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Reliability of linear and circular consecutive-k-out-of-n systems with shock model

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Abstract. A consecutive k-out-of-n system consists of an ordered sequence of n components, such that the system functions if and only if at least k ($k \leq n$) consecutive components function. The system is called linear (L) or circular (C) depending on whether the components are arranged on a straight line or form a circle. In the first part, we use a shock model to obtain the reliability function of consecutve-k-out-of-n systems with dependent and nonidentical components. In the second part, we treat some numerical examples to show the derive results and deduce the failure rate of each component and the system.

Résumé. Un système k-consécutifs-parmi-n est un système constitué de n composants, tel que ce système fonctionne si et seulement si au moins k ($k \leq n$) composants consécutifs fonctionnent. Le système est dit linéaire (L) ou circulaire (C) suivant la disposition des composants en ligne ou en cercle. Dans la première section, en utilisant le modèle de chocs, on établit la fiabilité du système en question ayant des composants dépendants et non identiques. Dans la deuxième section, on traite des exemples numériques qui illustrent les résultats obtenus tout en déduisant le taux de panne de chaque composant et du système.

Key words: Linear and circular consecutive-k-out-of-n system; Feliability function; Failure rate; Shock model.
AMS 2010 Mathematics Subject Classification :62N05; 68M15; 68M20; 90B25; 60K10.

1. Introduction

In some environments, the failure of the system depends not only on the time, but also upon the number of random shocks. So many applications in reliability analysis can be

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described by shock models. Shocks may refer for example to damage caused to biological organs by illness or environmental causes of damage acting on a technical system, see e.g Hameed and Proschan (1973). Also, as an example, a press machine produces external frames in automobiles or refrigerators. The machine can fail due to unexpected causes such as accidental changes in the temperature or electrical voltage, defective raw materials, or errors by human operators.

In the literature, the concept of models shock on systems was treated by many researchers, we can quote some of them. Marshall and Olkin (1967) considered forms of shock to deduce the bivariate exponential distribution. Where, their main objective is stated that two components are subjected to shocks from three different independent sources. One shock destroys one component and the third shock destroys both components. Shock model is also used by Grabski and Sarhan (1995) and Sarhan (1996) to obtain the reliability measures estimations for series and parallel systems with two nonindependent and nonidentical components. They assumed that the times at which the shocks occur are exponentially distributed. Sarhan, A.M. and Abouammoh (2000) used the shock model to derive the reliability function of k-out-of-n systems with nonindependent and nonidentical components. They assumed that a system is subjected to $n+m$ independent types of shocks.

Liu *et al.* (2008) proposed a model to evaluate the reliability function of series and parallel systems with degradation and random shocks. In their model, the system is assumed to be failed when internal degradation or cumulative damage from random shocks exceed random life thresholds.

There are other several types of the shock models which have been considered for the failure of a system as: extreme shock model, cumulative shock model, run shock model, or δ-shock model. For example, δ-shock model is based on the length of the time between successive shocks. In this model, the system fails when the time between two consecutive shocks falls below a fixed threshold δ. This shock model has been studied by Li and Zhao (2007) and Eryilmaz (2012).

The main objective in this paper is to use a shock model to obtain the reliability function of consecutive-k-out-of-n systems, which have a wide range of applications; e.g: telecommunications, gas and oil pipelines, transport network...etc, with dependent and nonidentical components in the two topologies, linear and circular configurations. Then to deduce the failure rate of the system and also the reliability and the failure rate of each component in the system. In the end, we treat some examples to illustrate the provided results.

2. Notations and assumptions

Notations

n: number of components in a system.
k: the minimum number of consecutive components required to be good for the system to be good.
$n+m(m \geq 1)$: number of independent sources of shocks.
s_i: the source i, c_i: the component i.
$S = \{s_1, \ldots, s_n, \ldots, s_{n+m}\}$: the set of all sources.
Ci: the set of sources which destroy component i.

$J = \begin{cases} n-k+1 & \text{In the linear case} \\ n & \text{In the circular case} \end{cases}$

$I(i) = \{c_i, \ldots, c_{i+k-1}\}$, $i = 1, \ldots, J$, the $(i)th$ minimal path.

$S_i = \bigcup_{l=i}^{i+k-1} C_l,\ \ i = 1, ..., J.$

$S_i^{(j)} = \begin{cases} S_i & \text{if } \ j = 1 \\ \cup_{l=1}^{j} S_{i_l} & \text{if } \ \ j = 2, \ldots, J \end{cases}$ where: $1 \leq i_1 < \cdots < i_l < \cdots < i_j \leq J$.

