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PERIODIC CONSTANT MEAN CURVATURE SURFACES IN $\mathbb{H}^2 \times \mathbb{R}^*$

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1. Introduction. A properly embedded surface Σ in $\mathbb{H}^2 \times \mathbb{R}$, invariant by a non-trivial discrete group of isometries of $\mathbb{H}^2 \times \mathbb{R}$, will be called a periodic surface. We will discuss periodic minimal and constant mean curvature surfaces. At this time, there is little theory of these surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and other homogeneous 3-manifolds, with the exception of the space forms.

The theory of doubly periodic minimal surfaces (invariant by a $\mathbb{Z}^2$ group of isometries) in $\mathbb{R}^3$ is well developed. Such a surface in $\mathbb{R}^3$, not a plane, is given by a properly embedded minimal surface in $\mathbb{T} \times \mathbb{R}$, $\mathbb{T}$ some flat 2-torus. One main theorem is that a finite topology complete embedded minimal surface in $\mathbb{T} \times \mathbb{R}$ has finite total curvature and one knows the geometry of the ends [11]. It is very interesting to understand this for such minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$, $\mathbb{M}^2$ a closed hyperbolic surface.

In this paper we will consider periodic surfaces in $\mathbb{H}^2 \times \mathbb{R}$. The discrete groups G of isometries of $\mathbb{H}^2 \times \mathbb{R}$ we consider are generated by horizontal translations ϕ_l along geodesics γ of $\mathbb{H}^2$ and/or a vertical translation $T(h)$ by some $h > 0$. We denote by $\mathbb{M}$ the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by G.

In the case G is the $\mathbb{Z}^2$ subgroup of the isometry group generated by ϕ_l and $T(h)$, $\mathbb{M}$ is diffeomorphic but not isometric to $\mathbb{T} \times \mathbb{R}$. Moreover $\mathbb{M}$ is foliated by the family of tori $\mathbb{T}(s) = (d(s) \times \mathbb{R})/G$ (here $d(s)$ is an equidistant to γ). All the $\mathbb{T}(s)$ are intrinsically flat and have constant mean curvature; $\mathbb{T}(0)$ is totally geodesic. In Section 3, we will prove an Alexandrov-type theorem for doubly periodic H-surfaces, i.e., an analysis of compact embedded constant mean curvature surfaces in such a $\mathbb{M}$ (Theorem 3.1).

The remainder of the paper is devoted to construct examples of periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

The first example we want to illustrate is the singly periodic Scherk minimal surface. In $\mathbb{R}^3$, it can be understood as the desingularization of two orthogonal planes. H. Karcher [5] has generalized this to desingularize k planes of $\mathbb{R}^3$ meeting along a line at equal angles, these are called Saddle Towers. In $\mathbb{H}^2 \times \mathbb{R}$, two situations are similar to these examples: the intersection of a vertical plane with the horizontal slice $\mathbb{H}^2 \times \{0\}$ and the intersection of k vertical planes meeting along a vertical geodesic at equal angles. These surfaces, constructed in Section 4, are singly periodic and called,

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respectively, “horizontal singly periodic Scherk minimal surfaces” and “vertical Saddle Towers”. For vertical intersections, the situation is in fact more general and was treated by F. Morabito and the second author in [13]; here we give another approach which is more direct (see also J. Pyo [17]).

In Section 5, we construct doubly periodic minimal examples. The first examples we obtain, called “doubly periodic Scherk minimal surfaces” bounded by four horizontal geodesics; two at height zero, and two at height $h > \pi$. The latter two geodesics are the vertical translation of the two at height zero. Each one of these Scherk surfaces has two “left-side” ends asymptotic to two vertical planar strips, and two “right-side” ends, asymptotic to the horizontal slices at heights zero and h. By recursive rotations by π about the horizontal geodesics, we obtain a doubly periodic minimal surface.

The other doubly periodic minimal surfaces of $\mathbb{H}^2 \times \mathbb{R}$ constructed in Section 5 are analogous to some of Karcher's Toroidal Halfplane Layers of $\mathbb{R}^3$ (more precisely, the ones denoted by $M_{\theta,0,\pi/2}$, $M_{\theta,\pi/2,0}$ and $M_{\theta,0,0}$ in [19]). The examples we construct, also called Toroidal Halfplane Layers, are all bounded by two horizontal geodesics at height zero, and its translated copies at height $h > 0$. Each ot these Toroidal Halfplane Layers has two “left-side” ends and two “right-side” ends, all of them asymptotic to either vertical planar strips or horizontal strips, bounded by the horizontal geodesics in its boundary. By recursive rotations by π about the horizontal geodesics, we obtain a doubly periodic minimal surface. In the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by a horizontal hyperbolic translation and a vertical translation leaving invariant the surface, we get a finitely punctured minimal torus and Klein bottle in $\mathbb{T} \times \mathbb{R}$, $\mathbb{T}$ some flat 2-torus.

Finally, in Section 6, we construct a periodic minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ analogous to the most symmetric Karcher's Toroidal Halfplane Layer in $\mathbb{R}^3$ (denoted by $M_{\theta,0,0}$ in [19]). A fundamental domain of this latter surface can be viewed as two vertical strips with a handle attached. This piece is a bigraph over a domain Ω in the parallelogram of the $\mathbb{R}^2 \times \{0\}$ plane whose vertices are the horizontal projection of the four vertical lines in the boundary of the domain, and the upper graph has boundary values 0 and $+\infty$: The trace of the surface on $\mathbb{R}^2 \times \{0\}$ is the two concave curves in the boundary of Ω. They are geodesic lines of curvature on the surface and their concavity makes the construction of these surfaces delicate. We refer to [5, 11, 19], where they are constructed by several methods. The complete surface is obtained by rotating by π about the vertical lines in the boundary. Considering the quotient of $\mathbb{R}^3$ by certain horizontal translations leaving invariant the surface, yields finitely punctured minimal tori and Klein bottles in $\mathbb{T} \times \mathbb{R}$.

The surface we construct in $\mathbb{H}^2 \times \mathbb{R}$ will have a fundamental domain Σ which may be viewed as k vertical strips ($k \geq 3$) to which one attaches a sphere with k disks removed. Σ is a vertical bigraph over a domain $\Omega \subset \mathbb{H}^2 \times \{0\} \equiv \mathbb{H}^2$; $\partial\Omega$ has $2k$ smooth arcs $A_1, B_1, \cdots, A_k, B_k$ in that order. Each A_i is a geodesic and each B_j is concave towards Ω. The A_i's are of equal length and the B_j's as well. The convex hull of the vertices of Ω is a polygonal domain $\widetilde{\Omega}$ that tiles $\mathbb{H}^2$; the interior angles of the vertices of $\widetilde{\Omega}$ are $\pi/2$. Thus Σ extends to a periodic minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ by symmetries: rotation by π about the vertical geodesic lines over the vertices of $\partial\Omega$.

The surface $\Sigma_+ = \Sigma \cap (\mathbb{H}^2 \times \mathbb{R}^+)$ is a graph over Ω with boundary values as indicated in Figure 1 (here $k = 4$). Σ_+ is orthogonal to $\mathbb{H}^2 \times \{0\}$ along the concave arcs B_j so Σ is the extension of Σ_+ by symmetry through $\mathbb{H}^2 \times \{0\}$.

Σ will be constructed by solving a Plateau problem for a certain contour and taking the conjugate surface of this Plateau solution. The result will be the part of

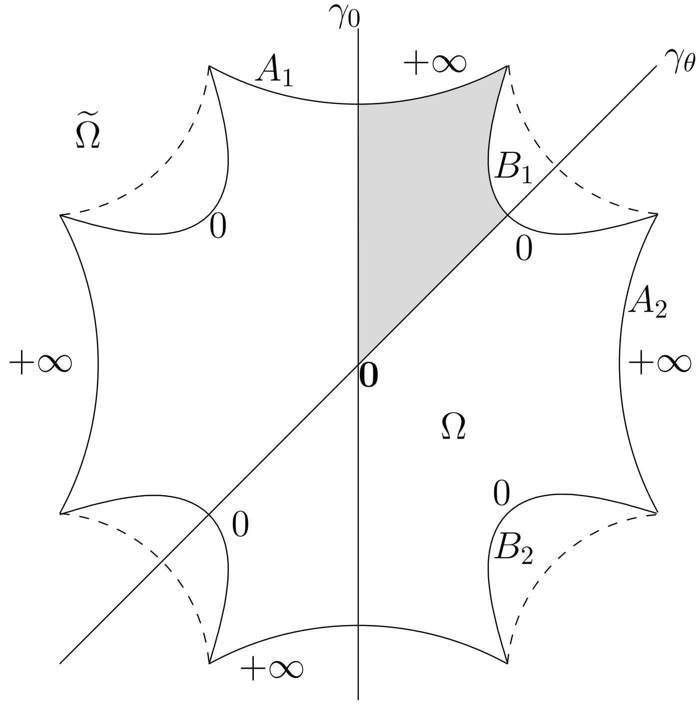

FIG. 1. *The domain* Ω

Σ_+ which is a graph over the shaded region on Ω in the Figure 1. This graph meets the vertical plane over γ_0 and γ_θ orthogonally, so extends by symmetry in these vertical planes. Σ_+ is then obtained by going around $\mathbf{0}$ by k symmetries.

2. Preliminaries.

2.1. Notation. In this paper, the Poincaré disk model is used for the hyperbolic plane, i.e.

$$\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

with the hyperbolic metric $g_{-1} = \frac{4}{(1-x^2-y^2)^2} g_0$, where g_0 is the Euclidean metric in $\mathbb{R}^2$. Thus x and y will be used as coordinates in the hyperbolic plane. We denote by $\mathbf{0}$ the origin $(0,0)$ of $\mathbb{H}^2$. In this model, the asymptotic boundary $\partial_\infty \mathbb{H}^2$ of $\mathbb{H}^2$ is identified with the unit circle. So any point in the closed unit disk is viewed as either a point in $\mathbb{H}^2$ or a point in $\partial_\infty \mathbb{H}^2$.

Let $\theta \in \mathbb{R}$. In $\mathbb{H}^2$, we denote by γ_θ the geodesic line $\{-x\cos\theta + y\sin\theta = 0\}$ and by γ_θ^+ the half geodesic line from $\mathbf{0}$ to $(\sin\theta, \cos\theta)$. We also denote by T_θ the hyperbolic angular sector $\{(r\sin u, r\cos u) \in \mathbb{H}^2, r \in [0,1), u \in [0,\theta]\}$.

For $\mu \in (-1,1)$ we denote by $g(\mu)$ the complete geodesic of $\mathbb{H}^2$ orthogonal to γ_0 at $q_\mu = (0,\mu)$. We have $g(0) = \gamma_{\pi/2}$. We also denote $g^+(\mu) = g(\mu) \cap \{x > 0\}$.

Fixed $\theta \in \mathbb{R}$, there exists a Killing vector field Y_θ which has length 1 along γ_θ and generated by the hyperbolic translation along γ_θ with $(\sin\theta, \cos\theta)$ as attractive fixed point at infinity. For $l \in (-1,1)$, we denote by ϕ_l the hyperbolic translation along γ_θ with $\phi_l(\mathbf{0}) = (l\sin\theta, l\cos\theta)$. $(\phi_l)_{l\in(-1,1)}$ is called the "flow" of Y_θ, even though

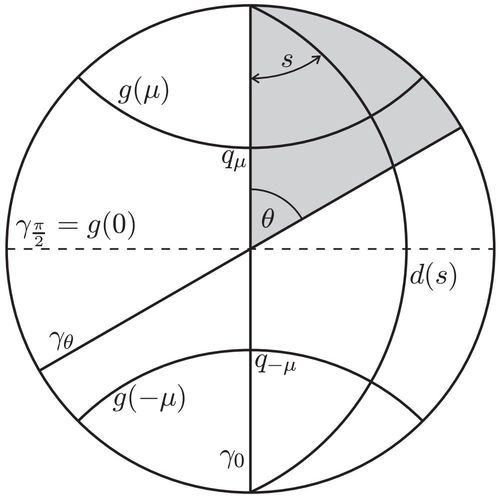

FIG. 2. *The hyperbolic angular sector T_θ corresponds to the shadowed domain.*

the family $(\phi_l)_{l\in(-1,1)}$ is not parameterized at the right speed. We notice that, if $(\phi_l)_{l\in(-1,1)}$ is the flow of Y_0, $g(\mu) = \phi_\mu(g(0))$.

For $\theta \in \mathbb{R}$, there is another interesting vector field that we denote by Z_θ. This vector field is the unit vector field normal to the foliation of $\mathbb{H}^2$ by the equidistant lines to $\gamma_{\theta+\pi/2}$ such that $Z_\theta(\mathbf{0}) = (1/2)(\sin\theta\partial_x + \cos\theta\partial_y)$. We notice that Z_θ is not a Killing vector field. This time, we define $(\psi_s)_{s\in\mathbb{R}}$ the flow of Z_θ (with the right speed). If $(\psi_s)_{s\in\mathbb{R}}$ is the flow of $Z_{\pi/2}$, we define $d(s) = \psi_s(\gamma_0)$ for s in $\mathbb{R}$. $d(s)$ is one of the equidistant lines to γ_0 at distance $|s|$. We remark that $Z_{\pi/2}$ is tangent to the geodesic lines $g(\mu)$.

In the sequel, we denote by t the height coordinate in $\mathbb{H}^2 \times \mathbb{R}$. Besides, we will often identify the hyperbolic plane $\mathbb{H}^2$ with the horizontal slice $\{t = 0\}$ of $\mathbb{H}^2 \times \mathbb{R}$. The Killing vector field Y_θ and its flow naturally extend to a horizontal Killing vector field and its flow in $\mathbb{H}^2 \times \mathbb{R}$. The same occurs for Z_θ and its flow.

We denote by $\pi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2$ the vertical projection and by $T(h)$ the vertical translation by h. Given two points p and q of $\mathbb{H}^2$ or $\mathbb{H}^2 \times \mathbb{R}$, we denote by $\overline{pq}$ the geodesic arc between these two points.

