le 8$. The generic superpotential for dP_5 and dP_6 was derived in [42], and for dP_7 and dP_8, the explicit, generic, superpotential is still not known. In [38, 43], all the baryonic and R-charges are explicitly listed for dP_n up to $n = 6$. It is simple to compute, using these data, the cubic 't Hooft anomalies and to match with our geometrical findings in Section 4.2.

In [27], the R- and baryonic charges of the dibaryons were analyzed through the framework of the exceptional collections on the del Pezzo surfaces. In particular, it was shown that the triangle anomalies among the R-symmetry and two baryonic symmetries, $\mathrm{tr}(RB_iB_j)$ are proportional to the intersection form of the two-cycles which are perpendicular to the Kähler class of the surface. It is easy to check that our formula in Section 4.2 naturally reproduces the result of [27].

6 Rolling down among SE vacua

The triangle anomalies in the CFT side and the CS coefficients of the gravity side showed a remarkable behavior. Namely, for quiver theories for toric SE manifolds, the coefficient c_{IJK} is determined solely by the toric data $k_{I,J,K}$ and is independent of other k_L for $L \neq I, J, K$ (4.13). We would like to give a heuristic physical interpretation of this fact. The same consideration can be applied to the del Pezzo cases, and its manifestation is (4.10). We concentrate on the toric cases below.

Consider a toric SE X whose dual toric diagram has d edges. Each edge E_I naturally corresponds to a global symmetry Q_I in the quiver theory. There are bifundamental fields Φ^I with charge δ^I_J under Q_J. Then, we can form a dibaryon operator

$$B^I = \epsilon_{i_1 i_2 \ldots i_N} \epsilon^{j_1 j_2 \ldots j_N} \Phi^{I i_1}{}_{j_1} \Phi^{I i_2}{}_{j_2} \cdots \Phi^{I i_N}{}_{j_N}. \tag{6.1}$$

It has the charge $N\delta^I_J$ under Q_J, which is precisely the charge (3.2) of a D3-brane wrapping the three-cycle determined by E_I.

Now, let us give a vacuum expectation value (vev) to B_I. Since B_I is charged only with respect to Q_I and not to $Q_{J\neq I}$, the theory flows to a theory with $d - 1$ global symmetries. On the gravity side, the Higgsing means that there is an infinite number of D3-branes wrapping around C_I, which presumably shrinks it just as in the blackhole condensation [44], see figure 5. It is the blowdown of the toric divisor corresponding to E_I on the Calabi–Yau cone over X. This procedure was used in the determination of the del Pezzo quiver in [41].

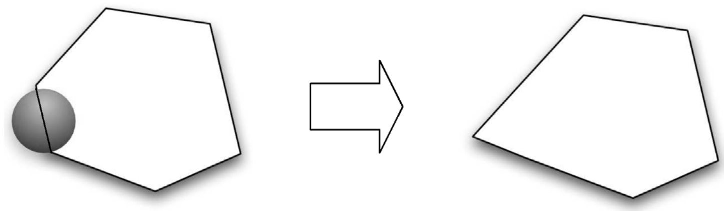

Figure 5: Schematic depiction of the dibaryon condensation. Each edge corresponds to a three-cycle in the toric SE around which D3-branes can be wrapped. Higgsing with the corresponding dibaryon operator in the quiver CFT eliminates that edge.

Recall that the same triangle anomaly can be calculated either in the ultraviolet or in the infrared. Thus, the triangle anomaly c_{JKL} among the global symmetries other than Q_I is the same before and after the Higgsing. Since the Higgsing eliminates the edge E_I, this means that c_{JKL} is independent of k_I. One can repeat the flow many times and we can reduce the toric diagram to a triangle, which is an orbifold of $\mathcal{N} = 4$ $SU(N)$ super Yang–Mills theory.

Let us consider the behavior of the central charge a along the flow. Consider a flow from the UV quiver theory to the IR quiver theory triggered by giving a vev to B_I. The IR theory also contains a free chiral scalar field which represents the fluctuation of the vev of B_I. Its contribution to a is of order $1/N^2$ compared to the contribution from the interacting part, so we can neglect them henceforth. Then, from the invariance of c_{IJK} along the flow (4.13), the central charge a in the IR theory can be obtained by maximizing the same function as that for the UV theory in a smaller region. Thus, a will presumably decrease, with the usual caveat on the fact that the trial function attains the maximum only locally.