U_i: the random time of shock from s_i.

$Q_i(t) = \mathbb{P}(U_i \leq t)$: the distribution function of U_i.

$\overline{Q_i}(t) = \mathbb{P}(U_i > t)$.

$T_i = \min_{j \in C_i} U_j$: the lifetime of the component i, $i = 1, 2, ..., n$.

A_i: the event $\{T_i > t\}$. $\quad B_i = \bigcap_{j=i}^{i+k-1} A_j$.

p: denotes L (linear) or C (circular).

T_p: the lifetime of "consecutive-k-out-of-n" system.

$R_p(t) = \mathbb{P}(T_p > t)$: the reliability function of "consecutive-k -out-of-n" system.

$\lambda_p(t) = \frac{-R_p^{'}(t)}{R_p(t)}$: the failure rate of "consecutive-k-out-of-n" system.

Assumptions

1. The T_i, $i = 1, \ldots, n$, are nonindependent and nonidentical distributed.
2. The system is subjected to $(n+m)$ independent sources of shocks.
3. s_i destroys c_i, $i = 1, \ldots, n$. At least one shock from the remaining m sources destroys all components, and shocks from the other sources $m-1$ destroy a group of components.

The system is subjected to a set of various shocks, where a shock from source i occurs at a random time U_i. The random variables U_i, $i = 1, \ldots, n+m$, are independent.

3. Reliability of Consecutive-k-out-of-n System

In this section, we consider both linear and circular consecutive-k -out-of- n systems with dependent and nonidentical components.

Definition 1. A consecutive k-out-of-n system consists of n linearly or circularly arranged components, this system works if and only if at least k consecutive components work. In other words, there is at least a minimal path which is working.

Definition 2. A minimal path vector is a set of minimum number of components in working state which ensures the system's functioning.
In the linear case, we have:

$$\begin{array}{rcl} I(1) & = & \{c_1, c_2, \ldots, c_{k-1}, c_k\} \\ I(2) & = & \{c_2, c_3, \ldots, c_k, c_{k+1}\} \\ & \vdots & \\ I(n-k+1) & = & \{c_{n-k+1}, c_{n-k+2}, \ldots, c_{n-1}, c_n\} \end{array}$$

But in the circular case, we have:

$$\begin{array}{ccc} I(1) & = & \{c_1, c_2, \ldots, c_{k-1}, c_k\} \\ I(2) & = & \{c_2, c_3, \ldots, c_k, c_{k+1}\} \\ & \vdots & \\ I(n-k+1) & = & \{c_{n-k+1}, c_{n-k+2}, \ldots, c_{n-1}, c_n\} \\ I(n-k+2) & = & \{c_{n-k+2}, c_{n-k+3}, \ldots, c_n, c_{n+1}\} \\ I(n-k+3) & = & \{c_{n-k+3}, c_{n-k+4}, \ldots, c_{n+1}, c_{n+2}\} \\ & \vdots & \\ I(n) & = & \{c_n, c_{n+1}, \ldots, c_{n+k-2}, c_{n+k-1}\} \end{array}$$

Where: $c_{n+i} = c_i$, $i = 1, \ldots, k-1$.

Theorem 1. *Let $R_p(t)$ be the reliability function of consecutive-k-out-of-n systems, and T_p denotes its lifetime. We have:*

$$R_p(t) = \sum_{j=1}^{J} \Big\{ (-1)^{j-1} \sum_{1 \le i_1 < \ldots < i_j \le J} \Big[\prod_{l \in S_i^{(j)}} \overline{Q_l}(t) \Big] \Big\} \tag{1}$$

– *If $p = L$, then $J = n - k + 1$.*
– *If $p = C$, then $J = n$.*

Proof. The linear or circular consecutive k-out-of-n system works if at least k consecutive components work. Then, we have:

$$\begin{aligned} R_p(t) = \mathbb{P}(T_p > t) &= \mathbb{P}\big(\cup_{i=1}^{J} \{T_i > t, T_{i+1} > t, \ldots, T_{i+k-1} > t\} \big) \\ &= \mathbb{P}\big(\cup_{i=1}^{J} \{\cap_{j=i}^{i+k-1} A_j\} \big) \\ &= \mathbb{P}\big(\cup_{i=1}^{J} B_i \big) \end{aligned}$$

we apply the addition theorem for J events:

$$\begin{aligned} R_p(t) &= \sum_{i=1}^{J} \mathbb{P}\big(B_i\big) - \sum_{1 \le i_1 < i_2 \le J} \mathbb{P}\big(B_{i_1} \cap B_{i_2}\big) \\ &+ \sum_{1 \le i_1 < i_2 < i_3 \le J} \mathbb{P}\big(B_{i_1} \cap B_{i_2} \cap B_{i_3}\big) \\ &+ \cdots + (-1)^{J-1} \mathbb{P}\big(B_1 \cap B_2 \cap \cdots \cap B_J\big) \end{aligned} \tag{2}$$

where,

$$\begin{aligned} \mathbb{P}\big(B_i\big) &= \mathbb{P}\big(T_i > t, T_{i+1} > t, \ldots, T_{i+k-1} > t\big) \\ &= \mathbb{P}\big(\min_{l \in C_i} U_l > t, \min_{l \in C_{i+1}} U_l > t, \ldots, \min_{l \in C_{i+k-1}} U_l > t \big) \\ &= \mathbb{P}\big(U_l > t \ \ \forall l \in C_i, U_l > t \ \ \forall l \in C_{i+1}, \ldots, U_l > t \ \ \forall l \in C_{i+k-1} \big) \\ &= \mathbb{P}\big(\{U_l > t, \ \ \forall l \in C_i \cup C_{i+1} \cup \ldots \cup C_{i+k-1}\} \big) \\ &= \mathbb{P}\big(\{ \min_{l \in C_i \cup \cdots \cup C_{i+k-1}} U_l > t \} \big) \\ &= \prod_{l \in \cup_{j=i}^{i+k-1} C_j} \mathbb{P}\big(U_l > t\big) = \prod_{l \in S_i} \overline{Q_l}(t) = \prod_{l \in S_i^{(1)}} \overline{Q_l}(t) \end{aligned} \tag{3}$$

Similarly,

$$\mathbb{P}\big(B_{i_1} \cap B_{i_2}\big) = \prod_{l \in S_i^{(2)}} \overline{Q_l}(t) \tag{4}$$

$$\mathbb{P}\big(B_{i_1} \cap B_{i_2} \cap B_{i_3}\big) = \prod_{l \in S_i^{(3)}} \overline{Q_l}(t) \tag{5}$$

and

$$\mathbb{P}\big(B_1 \cap B_2 \cap \cdots \cap B_J\big) = \prod_{l \in S} \overline{Q_l}(t) \tag{6}$$

Substituting the equations (3), (4), (5), and (6) into equation (2) we obtain:

$$\begin{aligned} R_p(t) &= \sum_{i=1}^{J} \prod_{l \in S_i^{(1)}} \overline{Q_l}(t) - \sum_{1 \le i_1 < i_2 \le J} \prod_{l \in S_i^{(2)}} \overline{Q_l}(t) \\ &+ \sum_{1 \le i_1 < i_2 < i_3 \le J} \prod_{l \in S_i^{(3)}} \overline{Q_l}(t) + \cdots + (-1)^{J-1} \prod_{l \in S} \overline{Q_l}(t) \\ &= \sum_{j=1}^{J} \Big\{ (-1)^{j-1} \sum_{1 \le i_1 < \ldots < i_j \le J} \Big[\prod_{l \in S_i^{(j)}} \overline{Q_l}(t) \Big] \Big\}. \end{aligned} \tag{7}$$

4. Numerical examples

4.1. Linear consecutive-k-out-of-n system

In particular it is assumed that:

(A1) $\forall k > 1$, $\forall n$, and $\sharp s = 2n + 1$.
(A2) s_i destroys c_i, $i = 1, \ldots, n$.
(A3) the sources between n and $2n + 1$ destroy k consecutive components:

- s_{n+1} destroys $\{c_1, \ldots, c_k\} = I(1)$.
- ...
- s_{2n-k+1} destroys $\{c_{n-k+1}, \ldots, c_n\} = I(n - k + 1)$.
- s_{2n-k+2} destroys $\{c_{n-k+2}, \ldots, c_n, c_1\}$.
- ...
- s_{2n} destroys $\{c_n, c_1, \ldots, c_{k-1}\}$.