2.2. Conjugate minimal surface. B. Daniel [2] and L. Hauswirth, R. Sa Earp and E. Toubiana [4] have proved that minimal disks in $\mathbb{H}^2 \times \mathbb{R}$ have an associated family of locally isometric minimal surfaces. In this subsection we briefly recall how they are defined.

Let $X = (\varphi, h) : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal minimal immersion, with Σ a simply connected Riemann surface. Then h is a real harmonic function and $\varphi = \pi \circ X$ is a harmonic map to $\mathbb{H}^2$. Let h^* be the real harmonic conjugate function of h and $\mathcal{Q}_\varphi$ be the Hopf differential of φ. Since X is conformal, we have

$$\mathcal{Q}_\varphi = -4\left(\frac{\partial h}{\partial z}\right)^2 dz^2,$$

where z is a conformal parameter on Σ. In [2] and [4] it has been proved that, for any $\theta \in \mathbb{R}$, there exists a minimal immersion $X_\theta = (\varphi_\theta, h_\theta) : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ whose induced metric on Σ coincides with the one induced by X, and such that $h_\theta = \cos\theta h + \sin\theta h^*$

and the Hopf differential of φ_θ is $\mathcal{Q}_{\varphi_\theta} = e^{-2i\theta}\mathcal{Q}_\varphi$. If N (resp. N_θ) denotes the unit normal to X (resp. X_θ), then $\langle N, \partial_t\rangle = \langle N_\theta, \partial_t\rangle$ (i.e. their angle maps coincide).

All these immersions X_θ are well-defined up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$. The immersion $X_{\pi/2}$ is called the conjugate immersion of X (and $X_{\pi/2}(\Sigma)$ is usually called conjugate minimal surface of $X(\Sigma)$), and it is denoted by X^*.

The data for the conjugate surface are the same as for $X(\Sigma)$, except that one rotates S and T by $\pi/2$: $S^* = JS$, and $T^* = JT$. Here S (resp. S^*) denotes the symmetric operator on Σ induced by the shape operator of $X(\Sigma)$ (resp. $X^*(\Sigma)$); T (resp. T^*) is the vector field on Σ such that $dX(T)$ (resp. $dX^*(T^*)$) is the projection of ∂_t on the tangent plane of $X(\Sigma)$ (resp. $X^*(\Sigma)$); and J is the rotation of angle $\pi/2$ on $T\Sigma$. See [2] for more details.

For C a curve on Σ, the normal curvature of $X(C)$ in the surface $X(\Sigma)$ is $-\langle C', S(C')\rangle$, and the normal torsion is $\langle J(C'), S(C')\rangle$. Thus the normal torsion of $X^*(C)$ on the conjugate surface $X^*(\Sigma)$ is minus the normal curvature of $X(C)$ on $X(\Sigma)$, and the normal curvature of $X^*(C)$ on $X^*(\Sigma)$ is the normal torsion of $X(C)$ on $X(\Sigma)$. In particular, if $X(C)$ is a vertical ambient geodesic on $X(\Sigma)$, then $X^*(C)$ is a horizontal line of curvature on the conjugate surface $X^*(\Sigma)$ whose geodesic curvature in the horizontal plane is the normal torsion on $X(\Sigma)$. Arguing similarly, we get that the correspondence $X \leftrightarrow X^*$ maps:

- vertical geodesic lines to horizontal geodesic curvature lines along which the normal vector field of the surface is horizontal; and
- horizontal geodesics to geodesic curvature lines contained in vertical geodesic planes Π (i.e. $\pi(\Pi)$ is a geodesic of $\mathbb{H}^2$) along which the normal vector field is tangent to Π.

Moreover, this correspondence exchanges the corresponding Schwarz symmetries of the surfaces X and X^*. For more definitions and properties, we refer to [2, 4].

2.3. Some results about graphs. In $\mathbb{H}^2 \times \mathbb{R}$, there exist different notions of graphs, depending on the vector field considered.

If u is a function on a domain Ω of $\mathbb{H}^2$, the graph of u, defined as

$$\Sigma_u = \{(p, u(p)) \mid p \in \Omega\},$$

is a surface in $\mathbb{H}^2 \times \mathbb{R}$. This surface is minimal (a vertical minimal graph) if u satisfies the vertical minimal graph equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}\right) = 0, \tag{1}$$

where all terms are calculated with respect to the hyperbolic metric.

If u is a solution of equation (1) on a convex domain of $\mathbb{H}^2$, L. Hauswirth, R. Sa Earp and E. Toubiana have proved in [4] that the conjugate minimal surface Σ_u^* of Σ_u is also a vertical graph.

Assume Ω is simply connected. The differential on Ω of the height coordinate of Σ_u^* is the closed 1-form

$$\omega_u^*(X) = \langle \frac{\nabla u^\perp}{\sqrt{1+\|\nabla u\|^2}}, X\rangle_{\mathbb{H}^2}, \tag{2}$$

where $\nabla u^\perp$ is the vector ∇u rotated by $\pi/2$. The height coordinate of Σ_u^* is a primitive h_u^* of ω_u^* and is the conjugate function of h_u on Σ_u. The formula (2) comes from the following computation. Let h be the height function along the graph surface and h^* its conjugate harmonic function. Let (e_1, e_2) be an orthonormal basis of the tangent space to $\mathbb{H}^2$ and $X = x_1 e_1 + x_2 e_2$ a tangent vector. Then

$$\omega_u^*(X) = dh^*(X + \langle \nabla u, X\rangle_{\mathbb{H}^2}\partial_t) = dh(N_u \wedge (X + \langle \nabla u, X\rangle_{\mathbb{H}^2}\partial_t)),$$

where $N_u = (\nabla u - \partial t)/W$ (with $W = \sqrt{1 + \|\nabla u\|^2}$). If $\nabla u = u_1 e_1 + u_2 e_2$ we have

$$N_u \wedge (X + \langle \nabla u, X\rangle_{\mathbb{H}^2}\partial_t) = \frac{u_2\langle \nabla u, X\rangle_{\mathbb{H}^2} + x_2}{W} e_1 - \frac{u_1\langle \nabla u, X\rangle_{\mathbb{H}^2} + x_1}{W} e_2 + \frac{u_1 x_2 - u_2 x_1}{W}\partial_t.$$

Thus

$$\omega_u^*(X) = \frac{u_1 x_2 - u_2 x_1}{W} = \langle \frac{\nabla u^\perp}{\sqrt{1 + \|\nabla u\|^2}}, X\rangle_{\mathbb{H}^2}.$$

Let us now fix $\theta \in \mathbb{R}$. Recall that $(\phi_l)_{l\in(-1,1)}$ is the flow of the Killing vector field Y_θ. Let D be a domain in the vertical geodesic plane $\gamma_{\theta+\pi/2} \times \mathbb{R}$ (this plane is orthogonal to γ_θ, viewed as a geodesic of $\{t = 0\}$). Let v be a function on D with values in $(-1, 1)$. Then, the surface $\{\phi_{v(p)}(p) \mid p \in D\}$ is called a Y_θ-graph. It is a graph with respect to the Killing vector field Y_θ in the sense that it meets each orbit of Y_θ in at most one point. If such a surface is minimal, it is called a minimal Y_θ-graph. Let v' be a second function defined on a domain of $\gamma_{\theta+\pi/2} \times \mathbb{R}$. If $v' \geq v$ on the intersection of their domains of definition, we say that the Y_θ-graph of v' lies on the positive Y_θ-side of the Y_θ-graph of v.

The same notion can be defined for the vector field Z_θ. If D is a domain in the vertical geodesic plane $\gamma_{\theta+\pi/2} \times \mathbb{R}$ and v is a function on D with values in $\mathbb{R}$, the surface $\{\psi_{v(p)}(p) \mid p \in D\}$ is called a Z_θ-graph ($(\psi_s)_{s\in\mathbb{R}}$ is the flow of Z_θ). This surface is a graph with respect to Z_θ since it meets each orbit of Z_θ in at most one point.

3. The Alexandrov problem for doubly periodic constant mean curvature surfaces. Let $(\phi_l)_{l\in(-1,1)}$ be the flow of Y_0 and consider G the $\mathbb{Z}^2$ subgroup of $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$ generated by ϕ_l and $T(h)$, for some positive l and h. We denote by $\mathbb{M}$ the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by G. The manifold $\mathbb{M}$ is diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}$. Moreover, $\mathbb{M}$ is foliated by the family of tori $\mathbb{T}(s) = (d(s) \times \mathbb{R})/G$, $s \in \mathbb{R}$ (we recall that $d(s)$ is an equidistant to γ_0). All the $\mathbb{T}(s)$ are intrinsically flat and have constant mean curvature $\tanh(s)/2$; $\mathbb{T}(0)$ is totally geodesic.

In this section, we study compact embedded constant mean curvature surfaces in $\mathbb{M}$. The tori $T(s)$ are examples of such surfaces when $0 \leq H < 1/2$.

First, let us observe what happens in $(\mathbb{H}^2 \times \mathbb{R})/G'$, where G' is the subgroup generated by $T(h)$. This quotient is isometric to $\mathbb{H}^2 \times \mathbb{S}^1$. Let Σ be a compact embedded constant mean curvature H surface in $\mathbb{H}^2 \times \mathbb{S}^1$. The surface Σ separates $\mathbb{H}^2 \times \mathbb{S}^1$. Indeed, if it is not the case, there exists a smooth Jordan curve whose intersection number with Σ is 1 modulo 2. In $\mathbb{H}^2 \times \mathbb{S}^1$, this Jordan curve can be moved so that it does not intersect Σ any more, which is impossible since the intersection number modulo 2 is invariant by homotopy.

Now, we consider γ a geodesic in $\mathbb{H}^2$ and $(\ell_s)_{s\in\mathbb{R}}$ the family of geodesics in $\mathbb{H}^2$ orthogonal to γ that foliates $\mathbb{H}^2$. By the maximum principle using the vertical annuli $\ell_s \times \mathbb{S}^1$, we get that $H > 0$, since Σ is compact. We can apply the standard Alexandrov reflection technique with respect to the family $(\ell_s \times \mathbb{S}^1)_{s\in\mathbb{R}}$. We obtain that Σ is symmetric with respect to some $\ell_{s_0} \times \mathbb{S}^1$. Doing this for every γ, one proves that Σ is a rotational surface around a vertical axis $\{p\}\times\mathbb{S}^1$ ($p \in \mathbb{H}^2$). Σ is then either a constant mean curvature sphere coming from the spheres of $\mathbb{H}^2 \times \mathbb{R}$ or the quotient by G' of a vertical cylinder or unduloid of axis $\{p\} \times \mathbb{R}$. This proves that, necessarily, $H > 1/2$. These surfaces are the only ones in $\mathbb{H}^2 \times \mathbb{S}^1$ which have a compact projection on $\mathbb{H}^2$. In $\mathbb{H}^2 \times \mathbb{R}$, determining which properly embedded CMC surfaces have a bounded projection on $\mathbb{H}^2$ (*i.e.* is included in a vertical cylinder) is an open question. Laurent Mazet has made progress on this problem [8].

The spheres, the cylinders and the unduloids can also be quotiented by G, if they are well placed in $\mathbb{H}^2 \times \mathbb{R}$ with respect to $\gamma_0 \times \mathbb{R}$. They give examples of compact embedded CMC surfaces in $\mathbb{M}$ for $H > 1/2$.

We remark that the vector field $Z_{\pi/2}$ is invariant by the group G, so it is well defined in $\mathbb{M}$. Moreover its integral curves are the geodesics orthogonal to $\mathbb{T}(0)$. This implies that the notion of $Z_{\pi/2}$ graph is well defined in $\mathbb{M}$. We have the following answer to the Alexandrov problem in $\mathbb{M}$.

Theorem 3.1. *Let $\Sigma \subset \mathbb{M}$ be a compact constant mean curvature embedded surface. Then, Σ is either:*

1. *a torus $\mathbb{T}(s)$, for some s; or*
2. *a "rotational" sphere; or*
3. *the quotient of a vertical unduloid (in particular, a vertical cylinder over a circle); or*
4. *a $Z_{\pi/2}$-bigraph with respect to $\mathbb{T}(0)$.*

Moreover, if Σ is minimal, then $\Sigma = \mathbb{T}(0)$.

The first thing we have to remark is that the last item can occur. Let γ be an embedded compact geodesic in the totally geodesic torus $\mathbb{T}(0)$. From a result by R. Mazzeo and F. Pacard [10], we know that there exist embedded constant mean curvature tubes that partially foliate a tubular neighborhood of γ. So if γ is not vertical, these cmc surfaces can not be of one of the three first type. If fact, these surfaces can be also directly derived from [18] (see also [15]). They have mean curvature larger than 1/2.

The second remark is that we do not know if there exist constant mean curvature 1/2 examples. If they exist, they are of the fourth type.

Very recently, J.M. Manzano and F. Torralbo [6] construct, for each value of $H > 1/2$, a 1-parameter family of "horizontal unduloidal-type surfaces" in $\mathbb{H}^2 \times \mathbb{R}$ of bounded height which are invariant by a fixed ϕ_l. All these examples are embedded vertical bigraphs. The limit surfaces in the boundary of this family are a rotational sphere and a horizontal cylinder.

Proof. Let Σ be a compact embedded constant mean curvature surface in $\mathbb{M}$ and consider a connected component $\widetilde{\Sigma}$ of its lift to $\mathbb{H}^2 \times \mathbb{S}^1$. If $\widetilde{\Sigma}$ is compact, the above study proves that we are then in cases *2* or *3*. We then assume that $\widetilde{\Sigma}$ is not compact. Even if $\widetilde{\Sigma}$ is not compact, the same argument as above proves that it separates $\mathbb{H}^2 \times \mathbb{S}^1$ into two connected components. We also assume that $\widetilde{\Sigma} \neq \gamma_0 \times \mathbb{S}^1$ (otherwise we are in Case *1*). Then, up to a reflection symmetry with respect to $\gamma_0 \times \mathbb{S}^1$, we can assume that $\widetilde{\Sigma} \cap (\{x \geq 0\} \times \mathbb{S}^1)$ is non empty.

