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On testing for the presence of hidden periodicities in time series

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Abstract. We study the problem of test based on the periodogram, which can be used to test the null hypothesis H_0 that the series is composed of specified linear process against the alternative hypothesis H_1 that there is an additional deterministic periodic component.

Résumé. Nous étudions le problème de test basé sur le périodogramme, pour construire le test de l'hypothèse nulle H_0 que le modèle est composé d'un processus linéaire donné contre l'hypothèse alternative H_1 que le modèle contient en plus une composante périodique déterministe.

Key words: Linear process; Periodogram; Kolmogorov-Smirnov test.
AMS 2010 Mathematics Subject Classification : Primary 62F03; 62F05 : Secondary 60G10.

1. Introduction

We consider a discrete parameter time series $(X_t)_{t\in\mathbb{Z}}$ such that

$$X_t = m_t + Z_t \, , \tag{1}$$

where

$$m_t = E(X_t) = \sum_{r=1}^{K} \{A_r \cos(\omega_r t) + B_r \sin(\omega_r t)\} \, , \tag{2}$$

with A_r, B_r, $r = 1, \ldots, K$ are non-random real constants and ω_r, $r = 1, \ldots, K$ are specified frequencies;

$$Z_t = \sum_{j=0}^{\infty} \psi_j(\theta)\varepsilon_{t-j} \, , \tag{3}$$

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with the ε_t are independently and identically distributed with $E(\varepsilon_t) = 0$ and $E(\varepsilon_t^2) = \sigma^2 < \infty$, and the $\psi_j(\theta)$ are specified functions of a vector valued parameter $\theta = (\theta_1, \ldots, \theta_p)$, with $\psi_0(\theta) = 1$ and $\sum_{j=0}^{\infty} \psi_j^2(\theta) < \infty$ which is the condition required for Z_t to be stationary with finite variance. Note that the AR, MA, and $ARMA$ models may be regarded as special cases of the linear process model (3). The time series (X_t) is then composed from the sum of K sinusoidal components with angular frequencies ω_r and additive linear process (Z_t) which is a completely stationary series having spectral density

$$f_Z(\lambda, \theta) = \frac{\sigma^2}{2\pi} \left| \sum_{j=0}^{\infty} \psi_j(\theta) e^{-ij\lambda} \right|^2 . \tag{4}$$

Let $\{X_1, \ldots, X_N\}$ be a sample of N consecutive observations generated by the time series $(X_t)_{t\in\mathbb{Z}}$. The periodogram of the set of observations, which may be defined as a function $I_{N,X}$ of angular frequency with range $[-\pi, \pi]$ that is proportional to $|\sum_{t=1}^{N} X_t e^{-i\omega t}|^2$, plays an important part in methods of making inferences about the structure of (X_t), in particular its spectral density.

Our aim is to study the statistical tests based on the properties of the periodogram and consisting of the null hypothesis

$$H_0: \; A_r = B_r = 0\,, \; \forall\, r = 1, \ldots, K \quad \text{against} \quad H_1: \begin{cases} A_r,\, B_r,\; r = 1, \ldots, K \\ \text{are not all zero}\,. \end{cases}$$

Walker (1965) gives some asymptotic results for the periodogram and the asymptotic relation between the periodogram of a linear process and the periodogram of the corresponding residual process. Anderson (1993) studied the problem of testing the null hypothesis that completely specifies the pattern of dependence. Anderson (1993) compared the sample standardized spectral distribution with the process standardized spectral distribution by means of a good of fit criterion, such as the Cramer-von Mises criterion or the Kolmogorov-Smirnov criterion. Quinn (1986) studies the hypothesis test a time series is composed from s sinusoidal components, at unknown frequencies, with additive Gaussian noise, against the alternative that there are r other sinusoidal components present.

In this paper, we develop a method for testing the null hypothesis that a time series is composed of the linear process (3) against the alternative hypothesis that the time series has the form given by (1), that is composed from the sum of K sinusoidal components with angular frequencies ω_r and additive linear process (Z_t) of the form (3). In Section 2 we establish some asymptotic properties of the normalized periodogram of linear process. In Section 3 we give some applications and numerical examples. Finally proofs are deferred in Section 4.