Let us compare the process we saw in this section with the rolling among Calabi–Yau vacua [45]. There, theories on various topologically distinct Calabi–Yau manifolds are connected by adiabatically changing the moduli. Here, theories on various topologically distinct SE manifolds are connected by the renormalization-group flow induced by the Higgsing of the dibaryons. Both have the same number of supercharges, and both can be understood as the Higgsing. Thus, we suggest to dub the phenomenon we found as the "rolling among SE vacua," although the rolling is unidirectional.

More detailed analysis of the rolling is clearly necessary and will be interesting. We hope to revisit this problem in the future.

7 Conclusion

In this paper, we explored a particular aspect of the AdS_5/CFT_4 correspondence. Namely, the matching between the CS interaction in the 5d bulk and the triangle anomaly in the 4d boundary. More precisely, we derived a formula for the CS interactions in terms of three-forms in the Einstein manifold used in the compactification, and we also evaluated the formula for the circle bundles over the del Pezzo surfaces and for the toric SE manifolds. Furthermore, we successfully matched the resulting expression to the triangle anomaly from the dual field theory. Condensation of dibaryons was crucial in the physical understanding of the calculation of the triangle anomaly in both sides of the duality.

We also found that the charges of the D-branes wrapping various three-cycles in the SE naturally and non-trivially combine the angular momenta along the isometry directions and the baryonic charges.

There are several open problems that we would like to point out. One possible direction of further research is to extend the determination of the lowest derivatives terms in the AdS theory and to check the very special geometry of the vector multiplet scalars. Another direction will be the study of a more thorough understanding of the charges of D-branes wrapping inside the SE manifolds. The new ingredients came in mostly from the fact that the manifold comes with a group action. We made some comments in Appendix C. Finally, the physics of the rolling among the SE vacua should be studied more thoroughly. We hope to revisit these problems in the future.

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A Details of supergravity reduction

Our goal in this section is to perform a compactification of a 10d solution of IIB supergravity to five dimensions. It is worth stressing that we are not attempting a reduction to a 5d theory. In fact, there is an extensive literature on supergravity reduction on positively curved symmetric manifolds. For example, there are some constructions of full consistent non-linear ansatz for the reduction on the spheres [46]. Other interesting truncations are presented in [47] and references therein. In this subsection, we carry out the compactification of type IIB theory on generic 5d Einstein manifolds. As such, we are forced to the perturbative analysis and will not pursue full non-linear reduction in this paper. Indeed, it is known that consistent reductions are possible only for a restricted set of manifolds [48].

Consider type IIB theory compactified on an Einstein 5-manifold X to have a 5d theory on AdS_5. Let the coordinates of X and AdS be y^i and x^μ and their fünfbeine be e^i and f^μ, respectively.

Since the action of the self-dual five-form in ten dimensions is rather subtle, we carry out the Kaluza–Klein analysis at the level of equation of motion. Let us explain the main technical point before going into the details. Schematically, one first expands the fluctuation using the harmonics of the internal manifold X,

$$\phi(x,y) = \phi_0(x,y) + \delta\phi^{(i)}(x)\psi_{(i)}(y) + \cdots , \tag{A.1}$$

so that $\delta\phi^{(i)}$ are the mass eigenstates. Then, one can identify the cubic couplings, such as the CS coefficient, by finding the equation of motion of $\delta\phi^{(i)}$ in the form

$$(D - m^2)\delta\phi^{(i)} = C^{(i)}_{(j)(k)}\delta\phi^{(i)}\delta\phi^{(j)} + \cdots . \tag{A.2}$$

If one is only interested in obtaining certain parts of the cubic coupling, one can set to zero any fluctuation which does not multiply the couplings. It does not change the results, and at the same time it greatly reduces the calculational burden.

Another technical difficulty lies in maintaining the self-duality of the ansatz for the five-form. Suppose X has ℓ $U(1)$ isometries k^i_a, $a = 1, \ldots, \ell$, with period 2π. The ansatz for the metric is the usual one,

$$ds^2_X = \sum_i (e^i + k^i_a A^a)^2, \tag{A.3}$$

where e^i are the fünfbein forms of the Einstein manifold and $A^a = A^a_\mu \, dx^\mu$ are one-forms on AdS_5.