Let γ be an integral curve of $Z_{\pi/2}$, *i.e.* a geodesic orthogonal to $\gamma_0 \times \mathbb{S}^1$. We denote by $P(s)$ the totally geodesic vertical annulus of $\mathbb{H}^2 \times \mathbb{S}^1$ which is normal to γ and tangent to $d(s) \times \mathbb{S}^1$. Since $\widetilde{\Sigma}$ is a lift of the compact surface Σ, $\widetilde{\Sigma}$ stays at a finite distance from $\gamma_0 \times \mathbb{S}^1$. Far from γ, the distance from $P(s)$ to $\gamma_0 \times \mathbb{S}^1 = P(0)$ tends to $+\infty$, if $s \neq 0$. Thus $P(s) \cap \widetilde{\Sigma}$ is compact for $s \neq 0$, and it is empty if $|s|$ is large enough. So start with s close to $+\infty$ and let s decrease until a first contact point between $\widetilde{\Sigma}$ and $P(s)$, for $s = s_0 > 0$. If $\widetilde{\Sigma}$ is minimal, by the maximum principle we get $\widetilde{\Sigma} = P(s_0)$. But the quotient of $P(s_0)$ is not compact in $\mathbb{M}$. We then deduce that $\widetilde{\Sigma}$ is not minimal. This proves that the only compact embedded minimal surface in $\mathbb{M}$ is $\mathbb{T}(0)$.

By the maximum principle, we know that the (non-zero) mean curvature vector of $\widetilde{\Sigma}$ does not point into $\cup_{s\geq s_0} P(s)$. Let us continue decreasing s and start the Alexandrov reflection procedure for $\widetilde{\Sigma}$ and the family of vertical totally geodesic annuli $P(s)$. Suppose there is a first contact point between the reflected part of $\widetilde{\Sigma}$ and $\widetilde{\Sigma}$, for some $s_1 > 0$. Then $\widetilde{\Sigma}$ is symmetric with respect to $P(s_1)$. Since $s_1 > 0$, then $\widetilde{\Sigma} \cap (\cup_{s_1\leq s\leq s_0} P(s))$ is compact. We get that $\widetilde{\Sigma}$ is compact, a contradiction. Hence we can continue the Alexandrov reflection procedure until $s = 0$ without a first contact point. This implies that $\widetilde{\Sigma} \cap (\{x \geq 0\} \times \mathbb{S}^1)$ is a Killing graph above $\gamma_0 \times \mathbb{S}^1$, for the Killing vector field Y corresponding to translations along γ (we notice that, along γ, Y and $Z_{\pi/2}$ coincide). Hence γ has at most one intersection point p with $\widetilde{\Sigma} \cap (\{x \geq 0\} \times \mathbb{S}^1)$ and this intersection is transverse.

Since at the first contact point between $\widetilde{\Sigma}$ and $P(s)$ (for $s = s_0$) the mean curvature vector of $\widetilde{\Sigma}$ does not point into $\cup_{s\geq s_0} P(s)$, we have that, for any $s' \in (0, s_0]$, the mean curvature vector of $\widetilde{\Sigma}$ on $\widetilde{\Sigma} \cap P(s')$ does not point into $\cup_{s\geq s'} P(s)$. In particular, the mean curvature vector of $\widetilde{\Sigma}$ at p points to the opposite direction as $\mathbb{Z}_{\pi/2}$. Doing this for every geodesic γ orthogonal to $\gamma_0 \times \mathbb{S}^1$, we get that $\widetilde{\Sigma} \cap (\{x \geq 0\} \times \mathbb{S}^1)$ is a $Z_{\pi/2}$ graph.

Now let us suppose that $\widetilde{\Sigma}$ is included in $\{x \geq 0\} \times \mathbb{S}^1$, and let $s_2 \geq 0$ and $s_3 > 0$ be the minimum and the maximum of the distance from $\widetilde{\Sigma}$ to $\gamma_0 \times \mathbb{S}^1$, respectively. Thus $\widetilde{\Sigma}$ is contained between $d(s_2) \times \mathbb{S}^1$ and $d(s_3) \times \mathbb{S}^1$. Because of the orientation of the mean curvature vector at the contact points of $\widetilde{\Sigma}$ with $d(s_2) \times \mathbb{S}^1$ and $d(s_3) \times \mathbb{S}^1$, we get

$$H_{d(s_2)\times\mathbb{S}^1} \geq H_{\widetilde{\Sigma}} \geq H_{d(s_3)\times\mathbb{S}^1}.$$

But $H_{d(s_2)\times\mathbb{S}^1} \leq H_{d(s_3)\times\mathbb{S}^1}$, hence $s_2 = s_3$ and $\widetilde{\Sigma} = d(s_2) \times \mathbb{S}^1$. This is, we are in Case *1*.

Then we assume that $\widetilde{\Sigma} \cap (\{x < 0\} \times \mathbb{S}^1)$ is non empty. Using the totally geodesic vertical annuli $P(s)$ for $s \leq 0$, we prove as above that $\widetilde{\Sigma} \cap (\{x \leq 0\} \times \mathbb{S}^1)$ is a $Z_{\pi/2}$ graph. Moreover the mean curvature vector points in the same direction as $Z_{\pi/2}$. This implies that $\widetilde{\Sigma}$ is normal to $\gamma_0 \times \mathbb{S}^1$. Thus, in the Alexandrov reflection procedure, a first contact point between the reflected part of $\widetilde{\Sigma}$ and $\widetilde{\Sigma}$ occurs for $s = 0$. $\widetilde{\Sigma}$ is then symmetric with respect to $P(0) = \gamma_0 \times \mathbb{S}^1$: we are in Case *4*. □

4. Minimal surfaces invariant by a $\mathbb{Z}$ subgroup. In this section, we are interested in constructing minimal surfaces which are invariant by a $\mathbb{Z}$ subgroup of $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$. At this time, only few non-trivial singly periodic examples are known: There are examples invariant by a one-parameter group of isometries [14, 20, 18, 21, 15, 9]; invariant by a vertical translation [3, 13]; or invariant by a horizontal translation along a horizontal geodesic [16].

The subgroups we consider are those generated by a translation ϕ_l along a horizontal geodesic or by a vertical translation $T(h)$ along ∂_t. The surfaces we construct are similar to Scherk's singly periodic minimal surfaces and Karcher's Saddle Towers of $\mathbb{R}^3$.

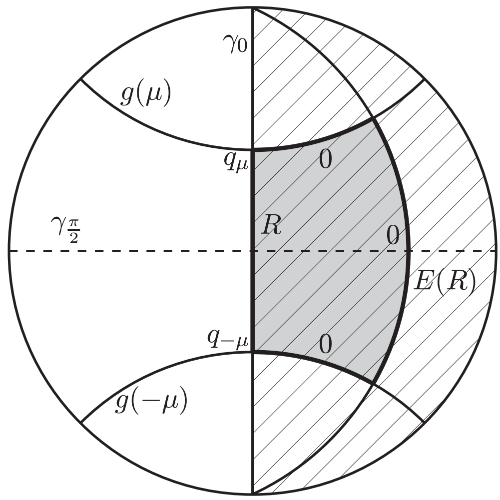

FIG. 3. *The shadowed domain is $\Omega(R)$, with the prescribed boundary data. The ruled region corresponds to D^+.*

4.1. Horizontal singly periodic Scherk minimal surfaces. In this subsection we construct a 1-parameter family of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, called "horizontal singly periodic Scherk minimal surfaces". Each of these surfaces can be seen as the desingularization of the intersection of a vertical geodesic plane and the horizontal slice $\mathbb{H}^2 \times \{0\}$, and it is invariant by a horizontal hyperbolic translation along the geodesic of intersection.

We fix $\mu \in (0,1)$ and define $q_\mu = (0,\mu)$ and $q_{-\mu} = (0,-\mu)$. Given $R > 0$, we denote by $\Omega(R)$ the compact domain in $\{x \geq 0\}$ between $E(R)$ and the geodesic lines $g(\mu)$, $g(-\mu)$, γ_0, where $E(R)$ is the arc contained in the equidistant line $d(R)$ which goes from $g(\mu)$ to $g(-\mu)$, see Figure 3. Let u_R be the solution to (1) over $\Omega(R)$ with boundary values zero on $\partial\Omega(R) \setminus \gamma_0$ and value R on $\overline{q_\mu q_{-\mu}}$ (minus its endpoints). By the maximum principle, $u_{R'} > u_R$ on Ω_R, for any $R' > R$.

Let us denote $D^+ = \{x \geq 0\}$ the hyperbolic halfplane bounded by γ_0. On D^+, we consider the solution v of (1) discovered by U. Abresch and R. Sa Earp, which takes value $+\infty$ on γ_0 and 0 on the asymptotic boundary $\partial_\infty D^+$ (see Appendix B). Such a v is a barrier from above for our construction, since we have $u_R \leq v$ for any R.

Since $(u_R)_R$ is a monotone increasing family bounded from above by v, we get that u_R converges as $R \to +\infty$ to a solution u of (1) on $\Omega(\infty) = \cup_{R>0}\Omega(R)$, with boundary values $+\infty$ over $\overline{q_\mu q_{-\mu}}$ (minus its endpoints) and 0 over the remaining boundary (including the asymptotic boundary $E(\infty)$ at infinity). In fact, this solution u, which is unique, can be directly derived from Theorem 4.9 in [9].

Let Σ_R be the minimal graph of u_R. Σ_R is in fact the solution to a Plateau problem in $\mathbb{H}^2 \times \mathbb{R}$ whose boundary is composed of horizontal and vertical geodesic arcs and the arc $E(R) \times \{0\}$. Let ϕ_l denote the flow of Y_0. Using the foliation of $\mathbb{H}^2 \times \mathbb{R}$ by the vertical planes $\phi_l(\gamma_{\pi/2} \times \mathbb{R}) = g(l) \times \mathbb{R}$, $l \in (-1,1)$, the Alexandrov reflection technique proves that Σ_R is a Y_0-bigraph with respect to $\gamma_{\pi/2} \times \mathbb{R}$. So $\Sigma_R^+ = \Sigma_R \cap \{y \geq 0\}$ is a Y_0-graph. Thus, the same is true for the minimal graph Σ of u and for $\Sigma^+ = \Sigma \cap \{y \geq 0\}$.

The boundary of Σ is composed of the vertical half-lines $\{q_\mu\} \times \mathbb{R}^+$, $\{q_{-\mu}\} \times \mathbb{R}^+$

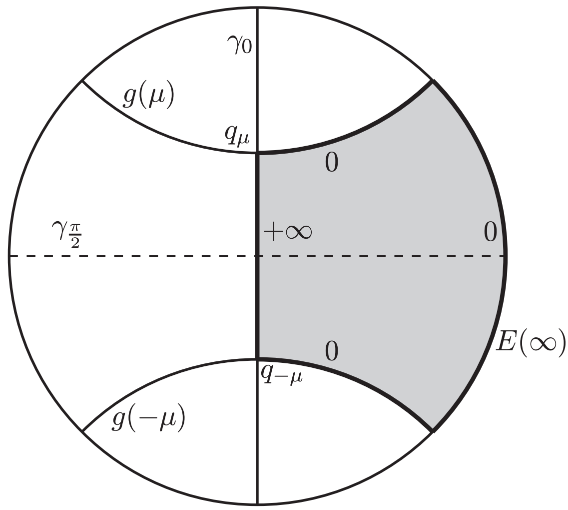

FIG. 4. *The domain* $\Omega(\infty)$ *with the prescribed boundary data.*

and the two halves $g^+(\mu), g^+(-\mu)$ of the horizontal geodesics $g(\mu), g(-\mu)$. The expected "horizontal singly periodic Scherk minimal surface" is obtained by rotating recursively Σ an angle π about the vertical and horizontal geodesics in its boundary. This "horizontal singly periodic Scherk minimal surface" is properly embedded, invariant by the horizontal translation ϕ_μ^4 along γ_0 and, far from $\gamma_0 \times \{0\}$, it looks like $(\gamma_0 \times \mathbb{R}) \cup \{t = 0\}$.

PROPOSITION 4.1. *For any* $\mu \in (0, 1)$, *there exists a properly embedded minimal surface* $\mathcal{M}_\mu$ *in* $\mathbb{H}^2 \times \mathbb{R}$ *invariant by the horizontal hyperbolic translation* ϕ_μ^4 *along* γ_0, *that we call* horizontal singly periodic Scherk minimal surface. *In the quotient by* ϕ_μ^4, $\mathcal{M}_\mu$ *is topologically a sphere minus four points corresponding to its ends: it has one top end asymptotic to* $(\gamma_0 \times \mathbb{R}^+)/\phi_\mu^4$, *one bottom end asymptotic to* $(\gamma_0 \times \mathbb{R}^-)/\phi_\mu^4$, *one left end asymptotic to* $\{t = 0, x < 0\}/\phi_\mu^4$, *and one right end asymptotic to* $\{t = 0, x > 0\}/\phi_\mu^4$. *Moreover,* $\mathcal{M}_\mu/\phi_\mu^4$ *contains the vertical lines* $\{q_{\pm\mu}\} \times \mathbb{R}$ *and the horizontal geodesics* $g(\pm\mu) \times \{0\}$, *and it is invariant by reflection symmetry with respect to the vertical geodesic plane* $\gamma_{\pi/2} \times \mathbb{R}$.

REMARK 4.2. **"Generalized horizontal singly periodic Scherk minimal surfaces".**

Consider the domain $\Omega(\infty)$ with prescribed boundary data $+\infty$ on $\overline{q_\mu q_{-\mu}}$, 0 on $g^+(\mu) \cup g^+(-\mu)$ and a continuous function f on the asymptotic boundary $E(\infty)$ of $\Omega(\infty)$ at infinity. By Theorem 4.9 in [9], we know there exists a (unique) solution to this Dirichlet problem associated to equation (1).

By rotating recursively such a graph surface an angle π about the vertical and horizontal geodesics in its boundary, we get a "generalized horizontal singly periodic Scherk minimal surface" $\mathcal{M}_\mu(f)$, which is properly embedded and invariant by the horizontal translation ϕ_μ^4 along γ_0. Such a $\mathcal{M}_\mu(f)$ can be seen as the desingularization of the vertical geodesic plane $\gamma_0 \times \mathbb{R}$ and a periodic minimal entire graph invariant by the horizontal translation ϕ_μ^4 along γ_0. Moreover, the surface $\mathcal{M}_\mu(f)$ contains the vertical lines $\{q_{\pm\mu}\} \times \mathbb{R}$ and the horizontal geodesics $g(\pm\mu) \times \{0\}$.