2. Asymptotic properties of the periodogram under the null hypothesis

We consider in this section the process (X_t) given by the model (1) and we study the asymptotic properties of the periodogram of the set observations $\{X_1, \ldots, X_N\}$ under the null hypothesis $H_0: \; A_r = B_r = 0\,, \; \forall\, r = 1, \ldots, K$. In this case, the process (X_t) is reduced to the linear process (Z_t) given by the model (3).

In the sequel, we denote f_Z and f_ε the spectral densities of the processes (Z_t) and (ε_t) respectively and f_X the spectral density of the process (X_t) under the null hypothesis H_0.

We know that f_Z of the process given by (3) is related to the spectral density f_ε of the process (ε_t) (see, e.g. Brockwell and Davis, 1991) by

$$f_Z(\omega) = \left| \sum_{j=0}^{\infty} \psi_j(\theta) e^{-ij\omega} \right|^2 f_\varepsilon(\omega), \qquad -\pi \le \omega \le \pi \quad . \tag{5}$$

Since (ε_t) is white noise, its spectral density function is

$$f_\varepsilon(\omega) = \frac{\sigma^2}{2\pi}, \qquad -\pi \le \omega \le \pi \quad . \tag{6}$$

Now, if we regard $I_{N,X}(\omega)/2\pi$ as the sample version of the spectral density function of $f_X(\omega)$ under the null hypothesis H_0, we might thought expect a similar relationship the periodograms $I_{N,X}(\omega)$ and $I_{N,\varepsilon}(\omega)$ of (X_t) and (ε_t) respectively. If $E(\varepsilon_t^4) < \infty$ and satisfying $\sum_{j=0}^{\infty} |\psi_j(\theta)||j|^\alpha < \infty$, $\alpha > 0$, this relationship is given, under the null hypothesis H_0, by

$$I_{N,X}(\omega) = \left| \sum_{k=0}^{\infty} \psi_k(\theta) e^{-ik\omega} \right|^2 I_{N,\varepsilon}(\omega) + R_N(\omega)\,,$$

where $E\left(|R_N(\omega)|^2\right) = O(1/N^{2\alpha})$ uniformly in $\omega \in [-\pi, \pi]$. (See e.g. Priestley, 1981, Theorem 6.2.2, page 424).

Remark 1. We have $R_N(\omega) \xrightarrow{\mathcal{P}} 0$ for every $\omega \in [-\pi, \pi]$, by Chebychev's inequality and hence $R_N(\omega)$ converges in distribution to 0.

We have the following results.

Theorem 1. *Let (Z_t) be a linear process defined in (3) such that $E(\varepsilon_t^4) < \infty$ and satisfying $\sum_{j=0}^{\infty} |\psi_j(\theta)||j|^\alpha < \infty$, $\alpha > 0$.*
If $f_Z(\lambda) > 0$ for all $\lambda \in [-\pi, \pi]$ and $0 < \lambda_1 < \cdots < \lambda_\nu < \pi$, then under the null hypothesis H_0, the random variables,

$$U_i = \frac{\sum_{k=1}^{i} \{I_{N,X}(\lambda_k)/f_Z(\lambda_k)\}}{\sum_{k=1}^{\nu} \{I_{N,X}(\lambda_k)/f_Z(\lambda_k)\}}, \qquad i = 1, \ldots, \nu - 1 \tag{7}$$

have asymptotically the same distribution as the independent order statistics $T_1, \ldots, T_{\nu-1}$, each one uniformly distributed on the interval $(0, 1)$.

Corollary 1. *Let the conditions of Theorem 1 be satisfied and define $U_0 = 0$, $U_\nu = 1$ and*

$$T_\nu = \max_{1\le i\le \nu}(U_i - U_{i-1}) = \max_{1\le i\le \nu} \frac{I_{N,X}(\lambda_i)/f_Z(\lambda_i)}{\sum_{k=1}^{\nu}\{I_{N,X}(\lambda_k)/f_Z(\lambda_k)\}} \,. \tag{8}$$

Then, under the null hypothesis H_0, as $N \to \infty$,

$$P(T_\nu \le t) \longrightarrow \sum_{j=0}^{\nu}(-1)^j \binom{\nu}{j} (1 - jt)_+^{\nu-1}, \tag{9}$$

where $x_+ = max(x, 0)$.