Let us abbreviate $\hat{e}^i = e^i + k^i_a A^a$. Then, the Hodge star exchanges

$$f^1, \ldots, f^5 \quad \longleftrightarrow \quad \hat{e}^1, \ldots, \hat{e}^5. \tag{A.4}$$

Thus, one can anticipate that the introduction of the following $\hat{}$ operation on differential forms of X defined by replacing e by $\hat{e}$,

$$\alpha^{(p)} = \alpha_{i_1 \cdots i_p} e^{i_1} \cdots e^{i_p} \longmapsto \hat{\alpha}^{(p)} \equiv \alpha_{i_1 \cdots i_p} \hat{e}^{i_1} \cdots \hat{e}^{i_p}, \tag{A.5}$$

greatly helps in maintaining the self-duality of the ansatz for F_5.

The following two formulae are useful in calculation. First is a formula for the $\hat{}$ operation using interior products:

$$\hat{\alpha} = \alpha + A^a \wedge \iota_{k_a} \alpha + \frac{1}{2} A^a \wedge A^b \wedge \iota_{k_b} \iota_{k_a} \alpha + \cdots . \tag{A.6}$$

Another is $*(\alpha^{(5-p)} \wedge \beta^{(p)}) = (-)^p (*\alpha) \wedge *\beta$, where the number in the parentheses in the superscript denotes the degree of the forms.

Let us carry out what we have just outlined. The equations of motion and the Bianchi identity in type IIB supergravity are:

$$R_{\mu\nu} = \frac{c}{24} F_{\mu \cdots} F_{\nu}{}^{\cdots}, \tag{A.7}$$

$$F = *F, \tag{A.8}$$

$$dF = 0, \tag{A.9}$$

where $R_{\mu\nu}$ is the Ricci curvature of the 10d metric and F is the self-dual five-form field strength. The right hand side of (A.7) should be contracted in a suitable way. c is a convention-dependent constant. We set other form fields and fermions to zero and dilaton to constant. In the following, we use the following convention when converting a p-form ω into its components $\omega_{\mu_1 \cdots \mu_p}$ by defining

$$\omega = \frac{1}{p!} \omega_{\mu_1 \mu_2 \cdots \mu_p} \, dx^{\mu_1} dx^{\mu_2} \cdots dx^{\mu_p}. \tag{A.10}$$

An example is

$$F = \frac{1}{120} F_{\mu\nu\rho\sigma\tau} \, dx^\mu \, dx^\nu \, dx^\rho \, dx^\sigma \, dx^\tau \tag{A.11}$$

for the self-dual five-form F.

The zero-th order solution is

$$ds^2 = L^2 \, ds^2_{\mathrm{AdS}} + L^2 \, ds^2_X, \quad F = \frac{2\pi N}{V} (\mathrm{vol}_X + \mathrm{vol}_{\mathrm{AdS}}). \tag{A.12}$$

We take the convention $R_{\mu\nu} = -4 g_{\mu\nu}$ for the AdS part and $R_{mn} = 4 g_{mn}$ for the SE part. Plugging (A.12) into the equation of motion of the metric, we get

$$4 = c\left(\frac{2\pi N}{V}\right)^2 L^{-8}. \tag{A.13}$$

Let us expand the fluctuation around the zero-th order solution in modes. One can consistently set to zero all the modes which are not invariant under the $U(1)$ isometries. We then take the ansatz for F_5 as

$$\frac{V}{2\pi N}F_5 = \hat{e}^1 \cdots \hat{e}^5 + B^a \wedge *k_a - F^I \wedge \hat{\omega}_I + *F^I \wedge \widehat{*\omega_I} + (*B^a) \wedge k_a + f^1 \cdots f^5, \tag{A.14}$$

where $k_a = g_{ij}k_a^i dy^j$, ω_I are three-forms to be identified shortly, $B^a = B^a_\mu dx^\mu$ and $F^I = F^I_{\mu\nu}dx^\mu dx^\nu/2$. We will see that this gives consistent equation of motion in five dimensions. Note that

$$(\omega_I \text{ elsewhere in the article}) = -\frac{2\pi}{V}(\omega_I \text{ here}). \tag{A.15}$$

This definition saves messy factors of powers of 2π.