In general, $\mathcal{M}_\mu(f)$ contains vertical geodesic arcs at the infinite boundary $\partial_\infty \mathbb{H}^2 \times$

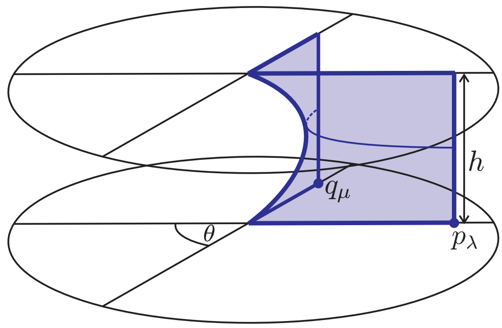

FIG. 5. *The embedded minimal disk $\Sigma_{h,\lambda,\mu}$ bounded by $\Gamma_{h,\lambda,\mu}$.*

$\mathbb{R}$ (over the endpoints of $g(\pm\mu)$ and their translated copies). To avoid such vertical segments, we take f vanishing on the endpoints of $E(\infty)$.

4.2. A Plateau construction of vertical Saddle Towers. In this section, we construct the 1-parameter family of most symmetric vertical Saddle Towers in $\mathbb{H}^2 \times \mathbb{R}$, which can be seen as the desingularization of n vertical planes meeting at a common axis with angle $\theta = \pi/n$, for some $n \geq 2$. When $n = 2$, the corresponding examples are usually called "vertical singly periodic Scherk minimal surfaces". For any fixed $n \geq 2$, these examples are included in the $(2n-3)$-parameter family of vertical Saddle Towers constructed by Morabito and the second author in [13]. These surfaces are all invariant by a vertical translation $T(h)$.

A fundamental piece of the Saddle Tower we want to construct is obtained by solving a Plateau problem. We now consider a more general Plateau problem, that will be also used in Sections 5 and 6.

Given an integer $n \geq 2$, we fix $\theta = \pi/n$. We consider in $\mathbb{H}^2$ the points

$$p_\lambda = (\lambda \sin\theta, \lambda\cos\theta) \quad \text{and} \quad q_\mu = (0, \mu),$$

for any $\lambda \in (0,1]$ and any $\mu \in (0,1]$ (see Figure 5). Given $h > 0$, we call $W_{h,\lambda,\mu} \subset \mathbb{H}^2 \times \mathbb{R}$ the triangular prism whose top and bottom faces are two geodesic triangular domains at heights 0 and h: the bottom triangle has vertices $(p_\lambda, 0), (\mathbf{0}, 0), (q_\mu, 0)$ and the top triangle is its vertical translation to height h.

If $\lambda < 1$ and $\mu < 1$, we consider the following Jordan curve in the boundary of $W_{h,\lambda,\mu}$:

$$\begin{aligned}\Gamma_{h,\lambda,\mu} = \; & \overline{(q_\mu,0)\,(\mathbf{0},0)} \cup \overline{(\mathbf{0},0)\,(p_\lambda,0)} \cup \overline{(p_\lambda,0)\,(p_\lambda,h)} \\ & \cup \overline{(p_\lambda,h)\,(\mathbf{0},h)} \cup \overline{(\mathbf{0},h)\,(q_\mu,h)} \cup \overline{(q_\mu,h)\,(q_\mu,0)}\end{aligned}$$

(see Figure 5). Since $\partial W_{h,\lambda,\mu}$ is mean-convex and $\Gamma_{h,\lambda,\mu}$ is contractible in $W_{h,\lambda,\mu}$, there exists an embedded minimal disk $\Sigma_{h,\lambda,\mu} \subset W_{h,\lambda,\mu}$ whose boundary is $\Gamma_{h,\lambda,\mu}$ (see Meeks and Yau [12]).

CLAIM 4.3. *$\Sigma_{h,\lambda,\mu}$ is the only compact minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ bounded by $\Gamma_{h,\lambda,\mu}$. Moreover, $\Sigma_{h,\lambda,\mu}$ is a minimal $Y_{\theta/2}$-graph and it lies on the positive $Y_{\theta/2}$-side of $\Sigma_{h,\lambda',\mu'}$, for any $\lambda' \leq \lambda$ and any $\mu' \leq \mu$.*

Proof. Let $\Sigma, \Sigma' \subset \mathbb{H}^2 \times \mathbb{R}$ be two compact minimal surfaces with $\partial\Sigma = \Gamma_{h,\lambda,\mu}$ and $\partial\Sigma' = \Gamma_{h,\lambda',\mu'}$, where $\lambda' \leq \lambda$ and $\mu' \leq \mu$. First observe that, by the convex hull

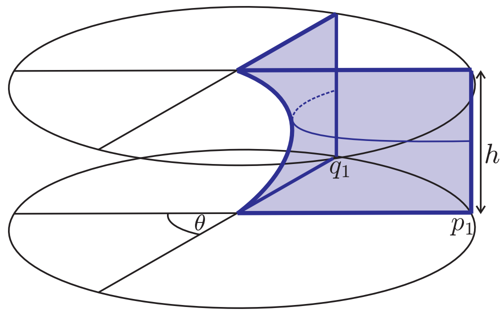

FIG. 6. *The embedded minimal disk* Σ_h *bounded by* Γ_h.

property (or by the maximum principle using vertical geodesic planes and horizontal slices), $\Sigma \subset W_{h,\lambda,\mu}$ and $\Sigma' \subset W_{h,\lambda',\mu'}$.

Let $(\phi_l)_{l\in(-1,1)}$ be the flow of $Y_{\theta/2}$. For l close to -1, $\phi_l(W_{h,\lambda',\mu'}) \cap W_{h,\lambda,\mu} = \emptyset$ and, for $-1 < l < 0$, $\phi_l(\Gamma_{h,\lambda',\mu'})$ and $W_{h,\lambda,\mu}$ do not intersect. So letting l increase from -1 to 0, we get by the maximum principle that $\phi_l(\Sigma')$ and Σ do not intersect until $l = 0$. When $\lambda = \lambda'$ and $\mu = \mu'$, this implies that $\Sigma = \Sigma'$ (hence $\Sigma = \Sigma_{h,\lambda,\mu}$) and it is a minimal $Y_{\theta/2}$-graph. Also this translation argument shows that Σ lies on the positive $Y_{\theta/2}$-side of Σ' when $\lambda' < \lambda$ and $\mu' < \mu$. □

From Claim 4.3, we deduce the continuity of $\Sigma_{h,\lambda,\mu}$ in the λ and μ parameters. The surfaces $\Sigma_{h,\lambda,\mu}$ will be used in Sections 5 and 6 for the construction of doubly periodic minimal surfaces and surfaces invariant by a subgroup of Isom($\mathbb{H}^2$). More precisely, in the following subsection we construct surfaces from $\Sigma_{h,\lambda,\mu}$ that we use in the sequel.

Now we only consider the $\lambda = \mu$ case. As $Y_{\theta/2}$-graphs, the surfaces $\Sigma_{h,\mu,\mu}$ form an increasing family in the μ parameter. So if we construct a "barrier from above", we could ensure the convergence of $\Sigma_{h,\mu,\mu}$ when $\mu \to 1$.

On the ideal triangular domain of vertices $\mathbf{0}, p_1, q_1$, there exists a solution u to the vertical minimal graph equation (1) which takes boundary values 0 on $\overline{q_1\mathbf{0}}$ and $\overline{\mathbf{0}p_1}$ and $+\infty$ on $\overline{p_1q_1}$. Let S_0 and S_h be, respectively, the graph surfaces of u and $h - u$.

Using the same argument as in Claim 4.3, we conclude that both S_0 and S_h are $Y_{\theta/2}$-graphs and lie on the positive $Y_{\theta/2}$-side of $\Sigma_{h,\mu,\mu}$, for any μ. They are the expected "barriers from above".

Using the monotonicity and the barriers, we conclude that there exists a limit Σ_h of the minimal $Y_{\theta/2}$-graphs $\Sigma_{h,\mu,\mu}$ when $\mu \to 1$. And it is also a minimal $Y_{\theta/2}$-graph. The surface Σ_h is a minimal disk bounded by

$$\Gamma_h = \overline{(q_1,0)(\mathbf{0},0)} \cup \overline{(\mathbf{0},0)(p_1,0)} \cup \overline{(q_1,h)(\mathbf{0},h)} \cup \overline{(\mathbf{0},h)(p_1,h)}.$$

In fact, applying the techniques of Claim 4.3, we get that Σ_h is the only minimal disk of $\mathbb{H}^2\times\mathbb{R}$ bounded by Γ_h which is contained in $W_{h,1,1}$. By uniqueness, Σ_h is symmetric with respect to the vertical plane $\gamma_{\theta/2} \times \mathbb{R}$ and the horizontal slice $\mathbb{H}^2 \times \{h/2\}$.

Now we can extend Σ_h by doing recursive symmetries along the horizontal geodesics in its boundary. The surface we obtain is properly embedded, invariant by the vertical translation $T(2h)$ and asymptotic to the n vertical planes $\gamma_{k\theta} \times \mathbb{R}$, $0 \le k \le n-1$, outside of a large vertical cylinder with axis $\{\mathbf{0}\} \times \mathbb{R}$.

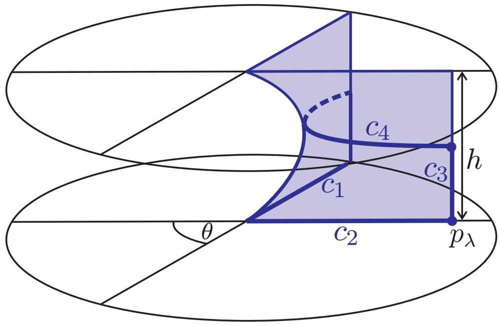

FIG. 7. *The minimal disk $\Sigma_{h,\lambda}$ bounded by $\Gamma_{h,\lambda}$, and the minimal vertical graph $M_{h,\lambda} = \Sigma_{h,\lambda} \cap \{0 \le t \le h/2\}$ bounded by $c_1 \cup c_2 \cup c_3 \cup c_4$.*

PROPOSITION 4.4. *For any natural $n \ge 2$ and any $h > 0$, there exists a properly embedded minimal surface $\mathcal{M}_h(n)$ in $\mathbb{H}^2 \times \mathbb{R}$ invariant by the vertical translation $T(2h)$ and asymptotic to the n vertical planes $\gamma_{\frac{k\pi}{n}} \times \mathbb{R}$, for $0 \le k \le n-1$, far from $\{\mathbf{0}\} \times \mathbb{R}$. Moreover, $\mathcal{M}_h(n)$ contains the horizontal geodesics $\gamma_{\frac{k\pi}{n}} \times \{0\}$, $0 \le k \le n-1$, and is invariant by reflection symmetry with respect to the vertical geodesic planes $\gamma_{(\frac{1}{2}+k)\frac{\pi}{n}} \times \mathbb{R}$, with $0 \le k \le n-1$, and respect to the horizontal slices $\mathbb{H}^2 \times \{\pm h/2\}$. We call such a surface* (most symmetric) vertical Saddle Tower.

4.3. The minimal surfaces $\Sigma_{h,\lambda}$ and $M_{h,\lambda}$. In order to prepare our work in Sections 5 and 6, we continue to study the solutions of the Plateau problem introduced in Subsection 4.2.

Recall that $n \ge 2$ is an integer, $\theta = \pi/n$ and $\lambda, \mu \in (0,1)$. We now fix λ and $h > 0$, and we consider the family of $Y_{\theta/2}$-graphs $\Sigma_{h,\lambda,\mu}$ as μ varies. This family is monotone increasing in the μ-parameter. And, for fixed h, the $Y_{\theta/2}$-graphs $\Sigma_{h,\lambda,\mu}$ are bounded from above by the surface Σ_h constructed in the preceding subsection. Thus $\Sigma_{h,\lambda,\mu}$ converges to a minimal $Y_{\theta/2}$-graph $\Sigma_{h,\lambda}$ when $\mu \to 1$. This surface is an embedded minimal disk bounded by

$$\Gamma_{h,\lambda} = \begin{array}{l} \overline{(q_1,0)\,(\mathbf{0},0)} \cup \overline{(\mathbf{0},0)\,(p_\lambda,0)} \cup \overline{(p_\lambda,0)\,(p_\lambda,h)} \\ \cup \overline{(p_\lambda,h)\,(\mathbf{0},h)} \cup \overline{(\mathbf{0},h)\,(q_1,h)}. \end{array}$$

In fact, applying the techniques of Claim 4.3, we conclude that $\Sigma_{h,\lambda}$ is the only minimal disk contained in $W_{h,\lambda,1}$ which is bounded by $\Gamma_{h,\lambda}$.

The Alexandrov reflection method with respect to horizontal slices shows that every $\Sigma_{h,\lambda,\mu}$ is a symmetric vertical bigraph with respect to $\mathbb{H}^2 \times \{h/2\}$ (see Appendix C). Hence this is also true for $\Sigma_{h,\lambda}$.

We consider

$$M_{h,\lambda} = \Sigma_{h,\lambda} \cap \{0 \le t \le h/2\},$$

which is a minimal vertical graph bounded by c_1, c_2, c_3, c_4 (see Figure 7), where:

- $c_1 = \overline{(q_1,0)\,(\mathbf{0},0)} = \gamma_0^+$ is half a complete horizontal geodesic line;
- $c_2 = \overline{(\mathbf{0},0)\,(p_\lambda,0)}$ is a horizontal geodesic of length $\ln\left(\frac{1+\lambda}{1-\lambda}\right)$, forming an angle θ with c_1 at $A_0 = (\mathbf{0},0)$;
- $c_3 = \overline{(p_\lambda,0)\,(p_\lambda,h/2)}$ is a vertical geodesic line of length $h/2$;

- $c_4 = M_{h,\lambda} \cap \{t = h/2\}$ is a horizontal geodesic curvature line with endpoints $(p_\lambda, h/2)$ and $(q_1, h/2)$.