Corollary 2. *Let the conditions of Theorem 1 be satisfied. Then, under the null hypothesis H_0, as $N \to \infty$,*

$$F_\nu(x) = \begin{cases} 0 & \text{if } x < U_1 \\ \dfrac{j}{\nu-1} & \text{if } U_j \le x < U_{j+1}, \quad j = 1,\dots,\nu-2 \\ 1 & \text{if } x \ge U_{\nu-1}\,, \end{cases} \tag{10}$$

converges to the empirical distribution function of a sample of size $(\nu - 1)$ from uniform distribution on the interval $(0, 1)$.

As application of Corollaries 1 and 2, we give some examples in the following section.

3. Applications

3.1. Fisher's test for hidden periodicities

Now, if we restrict our considerations to the harmonics frequencies $\omega_j = 2\pi j/N$, $j = 1, 2, \dots, \nu = \left[\frac{N-1}{2}\right]$; under the null hypothesis H_0, when N is large and if $f_Z(\omega)$ is known a prior, Corollary 1 may be used to construct an approximate test of the null hypothesis that (X_t) has spectral density f_X of linear process, against the alternative hypothesis that (X_t) contains an added deterministic period component of the form m_t defined by (2) of unspecified frequencies.

We evaluate the quantity $I_{N,X}(\lambda)/f_Z(\lambda)$ at the standard frequencies $\omega_j = 2\pi j/N$, $j = 1, \dots, \nu$ and test the null hypothesis H_0: $A_r = B_r = 0$ all $r = 1, \dots, K$. We reject the null hypothesis H_0 if the normalized periodogram $\{I_{N,X}(\omega_i)/f_Z(\omega_i)\}$ contains a value substantially larger than the average value; $\frac{1}{\nu}\sum_{i=1}^{\nu}\{I_{N,X}(\omega_i)/f_Z(\omega_i)\}$ i.e. if

$$\xi_\nu = \frac{\max_{1\le i\le \nu}\{I_{N,X}(\omega_i)/f_Z(\omega_i)\}}{\frac{1}{\nu}\sum_{j=1}^{\nu}\{I_{N,X}(\omega_j)/f_Z(\omega_j)\}} \tag{11}$$

is sufficiently large. We can apply the test, by computing the realized value x of ξ_ν from data $\{X_1, \dots, X_N\}$ and use (9) to approximate the value of $P(\xi_\nu \ge x)$, then we have

$$P(\xi_\nu \ge x) = P\left(T_\nu \ge \frac{x}{\nu}\right) \simeq 1 - \sum_{j=0}^{\nu}(-1)^j \binom{\nu}{j} \left(1 - \frac{jx}{\nu}\right)_+^{\nu-1} \tag{12}$$

and we reject the null hypothesis at level α if the value $P(\xi_\nu \ge x)$ is less than α.

Remark 2. We can also test the null hypothesis H_0 by referring the statistic

$$T_\nu = \frac{\max\limits_{1\le i\le\nu} \{I_{N,X}(\omega_i)/f_Z(\omega_i)\}}{\sum\limits_{j=1}^{\nu} \{I_{N,X}(\omega_j)/f_Z(\omega_j)\}} \tag{13}$$

to Fisher's distribution with ν degrees of freedom (see e.g. Priestley, 1981, chapter 8, page 618).

Example 1. Let $\{X_1, \ldots, X_N\}$, with the sample size $N = 512$, be the data generated by the process

$$X_t = 1.5\cos\left(\frac{8\pi}{9}t\right) + Z_t\,, \tag{14}$$

where (Z_t) is generated by $Z_t + 0.7Z_{t-1} = \varepsilon_t - 0.6\varepsilon_{t-1}$ with $\varepsilon_t \sim i.i.d.\mathcal{N}(0,1)$.

We test the null hypothesis

H_0: The spectral density of the process (X_t) is $f(\lambda) = \dfrac{1}{2\pi}\left|\dfrac{1-0.6e^{-i\lambda}}{1+0.7e^{-i\lambda}}\right|^2$.

In this case $\nu = \left[\frac{511}{2}\right] = 255$ and the realized value from the data of the test statistic given in (11) is $x = 5.2842$. Now from (12)

$$P\left(\xi_{255} \ge 5.2842\right) = 0.0015\ .$$

Then, we reject the null hypothesis at level 0.01 and 0.05.

3.2. The Kolmogorov-Smirnov test

Corollary 2 suggests, when N is large, an approximate test null hypothesis H_0, that (X_t) has completely specifed spectral density f_X.