(A4) s_{2n+1} destroys all components.
(A5) the random variables U_i, $i = 1, 2, \ldots, 2n + 1$ are Weibull distributed with parameters (α_i, β_i) respectively: $\overline{Q}_i(t) = \exp(-\alpha_i t^{\beta_i})$

Example 1: Linear consecutive-2-out-of-3 system

We consider a linear consecutive-2-out-of-3 system and the assumptions (A1)-(A5) are satisfied, we have:

– $S = \{s_1, ..., s_7\}$

- $C_1 = \{s_1, s_4, s_6, s_7\}$, $C_2 = \{s_2, s_4, s_5, s_7\}$, $C_3 = \{s_3, s_5, s_6, s_7\}$
- $I(1) = \{c_1, c_2\}$, $I(2) = \{c_2, c_3\}$.
- $S_1 = \cup_{j=1}^{2} C_j = S \setminus \{s_3\} = S_1^{(1)}$, $S_2 = \cup_{j=2}^{3} C_j = S \setminus \{s_1\} = S_2^{(1)}$
- $S_i^{(2)} = \cup_{l=1}^{2} S_{i_l} = S_{i_1} \cup S_{i_2} = S_1 \cup S_2 = S$, where $1 \leq i_1 < i_2 \leq 2$.

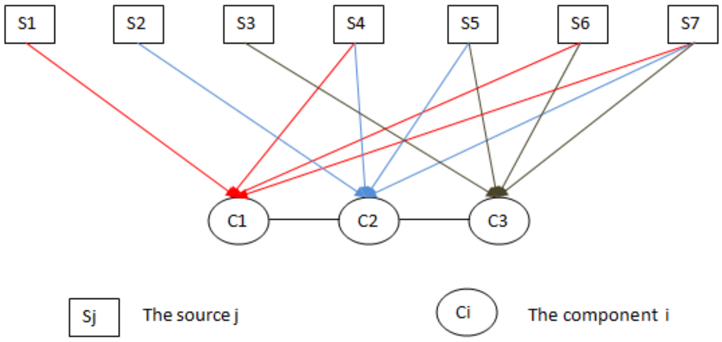

Fig. 1. the linear consecutive-2-out-of-3 system and 7 sources

Using equation (1), it follows that:

$$\begin{aligned} R_L(t) &= \sum_{j=1}^{2} \Big\{ (-1)^{j-1} \sum_{1 \leq i_1 < \ldots < i_j \leq 2} \Big[\prod_{l \in S_i^{(j)}} \overline{Q_l}(t) \Big] \Big\} \\ &= \prod_{l \in S_1^{(1)}} \overline{Q_l}(t) + \prod_{l \in S_2^{(1)}} \overline{Q_l}(t) - \prod_{l \in S} \overline{Q_l}(t) \\ &= \exp\Big(- \sum_{1 \leq l \leq 7} \alpha_l t^{\beta_l} \Big) \Big[\sum_{l \in \{1,3\}} \exp(-\alpha_l t^{\beta_l})^{-1} - 1 \Big], \end{aligned}$$

and $\lambda_L(t)$; the failure rate of the linear system is given by:

$$\lambda_L(t) = \sum_{1 \leq l \leq 7} \alpha_l \beta_l t^{\beta_l - 1} - \frac{\sum_{l \in \{1,3\}} \alpha_l \beta_l t^{\beta_l - 1} \exp(-\alpha_l t^{\beta_l})^{-1}}{\sum_{l \in \{1,3\}} \exp(-\alpha_l t^{\beta_l})^{-1} - 1}$$

In general, if $n = k + 1$, then:

$$R_L(t) = \exp\Big(- \sum_{1 \leq l \leq 2n+1} \alpha_l t^{\beta_l} \Big) \Big[\sum_{l \in \{1,n\}} \exp(-\alpha_l t^{\beta_l})^{-1} - 1 \Big] \tag{8}$$

$$\lambda_L(t) = \sum_{1 \leq l \leq 2n+1} \alpha_l \beta_l t^{\beta_l - 1} - \frac{\sum_{l \in \{1,n\}} \alpha_l \beta_l t^{\beta_l - 1} \exp(-\alpha_l t^{\beta_l})^{-1}}{\sum_{l \in \{1,n\}} \exp(-\alpha_l t^{\beta_l})^{-1} - 1}. \tag{9}$$

Example 2: Linear consecutive-3-out-of-5 system

For this system and the assumptions (A1)-(A5), we have:

- $S = \{s_1, ..., s_{11}\}$.
- $C_1 = \{s_1, s_6, s_9, s_{10}, s_{11}\}$, $C_2 = \{s_2, s_6, s_7, s_{10}, s_{11}\}$, $C_3 = \{s_3, s_6, s_7, s_8, s_{11}\}$, $C_4 = \{s_4, s_7, s_8, s_9, s_{11}\}$, $C_5 = \{s_5, s_8, s_9, s_{10}, s_{11}\}$.
- $I(1) = \{c_1, c_2, c_3\}$, $I(2) = \{c_2, c_3, c_4\}$, $I(3) = \{c_3, c_4, c_5\}$.
- $S_1 = S \setminus \{s_4, s_5\} = S_1^{(1)}$, $S_2 = S \setminus \{s_1, s_5\} = S_2^{(1)}$, $S_3 = S \setminus \{s_1, s_2\} = S_3^{(1)}$.
- $S_i^{(2)} = \cup_{l=1}^{2} S_{i_l} = S_{i_1} \cup S_{i_2}$, where $1 \leq i_1 < i_2 \leq 3$, then:

$$S_i^{(2)} = \begin{cases} S_1 \cup S_2 = S \setminus \{s_5\} \\ S_1 \cup S_3 = S \\ S_2 \cup S_3 = S \setminus \{s_1\} \end{cases}$$

- $S_i^{(3)} = \cup_{l=1}^{3} S_{i_l}$, where $1 \leq i_1 < i_2 < i_3 \leq 3$, then: $S_i^{(3)} = S$.

Using equation (1), we obtain:

$$\begin{aligned} R_L(t) &= \exp\Big(-\sum_{\substack{1\leq l\leq 11\\ l\neq 5}} \alpha_l t^{\beta_l}\Big)\Big[\sum_{l\in\{1,4\}} \exp(-\alpha_l t^{\beta_l})^{-1} - 1\Big] \\ &+ \exp\Big(-\sum_{2\leq l\leq 11} \alpha_l t^{\beta_l}\Big)\Big[\exp(-\alpha_2 t^{\beta_2})^{-1} - 1\Big], \end{aligned}$$

and the the failure rate of the system is as following:

$$\begin{aligned} \lambda_L(t) &= -\exp\Big(-\sum_{\substack{1\leq l\leq 11\\ l\neq 5}} \alpha_l t^{\beta_l}\Big)\Big[\sum_{\substack{1\leq l\leq 11\\ l\neq 5}} \alpha_l\beta_l t^{\beta_l-1}\Big(\sum_{l\in\{1,4\}} \exp(-\alpha_l t^{\beta_l})^{-1} - 1\Big) \\ &+\Big(\sum_{l\in\{1,4\}} \alpha_l\beta_l t^{\beta_l-1}\exp(-\alpha_l t^{\beta_l})^{-1}\Big)\Big] - \exp\Big(-\sum_{2\leq l\leq 11} \alpha_l t^{\beta_l}\Big) \\ &\Big[\sum_{2\leq l\leq 11} \alpha_l\beta_l t^{\beta_l-1}\Big(\exp(-\alpha_2 t^{\beta_2})^{-1} - 1\Big) + \alpha_2\beta_2 t^{\beta_2-1}\exp(-\alpha_2 t^{\beta_2})^{-1}\Big] \end{aligned}$$

In general, if $n = k + 2$, then:

$$\begin{aligned} R_L(t) &= \exp\Big(-\sum_{\substack{1\leq l\leq 2n+1\\ l\neq n}} \alpha_l t^{\beta_l}\Big)\Big[\sum_{l\in\{1,n-1\}} \exp(-\alpha_l t^{\beta_l})^{-1} - 1\Big] \\ &+ \exp\Big(-\sum_{2\leq l\leq 2n+1} \alpha_l t^{\beta_l}\Big)\Big[\exp(-\alpha_2 t^{\beta_2})^{-1} - 1\Big], \end{aligned} \tag{10}$$

and:

$$\begin{aligned}\lambda_L(t) &= -\exp\Big(-\sum_{\substack{1\le l\le 2n+1\\ l\neq n}}\alpha_l t^{\beta_l}\Big)\Big[\sum_{\substack{1\le l\le 2n+1\\ l\neq n}}\alpha_l\beta_l t^{\beta_l-1}\Big(\sum_{l\in\{1,n-1\}}\exp(-\alpha_l t^{\beta_l})^{-1}-1\Big)\\ &\quad+\Big(\sum_{l\in\{1,n-1\}}\alpha_l\beta_l t^{\beta_l-1}\exp(-\alpha_l t^{\beta_l})^{-1}\Big)\Big]-\exp\Big(-\sum_{2\le l\le 2n+1}\alpha_l t^{\beta_l}\Big)\\ &\quad\Big[\sum_{2\le l\le 2n+1}\alpha_l\beta_l t^{\beta_l-1}\Big(\exp(-\alpha_2 t^{\beta_2})^{-1}-1\Big)+\alpha_2\beta_2 t^{\beta_2-1}\exp(-\alpha_2 t^{\beta_2})^{-1}\Big].\end{aligned}\tag{11}$$