The domain Ω_0 over which $M_{h,\lambda}$ is a graph is included in the triangular domain of vertices $\mathbf{0}, p_\lambda, q_1$, and it is bounded by $\overline{q_1\mathbf{0}}$, $\overline{\mathbf{0}p_\lambda}$ and $\pi(c_4)$. The latter curve goes from p_λ to q_1 and is concave with respect to Ω_0 because of the boundary maximum principle using vertical geodesic planes, which implies that the mean curvature vector of $\pi(c_4) \times \mathbb{R}$ points outside $\Omega_0 \times \mathbb{R}$.

On $M_{h,\lambda}$, we fix the unit normal vector field N whose associated angle function $\nu = \langle N, \partial_t \rangle$ is non-negative. The vector field N extends smoothly to $\partial M_{h,\lambda}$ (by Schwarz symmetries). It is not hard to see that ν only vanishes on $c_3 \cup c_4$, and $\nu = 1$ at $A_0 = (\mathbf{0}, 0)$.

Since $\Sigma_{h,\lambda}$ is a $Y_{\theta/2}$-graph, then it is stable, so it satisfies a curvature estimate away from its boundary. Hence the curvature is uniformly bounded on $M_{h,\lambda}$ away from c_1, c_2 and c_3. Besides, $M_{h,\lambda}$ can be extended by symmetry along c_1 and c_2 as a vertical graph, thus as a stable surface. Hence, on $M_{h,\lambda}$, the curvature is uniformly bounded away from c_3.

Because of this curvature estimate and since $M_{h,\lambda} \subset W_{h,\lambda,1}$, the angle function ν goes to zero as we approach $q_1 \times [0, h/2]$, and the asymptotic intrinsic distance from c_1 to c_4 is $h/2$.

5. Doubly periodic minimal surfaces. In this section, we construct doubly periodic minimal surfaces, *i.e.* properly embedded minimal surfaces invariant by a subgroup of $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$ isomorphic to $\mathbb{Z}^2$. In fact, we only consider subgroups generated by a hyperbolic translation along a horizontal geodesic and a vertical translation. More precisely, let $(\phi_l)_{l \in (-1,1)}$ be the flow of Y_0. We are interested in properly embedded minimal surfaces which are invariant by the subgroup of $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$ generated by ϕ_l and $T(h)$, for fixed l and h. We notice that the quotient $\mathbb{M}$ of $\mathbb{H}^2 \times \mathbb{R}$ by this subgroup is diffeomorphic to $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T}$ is a 2-torus.

One trivial example of a doubly periodic minimal surface is the vertical plane $\gamma_0 \times \mathbb{R}$. The quotient surface is topologically a torus and it is in fact the only compact minimal surface in the quotient (see Theorem 3.1). Other trivial examples are given by the quotients of a horizontal slice $\mathbb{H}^2 \times \{t_0\}$ or a vertical totally geodesic minimal plane $g(\mu) \times \mathbb{R}$. Both cases give flat annuli in the quotient.

In the following subsections, we construct non-trivial examples, that are similar to minimal surfaces of $\mathbb{R}^3$ built by H. Karcher in [5]. Their ends are asymptotic to the horizontal and/or the vertical flat annuli described above.

5.1. Doubly periodic Scherk minimal surfaces. In this subsection we construct minimal surfaces of genus zero in $\mathbb{M}$ which have two ends asymptotic to two vertical annuli and two ends asymptotic to two horizontal annuli in the quotient. These examples are similar to the doubly periodic Scherk minimal surface in $\mathbb{R}^3$.

Let A_R, B_R in $g(-\mu)$ and C_R, D_R in $g(\mu)$ at distance R from γ_0 such that A_R and D_R are in $\{x < 0\}$ and B_R and C_R are in $\{x > 0\}$, see Figure 8. We fix $h > \pi$ and consider the following Jordan curve:

$$\begin{aligned}\Gamma_R = \ & \overline{(A_R,0)\,(B_R,0)} \cup (E(R) \times \{0\}) \cup \overline{(C_R,0)\,(D_R,0)} \\ & \cup \overline{(D_R,0)\,(D_R,h)} \cup \overline{(D_R,h)\,(C_R,h)} \cup (E(R) \times \{h\}) \\ & \cup \overline{(B_R,h)\,(A_R,h)} \cup \overline{(A_R,h)\,(A_R,0)},\end{aligned}$$

where $E(R)$ is the subarc of the equidistant $d(R)$ to γ_0 that joins B_R to C_R. We consider a least area embedded minimal disk Σ_R with boundary Γ_R.

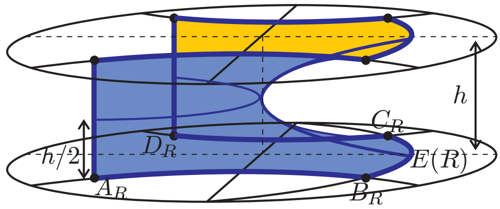

FIG. 8. *The Jordan curve* Γ_R *and the embedded minimal disk* Σ_R *bounded by* Γ_R.

Using the Alexandrov reflection technique with respect to horizontal slices, one proves that Σ_R is a vertical bigraph with respect to $\{t = h/2\}$ (see Appendix C).

Since Σ is area-minimizing, it is stable. This gives uniform curvature estimates far from the boundary. Besides $\Sigma_R \cap \{0 \le t \le h/2\}$ is a vertical graph that can be extended by symmetry with respect to $\overline{(A_R, 0)\,(B_R, 0)}$ to a larger vertical graph. Thus we also obtain uniform curvature estimates in a neighborhood of $\overline{(A_R, 0)\,(B_R, 0)}$. This is also true for the three other horizontal geodesic arcs in Γ_R.

Let A_∞ and D_∞ be the endpoints of $g(-\mu)$ and $g(\mu)$, that are limits of A_R and D_R as $R \to +\infty$. For any R, Σ_R is on the half-space determined by $\overline{A_\infty D_\infty} \times \mathbb{R}$ that contains Γ_R.

Since $h > \pi$, we can consider the surface S_h described in Appendix B: $S_h \subset \mathbb{H}^2 \times (0, h)$ is a vertical bigraph with respect to $\{t = h/2\}$ which is invariant by translations along γ_0 and whose boundary is $(\alpha \times \{0\}) \cup \overline{(0,1,0)(0,1,h)} \cup (\alpha \times \{h\}) \cup \overline{(0,-1,h)(0,-1,0)}$, where $\alpha = \partial_\infty \mathbb{H}^2 \cap \{x > 0\}$. Let $(\chi_l)_{l\in(-1,1)}$ be the flow of the Killing vector field $Y_{\pi/2}$. For l close to 1, $\chi_l(S_h)$ does not meet Σ_R. Since $\overline{(D_R, 0)\,(D_R, h)}$ and $\overline{(A_R, h)\,(A_R, 0)}$ are the only part of Γ_R in $\mathbb{H}^2 \times (0, h)$, we can let l decrease until $l_R < 0$, where $\chi_{l_R}(S_h)$ touches Σ_R for the first time. Actually, there are two first contact points: $(A_R, h/2)$ and $(D_R, h/2)$. By the maximum principle, the surface Σ_R is contained between $\chi_{l_R}(S_h)$ and $\overline{A_\infty D_\infty} \times \mathbb{R}$. We notice that $l_R > l_{R'}$, for any $R' > R$, and $l_R \to l_\infty > -1$, where $\chi_{l_\infty}(\gamma_0) = \overline{A_\infty D_\infty}$.

We recall that $Z_{\pi/2}$ is the unit vector field normal to the equidistant surfaces to $\gamma_0 \times \mathbb{R}$.

CLAIM 5.1. $\Sigma_R \setminus \Gamma_R$ *is a* $Z_{\pi/2}$*-graph over the open rectangle* $\overline{A_0\,D_0} \times (0, h)$ *in* $\gamma_0 \times \mathbb{R}$.

Proof. It is clear that the projection of $\Sigma_R \setminus \Gamma_R$ over $\gamma_0 \times \mathbb{R}$ in the direction of $Z_{\pi/2}$ coincides with $\overline{A_0\,D_0} \times (0, h)$. Let us prove that $\Sigma_R \setminus \Gamma_R$ is transverse to $Z_{\pi/2}$. Assume that q is a point in $\Sigma_R \setminus \Gamma_R$ where Σ_R is tangent to $Z_{\pi/2}$. Thus there is a minimal surface P given by Appendix B which is invariant by translation along $Z_{\pi/2}$, passes through q and is tangent to Σ_R. Near q, the intersection $P \cap \Sigma_R$ is composed of $2n$ arcs meeting at q, with $n \ge 2$.

By definition of P and Γ_R, the intersection $P \cap \Gamma_R$ is composed either by two points, or by one point and one geodesic arc of type $\overline{(A_R, 0)\,(B_R, 0)}$, or by two arcs of type $\overline{(A_R, 0)\,(B_R, 0)}$ and $\overline{(D_R, h)\,(C_R, h)}$. Since Σ_R is a disk, we get that there exists a component of $\Sigma_R \setminus P$ which has all its boundary in P. This is impossible by the maximum principle, since $\mathbb{H}^2 \times \mathbb{R}$ can be foliated by translated copies of P. The surface Σ_R is then transverse to $Z_{\pi/2}$.

Now let q be a point in $\overline{A_0\,D_0} \times (0, h)$, and ℓ_q be the geodesic passing by q and generated by $Z_{\pi/2}$. The intersection of ℓ_q with Σ_R is always transverse, so the number of intersection points does not depend on q. For $q = (A_0, h/2)$, this number is 1. Therefore, $\Sigma_R \setminus \Gamma_R$ is a $Z_{\pi/2}$-graph over the open rectangle $\overline{A_0\, D_0} \times (0, h)$. □

Now let R tend to ∞. Because of the curvature estimates, and using that each Σ_R is a $Z_{\pi/2}$-graph bounded by $\chi_{l_R}(S_h)$ and $\overline{A_\infty D_\infty} \times \mathbb{R}$, we obtain that, the surfaces Σ_R converge to a minimal surface Σ_∞ satisfying the following properties:

- Σ_∞ lies in the region of $\{0 \le t \le h\}$ bounded by $g(-\mu) \times \mathbb{R}$, $g(\mu) \times \mathbb{R}$, $\overline{A_\infty D_\infty} \times \mathbb{R}$ and $\chi_{l_\infty}(S_h)$;
- $\partial\Sigma_\infty = (g(-\mu) \times \{0\}) \cup (g(\mu) \times \{0\}) \cup (g(\mu) \times \{h\}) \cup (g(-\mu) \times \{h\})$;
- $\Sigma_\infty \setminus \partial\Sigma_\infty$ is a vertical bigraph with respect to $\{t = h/2\}$ and a $Z_{\pi/2}$-graph over $\overline{A_0\, D_0} \times (0, h)$;
- $\Sigma_\infty \cap \{x \le 0\}$ is asymptotic to $g(-\mu) \times [0, h]$ and $g(\mu) \times [0, h]$; and $\Sigma_\infty \cap \{x \ge 0\}$ is asymptotic to $\{t = 0\}/\phi_\mu^2$ and $\{t = h\}/\phi_\mu^2$.

After extending Σ_∞ by successive symmetries with respect to the horizontal geodesics contained in its boundary, we obtain a surface Σ invariant by the subgroup generated by the horizontal hyperbolic translation ϕ_μ^4 and the vertical translation $T(2h)$. In the quotient by ϕ_μ^4 and $T(2h)$, this surface is topologically a sphere minus four points. Two of the ends of Σ are vertical and two of them are horizontal. This surface is similar to the doubly periodic Scherk minimal surface of $\mathbb{R}^3$.

PROPOSITION 5.2. *For any $h > \pi$ and any $\mu \in (0, 1)$, there exists a properly embedded minimal surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ which is invariant by the vertical translation $T(2h)$ and the horizontal hyperbolic translation ϕ_μ^4 along γ_0. In the quotient by $T(2h)$ and ϕ_μ^4, Σ is topologically a sphere minus four points, and it has two ends asymptotic to the quotients of $\{x > 0, t = 0\}$ and $\{x > 0, t = h\}$, and two ends asymptotic to the quotients of $(g(-\mu) \cap \{x < 0\}) \times [0, h]$ and $(g(\mu) \cap \{x < 0\}) \times [0, h]$. Moreover, Σ contains the horizontal geodesics $g(\pm\mu) \times \{0\}, g(\pm\mu) \times \{h\}$, and is invariant by reflection symmetry with respect to $\{t = h/2\}$ and $\gamma_{\pi/2} \times \mathbb{R}$. We call these examples* doubly periodic Scherk minimal surfaces. *Finally, we remark that Σ admits a non-orientable quotient by ϕ_μ^4 and $T(h) \circ \phi_\mu^2$.*

REMARK 5.3. When $h < \pi$ and μ is large enough, we can prove by using the maximum principle with vertical catenoids and a fundamental piece of the surface Σ described in Proposition 5.2, that the corresponding doubly periodic Scherk minimal surface does not exist.

On the other hand, when $h < \pi$ and μ is small enough, we can solve the Plateau problem above in the exterior of certain surface $\mathcal{M}(R, \widetilde{\mu})$ described in Proposition 5.8, to prove that the corresponding doubly periodic Scherk minimal surface Σ exists.

5.2. Doubly periodic minimal Klein bottle examples: horizontal and vertical Toroidal Halfplane Layers. In this subsection, we construct non-trivial families of examples of doubly periodic minimal surfaces.

Let us consider the surface $\Sigma_{h,\lambda}$ constructed in Subsection 4.3 for $n = 2$. By successive extensions by symmetry along its boundary we get a properly embedded minimal surface Σ which is invariant by the vertical translation $T(2h)$ and the horizontal translation χ_λ^2, where $(\chi_l)_{l\in(-1,1)}$ is the flow of $Y_{\pi/2}$. The quotient surface by the subgroup of isometries of $\mathbb{H}^2 \times \mathbb{R}$ generated by $T(2h)$ and χ_λ^2 is topologically a Klein bottle minus two points. The ends of the surface are asymptotic to vertical annuli. If we consider the quotient by the group generated by $T(2h)$ and χ_λ^4, we get topologically a torus minus four points. This example corresponds to the Toroidal Halfplane Layer of $\mathbb{R}^3$ denoted by $M_{\theta,0,\pi/2}$ in [19].