F_5 above satisfies $F_5 = *F_5$ by construction because the one-forms f^μ and $\hat{e}^i$ constitute the zehnbein of the metric. $dF_5 = 0$ requires

$$d\omega_I = c_I^a \iota_{k_a} \operatorname{vol}_X \tag{A.16}$$

for some constants c_I^a. We define $k_I \equiv c_I^a k_a$ for brevity. Note also that

$$(k_I \text{ elsewhere in the article}) = 2\pi(k_I \text{ here}). \tag{A.17}$$

Furthermore, we assume ω_I to be co-closed. Then, $dF_5 = 0$ imposes on B^a, F^I the equations

$$d(A^a + B^a) = c_I^a F^I, \tag{A.18}$$

$$dF^I = 0, \tag{A.19}$$

$$d(*F^I) \wedge *\omega_I = -(*B^a) \wedge dk_a + F^I \wedge F^J \wedge \iota_{k_J}\omega_I \tag{A.20}$$

where we kept the fluctuations up to the second order. Let us define ω_a by $*dk_a/8$. One has $d\omega_a = \iota_{k^a}$ vol by using the fact[5] that we have $*d * dk = 2tk$

[5]One can replace ∂_i by ∇_i in the definition of Lie derivative. Thus, $\nabla_i k_j + \nabla_j k_i = 0$. Then,

$$R_{lj}k^l = R^k_{lkj}k^l = [\nabla_k, \nabla_j]k^k = g^{kl}[\nabla_k, \nabla_j]k_l = -g^{kl}\nabla_k\nabla_l k_j - g^{kl}\nabla_j\nabla_k k_l = -\nabla^2 k_j.$$

Hence, for Einstein manifold with $R_{ij} = tg_{ij}$, we have

$$(*d * dk)_i = \nabla^j(\nabla_i k_j - \nabla_j k_i) = 2tk_i.$$

for any Killing vector k in an Einstein spaces with $R_{ij} = tg_{ij}$. Then we see, from (A.20),

$$d * F^I \wedge \omega_K \wedge *\omega_I = -8 * B^a \wedge \omega_K \wedge *\omega_a + F^I \wedge F^J \wedge \omega_K \wedge \iota_{k_J}\omega_I. \quad \text{(A.21)}$$

Another important EOM comes from the Ricci curvature $R_{\mu\hat{i}} f^\mu \hat{e}^i$ with one leg in the AdS and one leg in the SE. While

$$R_{\mu\hat{i}} = \frac{1}{2} k_{ia} \nabla_\nu (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \quad \text{(A.22)}$$

from (A.3), the right hand side of (A.7) is given by

$$\frac{c}{24} F_{\mu\cdots} F_{\hat{i}}^{\cdots} = \frac{c}{24} \left(\frac{2\pi N}{V}\right)^2 L^{-8} \left(48 B^a_\mu k_{ai} - 6(*F^I)_{\mu\nu\rho} (*\omega_I)_{..} F^{J\nu\rho} (\omega_J)_{\hat{i}}^{\cdot\cdot}\right) \quad \text{(A.23)}$$

$$= 8 B^a_\mu k_{ai} - 4(*F^I \wedge F^J)_\mu \frac{(\omega_I \iota_{e_i} \omega_J)}{\mathrm{vol}_X}. \quad \text{(A.24)}$$

Thus, we get

$$\frac{1}{16}(d * dA^a) \wedge (k_a \cdot k_K)\mathrm{vol}_X = *B^a \wedge (k_a \cdot k_K)\mathrm{vol}_X + \frac{1}{2} F^I \wedge F^J \wedge \omega_I \iota_{k_K} \omega_J, \quad \text{(A.25)}$$

where we define $(a \cdot b)$ for two one-forms $a = a_i\, dx^i$ and $b = b_i\, dx^i$ by $(a \cdot b) = a_i b_j g^{ij}$.

From (A.21), and (A.25), we see B^a are the massive eigenmodes under Kaluza–Klein expansion, hence, we need to set $B^a = 0$ to get the ansatz for the massless fluctuation. Let us add the both sides of equations (A.21) and (A.25) and integrate over the internal manifold X. Using $\int_X \omega_K \wedge *\omega_a = \int_X k_K k_a \,\mathrm{vol}_X/8$, the term including the massive mode B^a vanishes, and we finally obtain the EOM for massless fields :

$$d * F^I \int_X \left(\omega_K \wedge *\omega_I + \frac{1}{16}(k_K \cdot k_I)\mathrm{vol}_X\right) = \frac{1}{4} F^I \wedge F^J \int_X \omega_{\{I} \wedge \iota_{k_J} \omega_{K\}}, \quad \text{(A.26)}$$

where $\{IJK\} = IJK + IKJ + \cdots$ without 1/6. The factor which multiplies $d * dF_I$ exactly reproduces the combination $g^{-2KK}_{IJ} + g^{-2CC}_{IJ}$ which appeared in ref. [25], where it was derived in a slightly different way.