Let $U_1 \le U_2 \le \cdots \le U_{\nu-1}$ denote the random variables defined in Theorem 1, by Corollary 2, these random variables may be considered as an ordered sample from an uniform distribution on an interval $(0,1)$. We consider the empirical distribution function $F_\nu(x)$ computed from $U_1, U_2, \ldots, U_{\nu-1}$ i.e. the step function defined by

$$F_\nu(x) = \begin{cases} 0 & \text{if } x < U_1 \\ \dfrac{j}{\nu-1} & \text{if } U_j \le x < U_{j+1}, \quad j = 1, \ldots, \nu-2 \\ 1 & \text{if } x \ge U_{\nu-1}. \end{cases} \tag{15}$$

We plot this function and check its compatibility with the theoritical distribution for an uniform distribution on $(0,1)$, $F(x) = x$, $0 \le x \le 1$; using the Kolmogorov-Smirnov test. Let

$$D_\nu = \max_x |F_\nu(x) - F(x)|\ ,$$

it follows from the theory of the well known Kolmogorov-Smirnov test (see, e.g. Feller, 1948) for every fixed $a \ge 0$, that, as $\nu \longrightarrow \infty$ (or $N \longrightarrow \infty$), we have

$$P\left(\sqrt{\nu-1}D_\nu \le a\right) \longrightarrow L(a), \tag{16}$$

where

$$L(a) = 1 - 2\sum_{j=1}^{\infty}(-1)^{j-1}e^{-j^2a^2}. \tag{17}$$

For $a = 1.36$, $L(a) = 0.95$ while for $a = 1.63$, $L(a) = 0.99$. Then, when ν (or N) is large and if a_α is the critical value of D_ν for the significance level α, we reject the null hypothesis if for any x in $(0, 1)$, $D_\nu > a_\alpha\,(\nu - 1)^{-\frac{1}{2}}$.

For $\nu > 30$ (or for sample size $N > 62$), a good approximation to the level-α Kolmogorov test is to reject the null hypothesis if the empirical distribution function $F_\nu(x)$ exists from the bounds: $x \pm a_\alpha\,(\nu - 1)^{-\frac{1}{2}}$, $0 < x < 1$, where $a_{0.05} = 1.36$ and $a_{0.01} = 1.63$.

An equivalent approach is to plot the standardized cumulative "normalized periodogram",

$$K(x) = \begin{cases} 0 & \text{if } x < 1 \\ U_i & \text{if } i \le x < i+1, \quad i = 1, \dots, \nu - 1 \\ 1 & \text{if } x \ge \nu \end{cases} \tag{18}$$

and rejecting the null hypothesis H_0 at the level α if for any x in $[1, \nu]$, the function $K(x)$ exists from the boundaries $\frac{x-1}{\nu-1} \pm a_\alpha\,(\nu - 1)^{-\frac{1}{2}}$.

Example 2. Let $\{X_1, \dots, X_N\}$, with the sample size $N = 512$, be the data generated by the process by the model

$$X_t + aX_{t-1} = \varepsilon_t + b\varepsilon_{t-1}$$

with $\varepsilon_t \sim i.i.d.\mathcal{N}(0, 1)$. We test the null hypothesis

$$H_0: \ f_X(\lambda) = \frac{1}{2\pi}\left|\frac{1 - 0.6e^{-i\lambda}}{1 + 0.7e^{-i\lambda}}\right|^2 \quad \text{against} \quad H_1: \quad H_0 \text{ is false.}$$

Figure 1 and Figure 2 show the cumulative periodogram $K(x)$ and Kolmogorov-Smirnov boundaries for the data with the significance level of the test $\alpha = 0.05$ and $\alpha = 0.01$. We do not reject the null hypothesis even at level 0.05 and 0.01 in Figure 1. However we reject the null hypothesis in Figure 2. The data was in fact generated by $X_t + 0.7X_{t-1} = \varepsilon_t - 0.6\varepsilon_{t-1}$ and $X_t + 0.6X_{t-1} = \varepsilon_t - 0.6\varepsilon_{t-1}$ in Figure 1 and Figure 2 respectively.