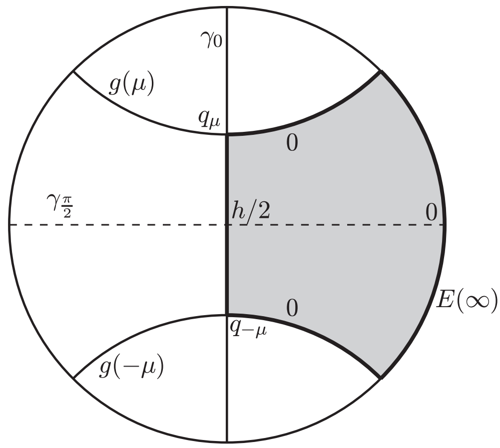

FIG. 9. *The minimal surface $\widetilde{\Sigma}$ of Proposition 5.5 is obtained from the vertical minimal graph w over $\Omega(\infty)$ (the shadowed domain) with the prescribed boundary data.*

PROPOSITION 5.4. *For any $h > 0$ and any $\lambda \in (0,1)$, there exists a properly embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ invariant by the vertical translation $T(2h)$ and the horizontal hyperbolic translation χ_λ^2 along $\gamma_{\pi/2}$, which is topologically a Klein bottle minus two points in the quotient by $T(2h)$ and χ_λ^2. The surface is invariant by reflection symmetry with respect to $\{t = h/2\}$, contains the geodesics $\gamma_0 \times \{0, h\}$, $\gamma_{\pi/2} \times \{0, h\}$ and $\{p_\lambda\} \times \mathbb{R}$, and its ends are asymptotic to the quotient of $\gamma_0 \times \mathbb{R}$. Moreover, the surface is topologically a torus minus four points when considered in the quotient by $T(2h)$ and χ_λ^4. We call these examples* horizontal Toroidal Halfplane Layers of type 1.

Let us see another example. This one is similar to the preceding one, but its ends are now asymptotic to horizontal slices. We use the notation introduced in Subsection 4.1. For $R > 0$, let w_R be the solution to (1) over $\Omega(R)$ with boundary values zero on $\partial\Omega(R) \setminus \gamma_0$ and $h/2$ on $\gamma_0 \cap \partial\Omega(R)$. By the maximum principle, $w_R < w_{R'} < v$ on Ω_R, for any $R' > R$, where v is the Abresch-Sa Earp barrier described in Appendix B. The graphs w_R converge as $R \to +\infty$ to the unique solution w of (1) on $\Omega(\infty)$ with boundary values $h/2$ on $\overline{q_\mu q_{-\mu}}$ minus its endpoints and 0 on the remaining boundary, including the asymptotic boundary at infinity. (By [9], we directly know that such a graph exists and is unique.)

By uniqueness, we know that such a graph is invariant by reflection symmetry with respect to the vertical geodesic plane $\gamma_{\pi/2} \times \mathbb{R}$. Moreover, the boundary of this graph is composed of two halves of $g(\mu)$ and $g(-\mu)$ and $\overline{(q_\mu, 0)(q_\mu, h/2)} \cup \overline{(q_\mu, h/2)(q_{-\mu}, h/2)} \cup \overline{(q_{-\mu}, h/2)(q_{-\mu}, 0)}$.

If we extend the graph of w by successive symmetries about the geodesic arcs in its boundary, we obtain a properly embedded minimal surface $\widetilde{\Sigma}$ which is invariant by the $\mathbb{Z}^2$ subgroup G_1 of isometries of $\mathbb{H}^2 \times \mathbb{R}$ generated by $T(h)$ and ϕ_μ^4. In the quotient by G_1, $\widetilde{\Sigma}$ is a Klein bottle with two ends asymptotic to the quotient by G_1 of the two horizontal annuli obtained in the quotient of $\mathbb{H}^2 \times \{0\}$. The quotient by the subgroup generated by $T(2h)$ and ϕ_μ^4 gives a torus minus four points. This example also corresponds to the Toroidal Halfplane Layer of $\mathbb{R}^3$ denoted by $M_{\theta,0,\pi/2}$ in [19].

Finally, we remark that taking limits of $\widetilde{\Sigma}$ as $h \to +\infty$, we get the horizontal

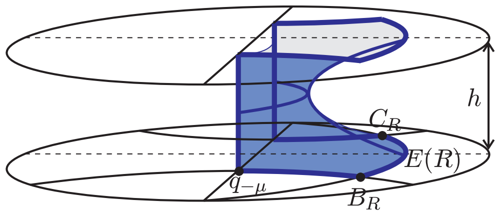

FIG. 10. *The embedded minimal disk Σ_R bounded by Γ_R (Subsection 5.3).*

singly periodic Scherk minimal surface constructed in Subsection 4.1.

PROPOSITION 5.5. *For any $h > 0$ and any $\mu \in (0, 1)$, there exists a properly embedded minimal surface $\widetilde{\Sigma}$ in $\mathbb{H}^2 \times \mathbb{R}$ which is invariant by the vertical translation $T(h)$ and the horizontal hyperbolic translation ϕ_μ^4 along γ_0. In the quotient by $T(h)$ and ϕ_μ^4, $\widetilde{\Sigma}$ is topologically a Klein bottle minus two points. The ends of $\widetilde{\Sigma}$ are asymptotic to the quotient of $\mathbb{H}^2 \times \{0\}$. The surface is invariant by reflection symmetry with respect to $\gamma_{\pi/2} \times \mathbb{R}$, and contains the geodesics $\gamma_0 \times \{h/2\}$, $\{q_{\pm\mu}\} \times \mathbb{R}$ and $g(\pm\mu) \times \{0\}$. Moreover, in the quotient by $T(2h)$ and ϕ_μ^4, the surface is topologically a torus minus four points corresponding to the ends of the surface (asymptotic to the quotient of the horizontal slices $\{t = 0\}$ and $\{t = h\}$). We call these examples* vertical Toroidal Halfplane Layers of type 1.

REMARK 5.6. **"Generalized vertical Toroidal Halfplane Layers of type 1".** Consider the domain $\Omega(\infty)$ with prescribed boundary data $h/2$ on $\overline{q_\mu q_{-\mu}}$ minus its endpoints, 0 on $(g(\mu) \cup g(-\mu)) \cap \{x > 0\}$ and a continuous function f on the asymptotic boundary $E(\infty)$ of $\Omega(\infty)$ at infinity, f vanishing on the endpoints of $E(\infty)$ and satisfying $|f| \leq h/2$. By Theorem 4.9 in [9], we know there exists a (unique) solution to this Dirichlet problem. By rotating recursively such a graph surface an angle π about the vertical and horizontal geodesics in its boundary, we get a *"generalized vertical Toroidal Halfplane Layers of type 1"*, which is properly embedded and invariant by the vertical translation $T(h)$ and the horizontal hyperbolic translation ϕ_μ^4 along γ_0. In the quotient by $T(h)$ and ϕ_μ^4, such a surface is topologically a Klein bottle minus two points corresponding to the ends of the surface, that are asymptotic to the quotient of a entire minimal graph invariant by ϕ_μ^4 which contains the geodesics $g(\mu) \times \{0\}$ and $g(-\mu) \times \{0\}$. In the quotient by $T(2h)$ and ϕ_μ^4, the surface is topologically a torus minus four points.

5.3. Other vertical Toroidal Halfplane Layers. The construction given in this subsection is very similar to the one considered in Subsection 5.1, and we use the notation introduced there. We consider $h > \pi$ and Γ_R the following Jordan curve:

$$\begin{aligned}\Gamma_R = \quad & \overline{(B_0,0)\,(B_R,0)} \cup (E(R) \times \{0\}) \cup \overline{(C_R,0)\,(C_0,0)} \\ & \cup \overline{(C_0,0)\,(C_0,h)} \cup \overline{(C_0,h)\,(C_R,h)} \cup (E(R) \times \{h\}) \\ & \cup \overline{(B_R,h)\,(B_0,h)} \cup \overline{(B_0,h)\,(B_0,0)}.\end{aligned}$$

Γ_R bounds an embedded minimal disk Σ_R with minimal area. As in Subsection 5.1, Σ_R is a vertical bigraph with respect to $\{t = h/2\}$. So the sequence of minimal surfaces Σ_R, as R varies, satisfies a uniform curvature estimate far from $\overline{(C_0,0)\,(C_0,h)}$, $\overline{(B_0,0)\,(B_0,h)}$, $E(R) \times \{0\}$ and $E(R) \times \{h\}$.

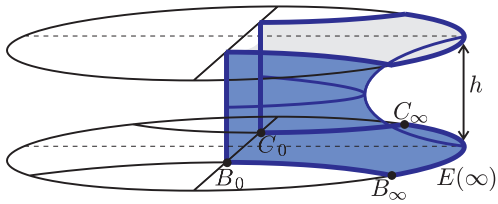

FIG. 11. *The embedded minimal disk Σ_∞ from which we obtain, after successive symmetries with respect to the geodesics in its boundary, the doubly periodic example described in Proposition 5.7.*

Using the Alexandrov reflection technique with respect to the vertical planes $g(\nu) \times \mathbb{R}$ as in Subsection 4.1, we prove that Σ_R is a Y_0-bigraph with respect to $g(0) \times \mathbb{R} = \gamma_{\pi/2} \times \mathbb{R}$. Thus extending Σ_R by symmetry with respect to $\overline{(B_0, 0)\,(B_R, 0)}$, $\overline{(B_0, 0)\,(B_0, h)}$ and $\overline{(B_0, h)\,(B_R, h)}$, we see that a neighborhood of $\overline{(B_0, 0)\,(B_0, h)}$ is a $Y_{\pi/2}$-graph. This neighborhood is then stable and we get curvature estimates there. Therefore, the minimal surfaces Σ_R satisfy a uniform curvature estimate far from $E(R) \times \{0\}$ and $E(R) \times \{h\}$.

The surface Σ_R is included in $\{x \geq 0\} \times [0, h]$. If S_h is the same surface as in Subsection 5.1 (described in Appendix B) and $(\chi_l)_{l \in (-1,1)}$ is the flow of $Y_{\pi/2}$, for l close to 1, $\chi_l(S_h)$ does not meet Σ_R. Since $\overline{(B_0, 0)\,(B_0, h)}$ and $\overline{(C_0, h)\,(C_0, 0)}$ are the only part of Γ_R in $\mathbb{H}^2 \times (0, h)$, we can let l decrease until $l_0 < 0$, where $\chi_{l_0}(S_h)$ touches $\partial\Sigma_R$ for the first time. Actually, l_0 does not depend on R, and there are two first contact points: $(B_0, h/2)$ and $(C_0, h/2)$. The surface Σ_R is then between $\chi_{l_0}(S_h)$ and $\gamma_0 \times \mathbb{R}$.

As in Subsection 5.1, $\Sigma_R \backslash \Gamma_R$ is a $Z_{\pi/2}$-graph over the open rectangle $\overline{B_0\, C_0} \times (0, h)$ in $\gamma_0 \times \mathbb{R}$. Then let R tend to $+\infty$. The surfaces Σ_R converge to a minimal surface Σ_∞ satisfying:

- Σ_∞ lies in the region of $\{0 \leq t \leq h\}$ bounded by $g(-\mu) \times \mathbb{R}$, $g(\mu) \times \mathbb{R}$, γ_0 and $\chi_{l_0}(S_h)$.
- Σ_∞ is bounded by four half geodesic lines: $\overline{(B_0, 0)\,(B_\infty, 0)}$, $\overline{(B_0, h)\,(B_\infty, h)}$, $\overline{(C_0, 0)\,(C_\infty, 0)}$, $\overline{(C_0, h)\,(C_\infty, h)}$, and by two vertical segments: $\overline{(B_0, 0)\,(B_0, h)}$ and $\overline{(C_0, 0)\,(C_0, h)}$. Here B_∞ and C_∞ are the limits of the B_R and C_R as $R \to +\infty$, contained in $\partial_\infty \mathbb{H}^2$.
- $\Sigma_\infty \setminus \partial\Sigma_\infty$ is a vertical bigraph with respect to $\{t = h/2\}$ and a $Z_{\pi/2}$-graph over $\overline{B_0\, C_0} \times (0, h)$.
- Σ_∞ is asymptotic to $\{t = 0\}$ and $\{t = h\}$.

By successive symmetries of Σ_∞ with respect to the geodesics in its boundary, we get an embedded minimal surface Σ invariant by the subgroup of isometries of $\mathbb{H}^2 \times \mathbb{R}$ generated by ϕ_μ^4 and $T(2h)$. The quotient surface is a torus minus four points. This example corresponds to a Toroidal Halfplane Layer of $\mathbb{R}^3$ denoted by $M_{\theta,\pi/2,0}$ in [19].

PROPOSITION 5.7. *For any $h > 0$ and any $\mu \in (0, 1)$, there exists a properly embedded minimal surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ which is invariant by the vertical translation $T(2h)$ and the horizontal hyperbolic translation ϕ_μ^4 along γ_0. In the quotient by $T(2h)$ and ϕ_μ^4, such a surface is topologically a torus minus four points. The ends of Σ are asymptotic to the quotient of the horizontal slices $\{t = 0\}$ and $\{t = h\}$. Moreover, Σ contains the geodesics $g(\pm\mu) \times \{0\}$, $g(\pm\mu) \times \{h\}$ and $\{q_{\pm\mu}\} \times \mathbb{R}$, and is invariant by reflection symmetry with respect to $\{t = h/2\}$ and $\gamma_{\pi/2} \times \mathbb{R}$. Finally, we remark that, in the quotient by ϕ_μ^4 and $T(h) \circ \phi_\mu^2$, Σ is topologically a Klein bottle minus two points* *removed. We call these examples* vertical Toroidal Halfplane Layers of type 2.

Finally, we observe that, as $h \to +\infty$, Σ converges to a horizontal singly periodic Scherk minimal surface described in Proposition 4.1.