Let us recapitulate what happens during the detailed calculation. If we reduce some higher dimensional form-field theory on an internal manifold without isometries, we need to have simultaneously closed and co-closed wavefunctions in the internal manifold to have a massless field in the non-compact dimensions. If the metric is the sole dynamical field, then upon reduction an isometry produces a gauge field through the ansatz (2.6). Through the coupling between the metric and five-form field, the gauge field from $g_{\mu\nu}$ and the gauge field from F_5 with co-closed but non-closed wavefunctions get off-diagonal components in the mass matrix, and precisely one linear combination remains massless for one Killing vector field. Thus, the number of massless gauge fields in AdS is

$$d = \ell + b^3, \tag{A.27}$$

where ℓ is the number of independent Killing vectors and b^3 is the dimension[6] of $H^3(X)$.

B Triangle anomaly for general toric quivers

In this appendix, we prove the formula

$$c_{IJK} = \frac{N^2}{2} |\det(k_I, k_J, k_L)| \tag{B.1}$$

for quiver gauge theories on the D3-branes probing the tip of a toric Calabi–Yau cone.

Let us denote by $k_I = (1, \vec{k}_I)$ $(I = 1, \ldots, d)$ the toric data of the toric Calabi–Yau manifold. We set $k_0 \equiv k_d$. One can express the same data using the language of the (p, q)-web, in which the direction of the i-th external leg is given by $(p_i, q_i) = \vec{k}_i - \vec{k}_{i-1}$. The field content of the corresponding quiver theory is summarized in Section 5.1, properties 1, 2, and 3. Let us consider a linear combination $Q = a^I Q_I$ of the $U(1)$ charges Q_I. Then, the charge of the superpotential under Q is $\sum a_I$ and the charge of the chiral superfields in $\mathcal{B}_{ij}$ is

$$\sum_{K=i}^{j-1} a^K = a_i + a_{i+1} + \cdots + a_{j-1}. \tag{B.2}$$

The number n_{ij} of chiral superfields in $\mathcal{B}_{ij}$ is given by the intersection number of the two (p, q)-legs, that is,

$$n_{ij} = \det(\vec{k}_j - \vec{k}_{j-1}, \vec{k}_i - \vec{k}_{i-1}), \tag{B.3}$$

[6] Forms which are closed and co-closed are automatically invariant under the isometry, and hence the number of harmonic three-forms is the same as the number of invariant harmonic three-forms.

while the number n_V of gauge groups is given by the area of the toric diagram

$$n_V = \sum \det(\vec{k}_I - \vec{k}_1, \vec{k}_{I+1} - \vec{k}_1). \quad \text{(B.4)}$$

Then the triangle anomaly among three Q's is given by

$$\frac{1}{N^2} c^{\text{CFT}}_{IJK} a^I a^J a^K = n_V \left(\frac{1}{2} \sum a^I \right)^3 + \sum_{I<J} n_{IJ} \left(\sum_{K=I}^{J-1} a^K - \frac{1}{2} \sum a^I \right)^3 . \quad \text{(B.5)}$$

This expression follows from the folded quiver picture of [11] and appeared explicitly in the work of Butti and Zaffaroni [18]. In the usual formula, we have 1 instead of $\sum a^I/2$; we would like to have the triangle anomaly including the global symmetry usually fixed by $\sum a^I = 2$, and so we resurrected that combination.

One can show, by mathematical induction, c^{CFT}_{IJK} only depends on $k_{I,J,K}$ and not on other k_L for $L \neq I, J, K$ nor on the number of edges. The proof goes as follows.