4. Proofs

Proof of Theorem 1. Under the null hypothesis $H_0: \ A_r = B_r = 0, \ \forall r = 1, \dots, K$, the process (X_t) is reduced to the linear process (Z_t) and then (X_t) and (Z_t) have the same spectral density ($f_X = f_Z$). Now, set $V_k = I_{N,X}(\lambda_k)/2\pi f_Z(\lambda_k)$, $k = 1, \dots, \nu$. The vector random $(V_1, \dots, V_\nu)$ converges in distribution to vector of independent components $(Y_1, \dots, Y_\nu)$, where $Y_i \sim \chi^2(2)/2$, $i = 1, \dots, \nu$, (see Brockwell and Davis, 1991, page 347).

Since the functions

$$\begin{array}{rrcl} h_i: & \mathbb{R}^\nu & \longrightarrow & \mathbb{R}, \qquad i = 1, \dots, \nu - 1 \\ & (V_1, \dots, V_\nu) & \longmapsto & U_i = h_i(V_1, \dots, V_\nu) \\ & & & = \left(\sum_{k=1}^{i} V_k\right) / \left(\sum_{k=1}^{\nu} V_k\right) \end{array}$$

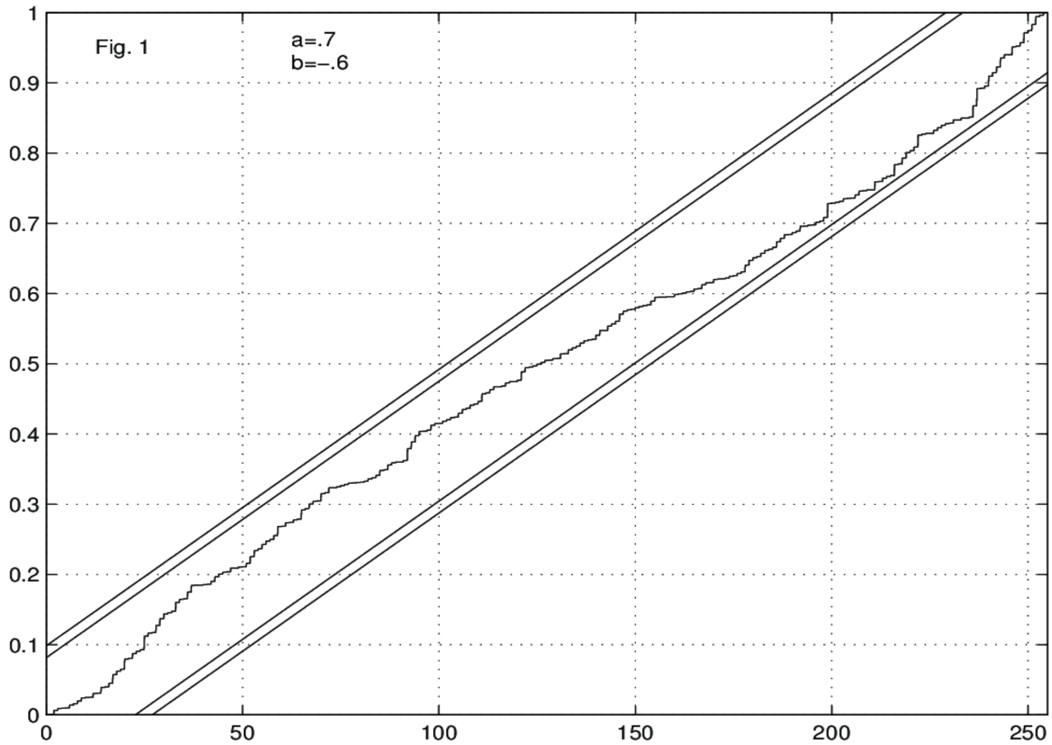

Fig. 1. The standardized cumulative normalized periodogram $K(x)$ and the Kolmogorov-Smirnov bounds with the significance level $\alpha = 0.05$ (inner) and $\alpha = 0.01$ (outer) for the Example 2. The data $\{X_1, \ldots, X_{512}\}$ is generated by the model $X_t + 0.7X_{t-1} = \varepsilon_t - 0.6\,\varepsilon_{t-1}$.

are continuous, then the $U_i = h_i(V_1, \ldots, V_\nu)$, $i = 1, \ldots, \nu - 1$ converge in distribution to $Z_i = h_i(Y_1, \ldots, Y_\nu)$, $i = 1, \ldots, \nu - 1$.