Example 3: Linear consecutive-5-out-of-8 system

We consider a linear consecutive-5-out-of-8 system and the assumptions (A1)-(A5) are satisfied, we have:

- $S=\{s_1,...,s_{17}\}$.
-

$$\begin{aligned}C_1 &= \{s_1,s_9,s_{13},s_{14},s_{15},s_{16},s_{17}\}, && C_2=\{s_2,s_9,s_{10},s_{14},s_{15},s_{16},s_{17}\},\\ C_3 &= \{s_3,s_9,s_{10},s_{11},s_{15},s_{16},s_{17}\}, && C_4=\{s_4,s_9,s_{10},s_{11},s_{12},s_{16},s_{17}\},\\ C_5 &= \{s_5,s_9,s_{10},s_{11},s_{12},s_{13},s_{17}\}, && C_6=\{s_6,s_{10},s_{11},s_{12},s_{13},s_{14},s_{17}\},\\ C_7 &= \{s_7,s_{11},s_{12},s_{13},s_{14},s_{15},s_{17}\}, && C_8=\{s_8,s_{12},s_{13},s_{14},s_{15},s_{16},s_{17}\}.\end{aligned}$$

- $I(1)=\{c_1,...,c_5\}$, $I(2)=\{c_2,...,c_6\}$, $I(3)=\{c_3,...,c_7\}$, $I(4)=\{c_4,...,c_8\}$.
- $S_1=S\setminus\{s_6,s_7,s_8\}=S_1^{(1)}$, $S_2=S\setminus\{s_1,s_7,s_8\}=S_2^{(1)}$, $S_3=S\setminus\{s_1,s_2,s_8\}=S_3^{(1)}$, $S_4=S\setminus\{s_1,s_2,s_3\}=S_4^{(1)}$,
- $S_i^{(2)}=\cup_{l=1}^2 S_{i_l}=S_{i_1}\cup S_{i_2}$, where $1\le i_1<i_2\le 4$, then:

$$S_i^{(2)}=\begin{cases}S_1\cup S_2=S\setminus\{s_7,s_8\}\\ S_1\cup S_3=S\setminus\{s_8\}\\ S_1\cup S_4=S\\ S_2\cup S_3=S\setminus\{s_1,s_8\}\\ S_2\cup S_4=S\setminus\{s_1\}\\ S_3\cup S_4=S\setminus\{s_1,s_2\}.\end{cases}$$

- $S_i^{(3)}=\cup_{l=1}^3 S_{i_l}=S_{i_1}\cup S_{i_2}\cup S_{i_3}=\begin{cases}S_1\cup S_2\cup S_3=S\setminus\{s_8\}\\ S_1\cup S_2\cup S_4=S\\ S_1\cup S_3\cup S_4=S\\ S_2\cup S_3\cup S_4=S\setminus\{s_1\}\end{cases}$
- $S_i^{(4)}=\cup_{l=1}^4 S_{i_l}=S$.

Using equation (1), we have:

$$\begin{aligned}R_L(t) &= \exp\Big(-\sum_{\substack{1\le l\le 17\\ l\neq 8, l\neq 7}}\alpha_l t^{\beta_l}\Big)\Big[\sum_{l\in\{1,6\}}\exp(-\alpha_l t^{\beta_l})^{-1}-1\Big]\\ &+\exp\Big(-\sum_{3\le l\le 17}\alpha_l t^{\beta_l}\Big)\Big[\sum_{l\in\{1,2\}}\exp(-\alpha_l t^{\beta_l})^{-1}-1\Big]\end{aligned}$$

and so:

$$\begin{aligned}
\lambda_L(t) \;=\; & -\exp\Big(-\sum_{\substack{1\leq l\leq 17\\ l\neq 7}} \alpha_l t^{\beta_l}\Big)\Big[\sum_{\substack{1\leq l\leq 17\\ l\neq 8, l\neq 7}} \alpha_l\beta_l t^{\beta_l-1} \\
& \Big(\sum_{l\in\{1,6\}} \exp(-\alpha_l t^{\beta_l})^{-1}-1\Big)+\Big(\sum_{l\in\{1,6\}} \alpha_l\beta_l t^{\beta_l-1}\exp(-\alpha_l t^{\beta_l})^{-1}\Big)\Big] \\
& -\exp\Big(-\sum_{3\leq l\leq 17} \alpha_l t^{\beta_l}\Big)\Big[\sum_{3\leq l\leq 17} \alpha_l\beta_l t^{\beta_l-1}\Big(\sum_{l\in\{1,2\}} \exp(-\alpha_l t^{\beta_l})^{-1}-1\Big) \\
& +\sum_{l\in\{1,2\}} \alpha_l\beta_l t^{\beta_l-1}\exp(-\alpha_l t^{\beta_l})^{-1}\Big].
\end{aligned}$$