Suppose $I, J, K \neq d$ and let us show c_{IJK} is independent of k_d. Consider two toric data, one is the original set $\{k_1, k_2, \ldots, k_d\}$ and the other is $\{k_1, \ldots, k_{d-1}\}$ without k_d. Let us distinguish various quantities for the latter by adding tilde above, e.g., $\tilde{n}_V$ and so on. Then, we have two relations

$$n_{I,d-1} + n_{I,d} = \tilde{n}_{I,d-1} \quad \text{(B.6)}$$

and

$$n_V - n_{d-1,d} = \tilde{n}_V. \quad \text{(B.7)}$$

Applying them to formula (B.5), we obtain

$$c^{\text{CFT}}_{IJK} a^I a^J a^K \big|_{a^N=0} = \tilde{c}^{\text{CFT}}_{IJK} a^I a^J a^K. \quad \text{(B.8)}$$

Thus, c_{IJK} for $I, J, K \neq d$ is independent of k_d. Inductively, we can show that c_{IJK} depends only on k_I, k_J, and k_K.

Hence, we can obtain c^{CFT}_{IJK} by considering the case of a triangle. One can easily show that, in this case,

$$n_V = n_{IJ} = n_{JK} = n_{KI} = |\det(k_I, k_J, k_K)|. \quad \text{(B.9)}$$

Plugging into formula (B.5), we finally obtain

$$c^{\text{CFT}}_{IJK} = \frac{N^2}{2} |\det(k_I, k_J, k_K)|. \quad \text{(B.10)}$$

It precisely agrees with the result from the supergravity analysis (4.17).

C More on the charge lattice

We would like to elaborate on the mathematics of the structure of the charges of the D3-branes.[7] The case for the toric SE manifolds was analyzed in ref. [12] mainly from the point of view of the toric geometry of the cone. We discuss the problem for arbitrary Einstein manifolds.

Let us denote the space of Killing vectors by N, which can be identified with the Lie algebra of $U(1)^\ell$. It comes with a natural integral structure by stating that $k \in N$ is one of the lattice points if and only if $e^{2\pi k} = id$. Denote the dual space of N by M. Integral points of M correspond to representations of $U(1)^\ell$. The Reeb vector $R \in N$ is given when we endow X with the Sasaki structure. If X is SE, all the toric data $k \in N$ should be on a plane. The plane is given by a distinguished element $P \in M$ as $\langle P, k\rangle = 1$, where P is the image of R under the identification $M \simeq N$ induced by the metric.

We deliberately used the letters M and N to evoke the connection with the toric geometry. Indeed, they are precisely M and N lattices of the cone over X, if $\ell = 3$.

We only consider the branes which wrap three-cycles invariant under the action of $U(1)^\ell$. As discussed in Section 3.1, two cycles are taken to be equivalent if they form the boundaries of an invariant four-chain. Let us call the group of the equivalence classes of such three-cycles as $HG_3(X)$, where G stands for giant gravitons. We also denote the space of linear combinations of ω_I by $HG^3(X)$, where ω_I are closed up to isometry (2.15).

We have an exact sequence

$$0 \longrightarrow H^3(X) \longrightarrow HG^3(X) \longrightarrow N \longrightarrow 0, \tag{C.1}$$

where the second arrow is just the inclusion, and the third arrow is given by (2.15). The exactness of the sequence is also obvious.

Correspondingly, we also have another exact sequence

$$0 \longrightarrow M \xrightarrow{\iota} HG_3(X) \xrightarrow{\pi} H_3(X) \longrightarrow 0, \tag{C.2}$$

where we assumed, as before, that we can take an invariant representative for all $H_3(X)$. Then, the third arrow π is just loosening of the equivalence relation. The second arrow ι is a bit tricky to define, so we postpone the

[7]The same analysis can be done for $(d-2)$-branes wrapping $(d-2)$-cycles in a d-dimensional manifold with isometry, since the mixing of the gauge fields coming from the metric and form-fields is a generic feature independent of the self-duality of the form-field, see [25]. We would like to thank A. Neitzke for raising this question.

discussion to the end of this section. In the toric case, the above sequence can be obtained from the usual sequence [49]

$$0 \longrightarrow M \longrightarrow \mathrm{Div}_T(C(X)) \longrightarrow \mathrm{Pic}(C(X)) \longrightarrow 0 \tag{C.3}$$

for the cone $C(X)$ over X, where Div_T denotes the group of toric divisors and Pic is the Picard group.

Two exact sequences have a nice physical interpretation. First, the relation between various gauge fields is given by (C.1). $H^3(X)$ is the wave-function for the purely "baryonic" gauge fields, i.e., gauge fields coming from F_5. The elements of N are the Killing vector fields of X, which give rise to the metric Kaluza–Klein gauge field. Formula (C.1) says that the total space of the gauge field is given by combining the metric and F_5 gauge fields and that there is generally no gauge fields which come purely from the metric.