Now, let us deal with the distribution of the random variables

$$Z_i = h_i(Y_1, \ldots, Y_\nu) = (\sum_{k=1}^{i} Y_k)/(\sum_{k=1}^{\nu} Y_k), \quad i = 1, \ldots, \nu - 1.$$

Since the Y_k, $k = 1, \ldots, \nu$ are independent and identically distributed as $\chi^2(2)/2$, the joint density function of $(Y_1, \ldots, Y_\nu)$ is

$$f_{Y_1 \ldots Y_\nu}(y_1, \ldots, y_\nu) = \left(e^{-\sum_{i=1}^{\nu} y_i}\right) \prod_{j=1}^{\nu} \mathbb{I}_{\{y_i \geq 0\}}\,, \tag{19}$$

where

$$\mathbb{I}_{\{y_j \geq 0\}} = \begin{cases} 1 \text{ if } y_i \geq 0 \\ 0 \text{ otherwise} . \end{cases}$$

Let $S_i = \sum_{k=1}^{i} Y_k$, $i = 1, \ldots, \nu$. This equivalent to write $Y_1 = S_1$ and $Y_i = S_i - S_{i-1}$, $i = 2, \ldots, \nu$.

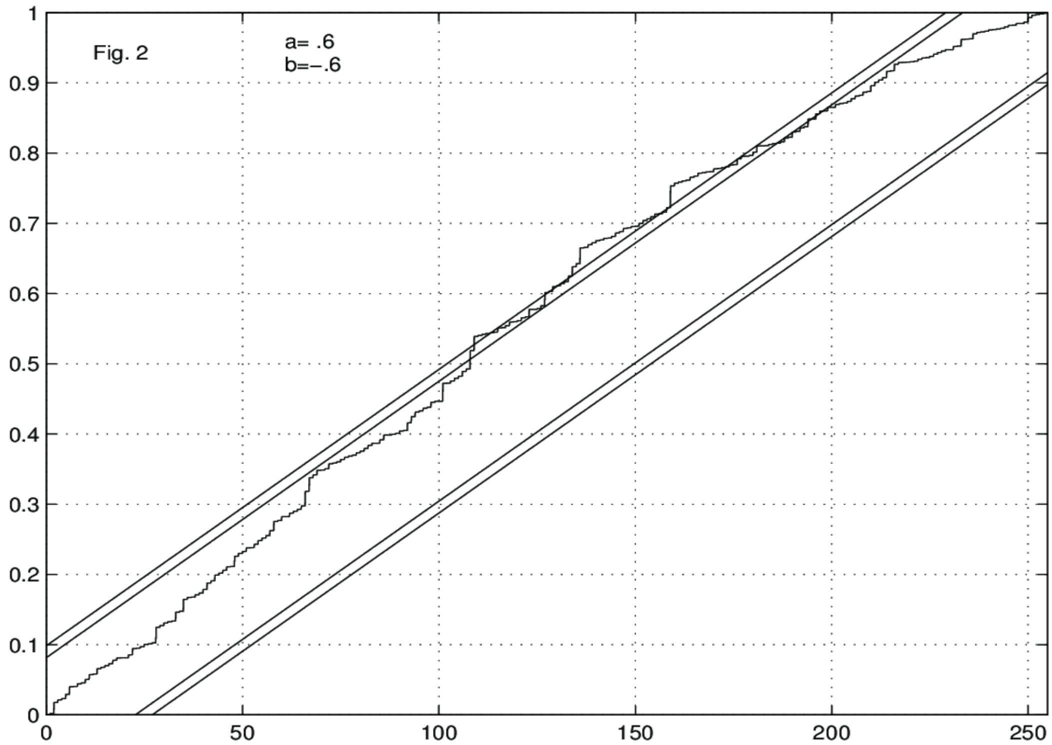

Fig. 2. The standardized cumulative normalized periodogram $K(x)$ and the Kolmogorov-Smirnov bounds with the significance level $\alpha = 0.05$ (inner) and $\alpha = 0.01$ (outer) for the Example 2. The data $\{X_1, \ldots, X_{512}\}$ is generated by the model $X_t + 0.6X_{t-1} = \varepsilon_t - 0.6\,\varepsilon_{t-1}$.