In general, if $n=k+3$, then:

$$\begin{aligned}
R(t) \;=\; & \exp\Big(-\sum_{\substack{1\leq l\leq 2n+1\\ l\neq n, l\neq n-1}} \alpha_l t^{\beta_l}\Big)\Big[\sum_{l\in\{1,n-2\}} \exp(-\alpha_l t^{\beta_l})^{-1}-1\Big] \\
& +\exp\Big(-\sum_{3\leq l\leq 2n+1} \alpha_l t^{\beta_l}\Big)\Big[\sum_{l\in\{1,2\}} \exp(-\alpha_l t^{\beta_l})^{-1}-1\Big]
\end{aligned} \tag{12}$$

$$\begin{aligned}
\lambda_L(t) \;=\; & -\exp\Big(-\sum_{\substack{1\leq l\leq 2n+1\\ l\neq n-1}} \alpha_l t^{\beta_l}\Big)\Big[\sum_{\substack{1\leq l\leq 2n+1\\ l\neq n, l\neq n-1}} \alpha_l\beta_l t^{\beta_l-1} \\
& \Big(\sum_{l\in\{1,n-2\}} \exp(-\alpha_l t^{\beta_l})^{-1}-1\Big)+\Big(\sum_{l\in\{1,n-2\}} \alpha_l\beta_l t^{\beta_l-1}\exp(-\alpha_l t^{\beta_l})^{-1}\Big)\Big] \\
& -\exp\Big(-\sum_{3\leq l\leq 2n+1} \alpha_l t^{\beta_l}\Big)\Big[\sum_{3\leq l\leq 2n+1} \alpha_l\beta_l t^{\beta_l-1}\Big(\sum_{l\in\{1,2\}} \exp(-\alpha_l t^{\beta_l})^{-1}-1\Big) \\
& +\sum_{l\in\{1,2\}} \alpha_l\beta_l t^{\beta_l-1}\exp(-\alpha_l t^{\beta_l})^{-1}\Big].
\end{aligned} \tag{13}$$

Remark 1. For all precedent cases, let $R_i(t)$ et $\lambda_i(t)$, $i=1,\dots,n$, denote respectively the reliability function and the failure rate of the component i, which are given by the following expressions:

$$R_i(t)=\exp\Big(-\sum_{l\in S_i} \alpha_l t^{\beta_l}\Big), \qquad \lambda_i(t)=\sum_{l\in S_i} \alpha_l\beta_l t^{\beta_l-1}.$$

4.2. Circular consecutive-k-out-of-n system

Example 4: Circular consecutive-2-out-of-3 system

We consider a circular consecutive-2-out-of-3 system and 7 independent sources act on the system (see fig.2).

- $S=\{s_1,s_2,s_3,s_4,s_5,s_6,s_7\}$
- $C_1=\{s_1,s_4,s_6,s_7\}$, $\quad C_2=\{s_2,s_4,s_5,s_7\}$, $\quad C_3=\{s_3,s_5,s_6,s_7\}$,

- $I(1) = \{c_1, c_2\}, \quad I(2) = \{c_2, c_3\}, \quad I(3) = \{c_3, c_1\}$.
- $S_1 = S \setminus \{s_3\} = S_1^{(1)}, \quad S_2 = S \setminus \{s_1\} = S_2^{(1)}, \quad S_3 = S \setminus \{s_2\} = S_3^{(1)}$,
- $S_i^{(2)} = \cup_{l=1}^{2} S_{i_l} = S_{i_1} \cup S_{i_2} = \begin{cases} S_1 \cup S_2 = S \\ S_1 \cup S_3 = S \\ S_2 \cup S_3 = S \end{cases}$
- $S_i^{(3)} = S$.