Secondly, sequence (C.2) relates various charges. Namely, M measures the Kaluza–Klein angular momenta, and $H_3(X)$ measures the D3-brane charges wrapping various cycles. The fact that $HG_3(X)$ is the extension of $H_3(X)$ by M tells us that although we can have excitations with purely Kaluza–Klein momenta and without D-brane charges, e.g., gravitons, generically any states with D-brane i.e., "baryonic" charges, also have angular momenta. It also matches nicely with the result in the recent works [21, 22] which studied the BPS states with no baryonic charges and their charge lattice through the analysis of the spectrum of the Laplacian. The states without D-brane charges also appear as the semiclassical strings moving along the null geodesics. The analysis for $Y^{p,q}$ was carried out in ref. [20].

In the literature on the SE/Quiver duality, relatively little attention is paid to the M part of the charges and the N part of the gauge fields, so it seems worthwhile to study further.

Let us now come back to the construction of the second arrow ι in (C.2). Take an integral basis of Killing vectors v_a, $a = 1, \ldots, \ell$, of N and take the dual basis u^a in M. The basic idea is first to remove subsets X_a from X so that $X \backslash X^a$ has a trivial S^1 bundle structure under the action of the vector field v_a, second to take a section of the bundle with its graph Z_a, and finally to set $\iota(v_a) \equiv \partial Z_a$.

The bundle structure is non-trivial, thus one cannot take a genuine section. The best one can do is to get a four-chain. Then, the boundary of the four-chain is the desired image under ι. To construct an element $\iota(u^a)$ in $HG_3(X)$ for u^a, first let us denote by Y^a the three-cycle where the Killing vector v_a degenerates. Define $B^a = (X \backslash Y^a)/U(1)_a$, where $U(1)_a$ is generated by v_a. Then, the orbit of v_a determines a genuine S^1 bundle

$$S^1 \longrightarrow X \backslash Y^a \stackrel{p}{\longrightarrow} B^a. \tag{C.4}$$

Consider the associated vector bundle over B^a obtained by the fiber S^1 by $\mathbb{C}$ and take a generic section of it. Let the zero locus of the section be given by $t^{ai}\gamma_i^a$, where γ_i^a is a 2d submanifold of b^a and t^{ai} is the multiplicity of the zero at γ_i^a. Then consider the bundle

$$S^1 \longrightarrow X \setminus \left(Y^a \cup \bigcup_a p^{-1}(\gamma_i^a) \right) \longrightarrow B^a \setminus \bigcup_a \gamma_i^a. \tag{C.5}$$

It is a trivial S^1 bundle because we removed γ_i^a and we can take a section Z^a of it.

Using Z_a, we define the image of u^a by ι as

$$\iota(u^a) \equiv \partial Z^a = Y^a + t^{ai} p^{-1}(\gamma_i^a). \tag{C.6}$$

As before, we assume that we can take Y^a and γ_i^a to be invariant under isometries.

The exactness of the sequence (C.2) is now obvious because the image is the boundary of the four-chain Z^a. Secondly, a D3-brane wrapping on ∂Z^a has angular momentum δ_b^a with respect to the isometry v_b. It is because

$$\int_{\partial Z^a} \omega_b = \int_{Z^a} d\omega_b = \int_{Z^a} \iota_{k_b} \operatorname{vol}^\circ = \delta_b^a. \tag{C.7}$$

For the sake of completeness, we would like to describe the second arrow ι in (C.2) and in (C.3) in the toric case. Let us denote the cone over X by $C(X)$, which is a toric variety. For $u \in M$, we can take a rational function χ^u on $C(X)$ satisfying

$$v^i \partial_i \chi^u = \sqrt{-1} \langle u, v \rangle \chi^u \tag{C.8}$$

for $v \in N$, where $\langle u, v \rangle$ is the natural pairing between M and N. It is unique up to multiplication by a complex number, since the torus action is dense in $C(X)$. Then the image is precisely the principal divisor $\operatorname{div}(\chi^u)$ determined by χ^u restricted on X, where the principal divisor $\operatorname{div}(f)$ of a rational function f is

$$\operatorname{div}(f) = \sum_\alpha n_\alpha C^\alpha, \tag{C.9}$$

with C^α the loci of the zeros and the poles of f and with n_α the degree of zeros or the negative of the degree of poles at n_α.

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