5.4. Other horizontal Toroidal Halfplane Layers. In this subsection, we also construct surfaces which are similar to some of Karcher's most symmetric Toroidal Halfplane Layers of $\mathbb{R}^3$. Now, its ends are asymptotic to vertical planes.

As in the preceding subsection, for $R \geq 0$, we consider the points B_R and C_R in $g(-\mu) \cap \{x \geq 0\}$ and $g(\mu) \cap \{x \geq 0\}$ at distance R from γ_0. Let $\mathcal{P}(R)$ be the polygonal domain in $\mathbb{H}^2$ with vertices B_0, B_R, C_R and C_0. Let u_n be the solution to (1) defined in $\mathcal{P}(R)$ with boundary value 0 on $\overline{C_R C_0} \cup \overline{C_0 B_0} \cup \overline{B_0 B_R}$ and n on $\overline{B_R C_R}$. The graph of u_n is bounded by a polygonal curve. As in Subsection 4.1, the sequence converges to a solution u_∞ of (1) on $\mathcal{P}(R)$ with boundary values 0 on $\overline{C_R C_0} \cup \overline{C_0 B_0} \cup \overline{B_0 B_R}$ and $+\infty$ on $\overline{B_R C_R}$ (by [14], we know that it exists and is unique). The graph of u_∞, denoted by Σ_R, is bounded by $(\{C_R\} \times \mathbb{R}^+) \cup \overline{C_R C_0} \cup \overline{C_0 B_0} \cup \overline{B_0 B_R} \cup (\{B_R\} \times \mathbb{R}^+)$ and is asymptotic to $\overline{C_R B_R} \times \mathbb{R}$.

By uniqueness of u_∞, Σ_R is symmetric with respect to $\gamma_{\pi/2} \times \mathbb{R}$. We denote by β_1 the geodesic curvature line of symmetry $\Sigma_R \cap (\gamma_{\pi/2} \times \mathbb{R})$, and by F_R the intersection point of $\gamma_{\pi/2}$ with $\overline{B_R C_R}$. We also consider the following points in the boundary of Σ_R:

$$p_1 = (\mathbf{0}, 0), \quad p_2 = (B_0, 0), \quad p_3 = (B_R, 0).$$

The boundary of $\Sigma_R \cap \{y \leq 0\}$ is composed of the union of the curves β_1, $\beta_2 = \overline{p_1 p_2}$, $\beta_3 = \overline{p_2 p_3}$ and $\beta_4 = \{B_R\} \times \mathbb{R}^+$.

The vertical coordinate of the conjugate surface to Σ_R is given by a function h^* defined on $\mathcal{P}_R$, which is a primitive of the closed 1-form ω^* defined by (2). We fix the primitive such that $h^*(B_R) = 0$ (we recall that the conjugate surface is well defined up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$. We can consider $h^*(B_R) = 0$ up to a vertical translation). By definition of ω^* and using the fact that $u_\infty \geq 0$ in $\mathcal{P}(R)$, we get that h^* increases from 0 to $h^*(B_0) > 0$ along $\overline{B_R B_0}$; it increases from $h^*(B_0)$ to $h_0 = h^*(\mathbf{0}) > h^*(B_0)$ along $\overline{B_0 \mathbf{0}}$; h^* is constant along $\overline{\mathbf{0} F_R}$; and finally h^* increases from 0 to h_0 along $\overline{B_R F_R}$. In fact, h_0 is equal to the distance from B_R to F_R , i.e. $h_0 = h_0(\mu, R) = \frac{1}{2}\mathrm{dist}_{\mathbb{H}^2}(B_R, C_R) > \ln \frac{1+\mu}{1-\mu}$.

We denote by Σ_R^* the conjugate minimal surface of $\Sigma_R \cap \{y \leq 0\}$. We have that $\partial \Sigma_R^* = \beta_1^* \cup \beta_2^* \cup \beta_3^* \cup \beta_4^*$, where each β_i^* corresponds by conjugation to β_i. We also denote by p_i^* the point in $\partial \Sigma_R^*$ corresponding by conjugation to p_i, $i = 1, 2, 3$.

Up to a vertical translation, we have fixed $p_3^* \in \{t = 0\}$. We can also take $p_2^* = (\mathbf{0}, h^*(B_0))$, after a horizontal translation.

On the other hand, we know from [4] that Σ_R^* is a vertical graph over a domain $\mathcal{P}(R)^*$, since $\mathcal{P}(R)$ is convex. In particular, Σ_R^* is embedded. We now use the properties conjugation introduced in Subsection 2.2 to describe the boundary of Σ_R^*:

- β_1^* is half a horizontal geodesic with endpoint p_1^*. Since $p_1^* = (\pi(p_1^*), h_0)$, then we conclude that β_1^* is contained in $\{t = h_0\}$.
- The arc β_2^* is a vertical geodesic curvature line of length $\ln \frac{1+\mu}{1-\mu}$ starting horizontally at p_2^* and finishing at p_1^*. In fact, β_2^* is the graph of a convex increasing function over the (oriented) horizontal geodesic segment $\overline{\mathbf{0}\, \pi(p_1^*)}$. Up to a rotation, we can assume $\overline{\mathbf{0}\, \pi(p_1^*)} \subset \gamma_0^+$. Since β_1 and β_2 meet orthogonally at p_1 and conjugate surfaces are isometric, we get that β_1^* is orthogonal to the vertical geodesic plane $\gamma_0 \times \mathbb{R}$. In particular, we can assume up to a

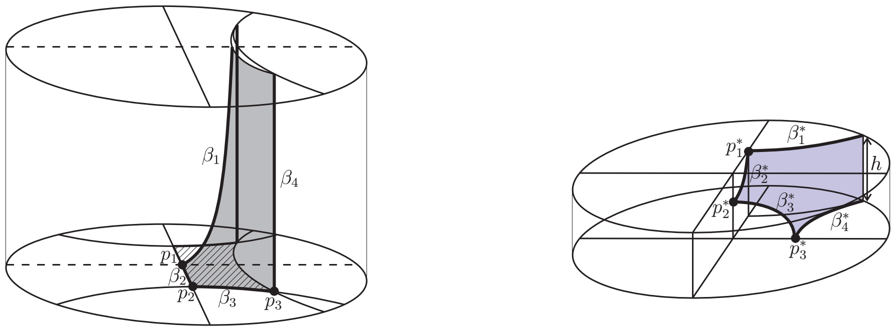

FIG. 12. *Left: $\Sigma_R \cap \{y \leq 0\}$. Right: The conjugate surface Σ_R^* of $\Sigma_R \cap \{y \leq 0\}$, from which we obtain after successive symmetries the doubly periodic example described in Proposition 5.8.*

reflection symmetry with respect to $\gamma_0 \times \mathbb{R}$ that $\beta_1^* = g^+(\nu) \times \{h_0\}$, for a certain $\nu \in (0, \mu)$.

- The curve β_3^* is a vertical curvature line of length R starting horizontally at p_2^* and finishing vertically at $p_3^* = (\pi(p_3^*), 0)$. Since β_2, β_3 meet orthogonally at p_2, the same happens to β_2^*, β_3^* at p_2^*. In particular, $\beta_3^* \subset \gamma_{\pi/2} \times \mathbb{R}$, and the normal to the surface along β_3^* is tangent to $\gamma_{\pi/2} \times \mathbb{R}$. Hence β_3^* is the graph of a strictly decreasing concave function over the (oriented) horizontal segment $\overline{\mathbf{0}\,\pi(p_3^*)} \subset \gamma_{\pi/2}$. Finally, since $\Sigma_R^* \subset \{x > 0\}$ in a neighborhood of β_2^*, we deduce $\overline{\mathbf{0}\,\pi(p_3^*)} \subset \gamma_{\pi/2}^+$.
- The curve $\beta_4^* \subset \{t = 0\}$ is a horizontal curvature line with non-vanishing geodesic curvature in $\{t = 0\} \equiv \mathbb{H}^2$. Since the normal to Σ_R^* points to the positive direction of the x-axis at p_3^* and $\Sigma_R^* \subset \{y > 0\}$ in a small neighborhood of β_3^*, we get that β_4^* is orthogonal to $\gamma_{\pi/2} \times \mathbb{R}$ and lies inside $\{y > 0\}$ near p_3^*. Moreover, the intrinsic distance in $\Sigma_R \cap \{y \leq 0\}$ between β_1 and β_4 is h_0 (which is the asymptotic distance at infinity), and $\Sigma_R \cap \{y \leq 0\}$ is isometric to Σ_R^*, then β_4^* is asymptotic to $g(\nu)$ at $\partial_\infty \mathbb{H}^2$. This is, Σ_R^* is asymptotic to $g(\nu) \times [0, h_0]$. Finally, we know by the maximum principle for surfaces with boundary that β_4^* is concave with respect to $\mathcal{P}(R)^*$. In particular, it is contained in $\{y > 0\}$.

By the maximum principle, $\Sigma_R^* \subset \{0 \leq t \leq h_0\}$. If we make reflection symmetries with respect to $\mathbb{H}^2 \times \{0\}$, $\gamma_0 \times \mathbb{R}$ and $\gamma_{\pi/2} \times \mathbb{R}$, we get a properly embedded minimal annulus bounded by the geodesics $g(\pm\nu) \times \{\pm h_0\}$. Then by successive symmetries with respect to these geodesic boundary lines, we get a doubly periodic minimal surface invariant by ϕ_ν^4 and $T(4h_0)$. In the quotient by ϕ_ν^4 and $T(4h_0)$, the surface is topologically a torus minus four points. In the quotient by $T(4h_0)$ and $T(2h_0) \circ \phi_\nu^2$, the surface is topologically a Klein bottle minus two points. These examples correspond to the Toroidal Halfplane Layers of $\mathbb{R}^3$ denoted by $M_{\theta,0,0}$ in [19]. We now have two free parameters instead of only one.

PROPOSITION 5.8. *For any $R > 0$ and any $\mu \in (0, 1)$, there exist $h_0 = h_0(R, \mu) > \ln \frac{1+\mu}{1-\mu}$ and $\nu = \nu(R, \mu) \in (0, \mu)$ for which there exists a properly embedded minimal surface $\mathcal{M}(R, \mu)$ in $\mathbb{H}^2 \times \mathbb{R}$ which is invariant by the vertical translation $T(4h_0)$ and the horizontal hyperbolic translation ϕ_ν^4 along γ_0. In the quotient by $T(4h_0)$ and ϕ_ν^4, $\mathcal{M}(R, \mu)$ is topologically a torus minus four points, whose ends are asymptotic to the quotient of $g(\pm\nu) \times \mathbb{R}$. Moreover, $\mathcal{M}(R, \mu)$ contains the horizontal geodesics*

$g(\pm\nu)\times\{\pm h_0\}$, and is invariant by reflection symmetry with respect to $\gamma_0\times\mathbb{R}$, $\gamma_{\pi/2}\times\mathbb{R}$ and $\{t=0\}$. In the quotient by $T(4h_0)$ and $T(2h_0)\circ\phi_\nu^2$, $\mathcal{M}(R,\mu)$ is topologically a Klein bottle minus two points. We call these examples horizontal Toroidal Halfplane Layers of type 2.

Remark 5.9. Up to a hyperbolic horizontal translation along γ_0, we can fix $B_0=\mathbf{0}$ in the construction above. Then the graph $u_\infty=u_\infty(\mu,R)$ converges as $\mu\to+\infty$ to the unique minimal graph w over the geodesic triangle of vertices $\mathbf{0},B_R,q_1=(0,1)$ with boundary values 0 over $\overline{B_R\,\mathbf{0}}\cup\overline{\mathbf{0},q_1}$ and $+\infty$ over $\overline{B_R\,q_1}$. Such a limit graph produces, after successive rotations about the horizontal geodesics $\overline{B_R\,\mathbf{0}}\cup\overline{\mathbf{0},q_1}$ and the vertical geodesic $\{B_R\}\times\mathbb{R}^+$ in its boundary, one of the "horizontal helicoids" $\mathcal{H}$ described by Pyo in [16]. Then the conjugate surfaces $\mathcal{M}(R,\mu)$ converge as $\mu\to+\infty$ to one of the "horizontal catenoids" constructed in [13, 16].

6. Minimal surfaces invariant by a subgroup of $\mathrm{Isom}(\mathbb{H}^2)$**.** In this section, we construct some examples of minimal surfaces invariant by a subgroup G of the isometries of $\mathrm{Isom}(\mathbb{H}^2\times\mathbb{R})$ that fix the vertical coordinate. We will say that such a G is a subgroup of the isometries of $\mathrm{Isom}(\mathbb{H}^2)$. In fact, the subgroups we consider come from tilings of the hyperbolic plane. We will use some notation that we introduce in Appendix A.

The horizontal slices are clearly invariant by any subgroup of the isometries of $\mathrm{Isom}(\mathbb{H}^2)$. The first non-trivial example is the following: We consider $n\geq 3$ and $\theta=\pi/n$. From Appendix A, there is $y\in\gamma_{\theta/2}$ such that the polygon $\mathcal{P}_y$ is a regular convex polygon in $\mathbb{H}^2$ with $2n$ edges of length $2h_n$ and inner angle $\pi/2$ at the vertices (see Appendix A for the definitions of $\mathcal{P}_y$ and h_n). On this polygon, there is a solution u of (1) with boundary values $\pm\infty$ alternately on each edge. The graph of u is a minimal surface bounded by $2n$ vertical lines over the vertices of $\mathcal{P}_y$. Since $\mathcal{P}_y$ is the fundamental piece of a colorable tiling of $\mathbb{H}^2$ (see Proposition A.2) the graph of u can be extended by successive symmetries along its boundary to a properly embedded minimal surface in $\mathbb{H}^2\times\mathbb{R}$. This surface is invariant by the subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ generated by the symmetries with respect to the vertices of the tiling.

We now construct other non-trivial examples of properly embedded minimal surfaces invariant by a subgroup of the isometries of $\mathrm{Isom}(\mathbb{H}^2)$. The construction of these surfaces is similar to the one for some of the most symmetric Karcher's Toroidal Halfplane Layers in $\mathbb{R}^3$.