The jacobian of the transformation

$$(Y_1, \ldots, Y_\nu) \longrightarrow \tau_\nu(Y_1, \ldots, Y_\nu) = (S_1, \ldots, S_\nu)$$

is equal to 1, it follows from (19), that the joint density function of $(S_1, \ldots, S_\nu)$ is

$$\begin{aligned} f_{S_1 \ldots S_\nu}(s_1, \ldots, s_\nu) &= (e^{-s_\nu}) \left(\mathbb{1}_{\{s_1 \geq 0\}} \prod_{j=2}^{\nu} \mathbb{1}_{\{s_i - s_{i-1} \geq 0\}} \right) \\ &= (e^{-s_\nu}) \mathbb{1}_{\{0 \leq s_1 \leq \cdots \leq s_\nu\}}. \end{aligned} \tag{20}$$

The marginal density of S_ν is the probability density function of the sum of ν independent standard $(\chi^2(2)/2)$ exponential random variables. Thus,

$$f_{S_\nu}(s_\nu) = \frac{s_\nu^{\nu-1}}{(\nu-1)!} (e^{-s_\nu}) \mathbb{1}_{\{s_\nu \geq 0\}} \;. \tag{21}$$

Using (20) and (21), the conditional density of $(S_1, \ldots, S_{\nu-1})$ given S_ν is

$$\begin{aligned} f_{S_1 \cdots S_{\nu-1}/S_\nu}(s_1, \ldots, s_{\nu-1}) &= \frac{f_{S_1 \ldots S_\nu}(s_1 \ldots, s_\nu)}{f_{S_\nu}(s_\nu)} \\ &= (\nu-1)! s_\nu^{1-\nu} \mathbb{1}_{\{0 \le s_1 \le \cdots \le s_\nu\}}. \end{aligned} \tag{22}$$

By definition $Z_i = S_i/S_\nu$, $i = 1, \ldots, \nu - 1$; the conditional density $f_{Z_1 \ldots Z_{\nu-1}/S_\nu}(z_1, \ldots, z_{\nu-1}/s_\nu)$ of $(Z_1, \ldots, Z_{\nu-1})$ given S_ν is obtained using the transformation $\eta : \mathbb{R}^{\nu-1} \longrightarrow \mathbb{R}^{\nu-1}$ defined by $(s_1, \ldots, s_{\nu-1}) \longmapsto (z_1, \ldots, z_{\nu-1}) = \eta(s_1, \ldots, s_{\nu-1})$ where $z_i = s_i/s_\nu$, $i = 1, \ldots, \nu - 1$ (s_ν fixed).

The jacobian of the transformation η is equal to $(s_\nu)^{\nu-1}$, it follows from (22) that the conditional density function of $(Z_1, \ldots, Z_{\nu-1})$ given S_ν is

$$\begin{aligned} f_{Z_1 \cdots Z_{\nu-1}/S_\nu}(z_1, \ldots, z_{\nu-1}/s_\nu) &= (s_\nu^{\nu-1})(\nu-1)! s_\nu^{1-\nu} \mathbb{1}_{\{0 \le s_\nu z_1 \le \cdots \le s_\nu z_{\nu-1} \le s_\nu\}} \\ &= (\nu-1)! \mathbb{1}_{\{0 \le z_1 \le \cdots \le z_{\nu-1} \le 1\}} . \end{aligned}$$

The expression of this conditional density function of $(Z_1, \ldots, Z_{\nu-1})$ given S_ν does not depend on s_ν, we can write the unconditional joint density of $(Z_1, \ldots, Z_{\nu-1})$ as

$$f_{Z_1 \cdots Z_{\nu-1}}(z_1, \ldots, z_{\nu-1}) = (\nu-1)! \mathbb{1}_{\{0 \le z_1 \le \cdots \le z_{\nu-1} \le 1\}}$$

which is precisely the joint density of the order statistics of random sample of size $(\nu - 1)$ from uniform distribution $(0, 1)$. □

Proof of Corollary 1. According to Theorem 1, the random variables U_i, $i = 1, \ldots, \nu - 1$ are asymptotically distributed as the order statistics of a sample $(\nu-1)$ independent random variables. Then under the null hypothesis H_0, T_ν is asymptotically distributed as the length of the largest subinterval of $(0, 1)$ obtained when the interval is randomly partitioned by $(\nu-1)$ points independently and uniformly distributed on $(0, 1)$. The explicit expression (9) of the distribution function of this length is shown by (Feller, 1971, page 29). □

Proof of Corollary 2. The result is an immediate consequence of Theorem 1. □

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