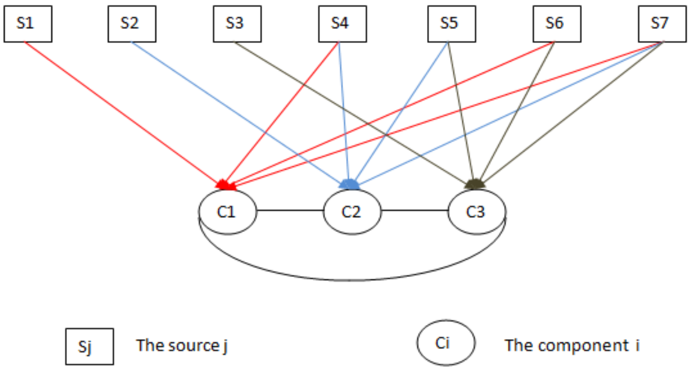

Fig. 2. The circular consecutive-2-out-of-3 system and 7 sources

$U_i \sim \text{Weib}(\alpha_i, \beta_i)$, $i = 1; \ldots; 7$, and using equation (1), it follows that:

$$R_C(t) = \exp\Big(-\sum_{1\le l\le 7} \alpha_l t^{\beta_l}\Big)\Big[\sum_{1\le l\le 3} \exp(-\alpha_l t^{\beta_l})^{-1} - 2\Big],$$

then $\lambda_C(t)$; the failure rate of the circular system, is given by:

$$\lambda_C(t) = \sum_{1\le l\le 7} \alpha_l \beta_l t^{\beta_l - 1} - \frac{\sum_{1\le l\le 3} \alpha_l \beta_l t^{\beta_l-1} \exp(-\alpha_l t^{\beta_l})^{-1}}{\sum_{1\le l\le 3} \exp(-\alpha_l t^{\beta_l})^{-1} - 2}.$$

Example 5: Circular consecutive-3 -out-of-5 system

We consider a circular consecutive-3-out-of-5 system and 11 independent sources act on the system.

- $S = \{s_1, ..., s_{11}\}$.
- $C_1 = \{s_1, s_6, s_9, s_{10}, s_{11}\}$, $C_2 = \{s_2, s_6, s_7, s_{10}, s_{11}\}$, $C_3 = \{s_3, s_6, s_7, s_8, s_{11}\}$, $C_4 = \{s_4, s_7, s_8, s_9, s_{11}\}$, $C_5 = \{s_5, s_8, s_9, s_{10}, s_{11}\}$.
- $I(1) = \{c_1, c_2, c_3\}$, $I(2) = \{c_2, c_3, c_4\}$, $I(3) = \{c_3, c_4, c_5\}$, $I(4) = \{c_4, c_5, c_1\}$, $I(5) = \{c_5, c_1, c_2\}$.
- $S_1 = S \setminus \{s_4, s_5\}$, $S_2 = S \setminus \{s_1, s_5\}$, $S_3 = S \setminus \{s_1, s_2\}$, $S_4 = S \setminus \{s_2, s_3\}$, $S_5 = S \setminus \{s_3, s_4\}$.

– $S_i^{(2)} = \cup_{l=1}^{2} S_{i_l} = S_{i_1} \cup S_{i_2}$, where $1 \leq i_1 < i_2 \leq 5$, then:

$$S_i^{(2)} = \begin{cases} S_1 \cup S_2 = S \setminus \{s_5\} \\ S_1 \cup S_5 = S \setminus \{s_4\} \\ S_2 \cup S_3 = S \setminus \{s_1\} \\ S_3 \cup S_4 = S \setminus \{s_2\} \\ S_4 \cup S_5 = S \setminus \{s_3\} \\ S \text{ for the other cases.} \end{cases}$$

– $S_i^{(3)} = \cup_{l=1}^{3} S_{i_l} = S$, where $1 \leq i_1 < i_2 < i_3 \leq 5$.
– $S_i^{(4)} = \cup_{l=1}^{4} S_{i_l} = S$, where $1 \leq i_1 < ... < i_4 \leq 5$. $S_i^{(5)} = S$.

Using equation (1), we obtain:

$$\begin{aligned} R_C(t) &= \sum_{1 \leq l \leq 5} \prod_{l \in \{i, i+1\}} \overline{Q_l}(t) + \prod_{1 \leq l \leq 11} \overline{Q_l}(t) \Big[1 - \sum_{1 \leq l \leq 5} \overline{Q_l}(t)^{-1}\Big], \quad \text{where } \overline{Q_6}(t) = \overline{Q_1}(t) \\ &= \sum_{1 \leq l \leq 5} \exp\Big(-\sum_{l \in \{i, i+1\}} \alpha_l t^{\beta_l}\Big) + \exp\Big(-\sum_{1 \leq l \leq 11} \alpha_l t^{\beta_l}\Big) \Big[1 - \sum_{1 \leq l \leq 5} \exp(-\alpha_l t^{\beta_l})^{-1}\Big]. \end{aligned}$$

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