Fix $n\geq 3$ and $h>h_n$. By Claim A.1 and Proposition A, there exist $\ell<h_n$ and a convex polygonal domain $\mathcal{P}(n,h)\subset\mathbb{H}^2$ with $2n$ edges of lengths h and ℓ, disposed alternately, whose inner angles are $\pi/2$. Such a domain $\mathcal{P}(n,h)$ produces by successive rotations about its vertices a colorable tiling of $\mathbb{H}^2$.

Consider the minimal graph Σ over $\mathcal{P}(n,h)$ with boundary values 0 over the edges of length h and $+\infty$ over the edges of length ℓ. Such a graph exists, by [14], and is unique. By uniqueness, Σ is invariant by reflection symmetry across the vertical geodesic planes passing through the origin of $\mathcal{P}(n,h)$ and the middle points of the edges of the polygon. We rotate Σ about the horizontal and vertical geodesics in its boundary, producing a properly embedded minimal surface $\mathcal{M}$ invariant by a subgroup of the group of isometries of the tiling produced from $\mathcal{P}(n,h)$. $\mathcal{M}$ projects vertically over the whole $\mathbb{H}^2$, and contains all the edges of the tiling coming from the edges of $\mathcal{P}(n,h)$ of length h (identifying them with the corresponding horizontal geodesics at height zero), and the vertical geodesics over the vertices of the tiling.

Proposition 6.1. *For any $n \geq 2$ and any $h > h_n$, there exists a properly embedded minimal surface $\mathcal{M}$ invariant by the group of isometries of the tiling produced by the polygon $\mathcal{P}(n,h)$ defined above. The vertical projection of $\mathcal{M}$ is the entire $\mathbb{H}^2$ and the ends of $\mathcal{M}$ are asymptotic to the vertical geodesic planes over the edges of the tiling coming from the edges of $\mathcal{P}(n,h)$ with length ℓ. Moreover, $\mathcal{M}$ contains all the edges of the tiling coming from the edges of length h and the vertical geodesics over the vertices of the tiling.*

In the following subsections, we prove:

Proposition 6.2. *For any $n \geq 3$ and any $h > h_n$, there exists a properly embedded minimal surface M invariant by the group of isometries of the tiling produced by the polygon $\mathcal{P}(n,h)$. M projects vertically over the tiles in black and its ends are asymptotic to the vertical geodesic planes over the edges of the tiling coming from the edges of $\mathcal{P}(n,h)$ of length h. Moreover, M is invariant by reflection symmetry across $\{t = 0\}$ and contains the vertical geodesics over the vertices of the tiling.*

6.1. The conjugate minimal surfaces $M^*_{h,\lambda}$. Let $n \geq 3$ be an integer and $\theta = \pi/n$. We consider $h > 0$ and $\lambda \in (0,1)$. In Subsection 4.3, we have constructed the minimal surface $M_{h,\lambda}$ which is bounded by the union of four curves: c_1, c_2, c_3 and c_4.

Let $M^*_{h,\lambda}$ be the conjugate minimal surface of $M_{h,\lambda}$. The aim of this subsection is to describe $M^*_{h,\lambda}$ and prove that it is embedded. We notice that $M^*_{h,\lambda}$ is well defined up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$. In the following, we will fix this isometry by making some hypotheses on $M^*_{h,\lambda}$.

The vertical coordinate h^* of $M^*_{h,\lambda}$ is defined on Ω_0 by a primitive of the closed 1-form ω^* defined in (2). Up to a vertical translation, we can assume $h^*(p_\lambda) = 0$. Because of the definition of ω^* and since $M_{h,\lambda} \subset \mathbb{H}^2 \times [0, h/2]$, h^* increases along $\pi(c_4)$ from p_λ to q_1, along c_2 from p_λ to $\mathbf{0}$ and along c_1 from $\mathbf{0}$ to q_1. Thus h^* is non-negative.

The surface $M^*_{h,\lambda}$ is bounded by $c_1^*, c_2^*, c_3^*, c_4^*$, where each c_i^* corresponds by conjugation to c_i. Let us give a first description of these curves (see Figure 13):

- c_1^* is a vertical geodesic curvature line lying on a vertical geodesic plane Π_1, with infinite length and endpoint A_0^*, the conjugate point to A_0. We can assume that A_0^* is the point $(\mathbf{0}, h^*(\mathbf{0}))$ and that Π_1 is the plane $\gamma_0 \times \mathbb{R}$. The unit tangent vector to c_1^* at A_0^* is horizontal and we assume it points to $\{y \geq 0\}$. The angle function ν^* is positive along c_1^* (as this was the case for the angle function ν of $M_{h,\lambda}$ along c_1) and the height function increases along c_1^* when starting from A_0^*. In the Euclidean plane Π_1, c_1^* is then the graph of a convex increasing function over a part $[\mathbf{0}, a_1)$ of γ_0^+ (a_1 could be a priori in the asymptotic boundary of $\mathbb{H}^2$).
- c_2^* is a vertical geodesic curvature line of length $\ln\left(\frac{1+\lambda}{1-\lambda}\right)$ lying on a vertical geodesic plane Π_2. Since, the angle between c_1 and c_2 is θ at A_0, we get that the angle between Π_1 and Π_2 is θ ($M^*_{h,\lambda}$ is horizontal at A_0^* and isometric to $M_{h,\lambda}$). We take Π_2 the vertical plane $\pi^{-1}(\gamma_\theta)$. Now $M^*_{h,\lambda}$ is uniquely defined. Starting from A_0^*, the height function decreases along c_2^* from $h^*(\mathbf{0})$ to $h^*(p_\lambda) = 0$. In the Euclidean plane Π_2, c_2^* is then the graph of a concave decreasing function over a part of the geodesic γ_θ^+. We denote by A_2^* the endpoint of c_2^* which is different from A_0^*. We have $A_2^* = (a_2, 0)$, with $a_2 \in \gamma_\theta^+$.

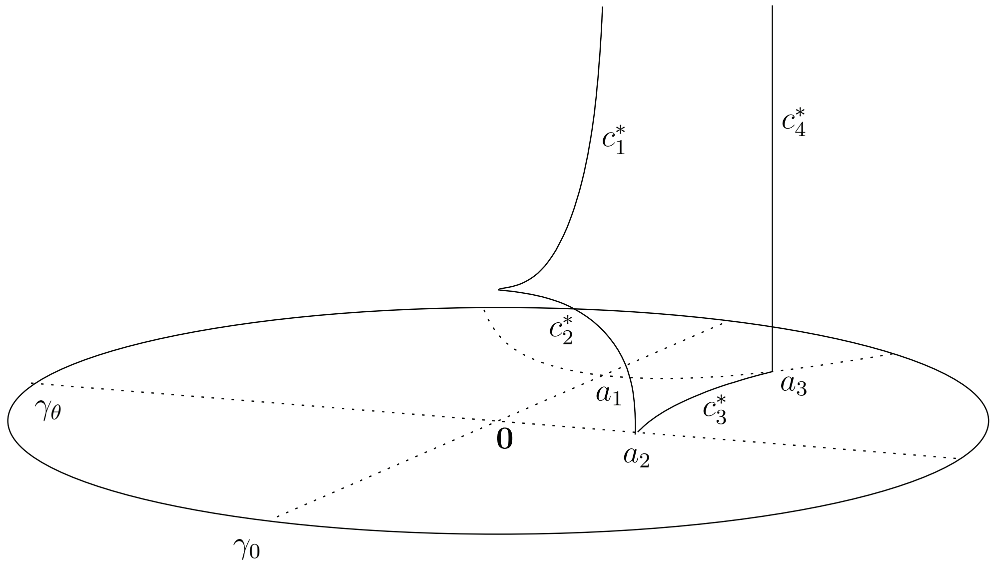

FIG. 13. *The boundary of* $M^*_{h,\lambda}$

- c_3^* is a horizontal geodesic curvature line of length $h/2$ at height zero, going from A_2^* to a point $A_3^* = (a_3, 0)$. The unit tangent vector to c_3^* at A_2^* is orthogonal to Π_2 and points into the side of Π_2 that contains c_1^*. As a curve of $\mathbb{H}^2 \times \{0\}$, the geodesic curvature of c_3^* never vanishes. In fact, since the normal vector field of $M_{h,\lambda}$ rotates less than π along c_3, the total geodesic curvature of $\pi(c_3^*) \subset \mathbb{H}^2$ is less than π. This implies that $\pi(c_3^*)$ and c_3^* are embedded and $\pi(c_3^*)$ does not intersect $\overline{\mathbf{0}a_2}$.
- c_4^* is the half vertical geodesic line $\{a_3\} \times \mathbb{R}^+$.

We know that the distance between c_1 and c_4 is uniformly bounded (in the sense that if c_1 and c_4 are parameterized by arc-length then the distance between $c_1(t)$ and $c_4(t)$ is bounded) and the surface is isometric to its conjugate, so the same is true for c_1^* and c_4^*. Thus the distance between a_1 and a_3 is bounded. This is, a_1 is in $\mathbb{H}^2$, not in $\partial_\infty \mathbb{H}^2$. Then, in the Euclidean plane Π_1, c_1^* is the graph of a convex increasing function over a part $[\mathbf{0}, a_1)$ of γ_0^+ with limit $+\infty$ at a_1.

Because of the asymptotic behaviour of $M_{h,\lambda}$ near q_1, $M^*_{h,\lambda}$ is asymptotic to $\overline{a_1a_3} \times \mathbb{R}$, and the geodesic $\overline{a_1a_3}$ has length $h/2$. Besides, since the normal vector to $M^*_{h,\lambda}$ lies in Π_1 along c_1^*, the geodesic $\overline{a_1a_3}$ is orthogonal to γ_0 at a_1, and a_3 lies in $\{x \geq 0\}$.

Let $(\phi_l)_{l\in(-1,1)}$ be the flow given by Y_0. Let γ be the complete geodesic of $\mathbb{H}^2$ that contains a_1 and a_3. We know that γ is orthogonal to γ_0. We consider the foliation of $\mathbb{H}^2 \times \mathbb{R}$ by the vertical geodesic planes $\phi_l(\gamma \times \mathbb{R})$. Since every point in $M^*_{h,\lambda}$ is at a bounded distance from its boundary, for l close to 1 we have $\phi_l(\gamma \times \mathbb{R}) \cap M^*_{h,\lambda} = \emptyset$. Let l decrease until a first contact point for $l = l_0$. Since $M^*_{h,\lambda}$ is asymptotic to $\overline{a_1a_3} \times \mathbb{R}$, either $l_0 = 0$ or $l_0 > 0$. Let us assume $l_0 > 0$ and reach a contradiction. We have two cases: the first contact point is contained on c_3^* or it coincides with A_2^*. In the first case we get a contradiction using the maximum principle, since the normal vector field of the surface is horizontal along c_3^* and $\phi_{l_0}(\gamma \times \mathbb{R})$ is on one side of $M^*_{h,\lambda}$. Let us now assume that the first contact point is A_2^*. The unit tangent vector to c_3^* points into $\cup_{l \geq l_0} \phi_l(\gamma \times \mathbb{R})$ at A_2^*, this contradicts that we have the first contact point for $l = l_0$. So $M_{h,\lambda}^*$ never intersects $\phi_l(\gamma \times \mathbb{R})$ until $l = 0$. This implies that a_2 and c_3^* are in the half hyperbolic space bounded by γ which contains $\mathbf{0}$.

Let γ' be the geodesic passing through a_3 and orthogonal to γ_θ. Using a similar argument as above with the corresponding foliation by vertical geodesic planes, we can prove that:

- c_3^* is in the half hyperbolic plane $\{x \geq 0\}$;
- c_3^* and a_2 are in the half hyperbolic plane bounded by γ' which contains $\mathbf{0}$.

For the second item, we need to extend $M_{h,\lambda}^*$ by symmetry along c_2^*.

Let Ω be the domain of $\mathbb{H}^2$ bounded by $\overline{a_3 a_1}$, $\overline{a_1 \mathbf{0}}$, $\overline{\mathbf{0} a_2}$ and c_3^*. Since the angle function ν^* never vanishes outside $c_3^* \cup c_4^*$, we conclude $M_{h,\lambda}^* \subset \Omega \times \mathbb{R}$. In fact, since A_0^* is the only point in $M_{h,\lambda}^*$ that projects on $\mathbf{0}$, $M_{h,\lambda}^*$ is a vertical graph over Ω. This implies that $M_{h,\lambda}^*$ is embedded.

6.2. Symmetry and the period problem. We recall that $n \geq 3$. From now on, we assume that $h > h_n$, where h_n is defined in Appendix A (h_n is the length of the edges of the regular geodesic polygon with $2n$ edges with interior angles $\pi/2$). We want to find a value for the parameter λ for which we can construct an embedded minimal surface extending $M_{h,\lambda}^*$ by symmetry along its boundary.

Let us consider the surface $\Sigma_{h,\lambda}$ described in Subsection 4.3. The same argument as in this subsection proves that $\Sigma_{h,\lambda}$ converges when $\lambda \to 1$. By uniqueness, we get that this limit minimal surface must be Σ_h, described in Subsection 4.2 (see Figure 6). Moreover, the surfaces $\Sigma_{h,\lambda}$ depend continuously on the parameter λ. Thus a_1, a_2 and a_3 depend continuously on λ as well.

We define $M_h = \Sigma_h \cap \{0 \leq t \leq h/2\}$, and M_h^* its conjugate surface. As both $M_{h,\lambda}$ and $M_{h,\lambda}^*$ are vertical minimal graphs and $M_{h,\lambda}$ converges to M_h as $\lambda \to 1$, we can conclude as in [13] that the graphs $M_{h,\lambda}^*$ converge to M_h^* when $\lambda \to 1$.

We translate vertically M_h^* so that $A_0^* = (\mathbf{0}, 0)$. The curve $M_h \cap \{t = h/2\}$ corresponds by conjugation to a vertical geodesic $\{a'\} \times \mathbb{R}$, where a' is the limit of the points a_3 when $\lambda \to 1$. Since M_h is invariant by the reflection symmetric with respect to the plane $\gamma_{\theta/2} \times \mathbb{R}$, then M_h^* is invariant by the rotation of angle π about the geodesic $\gamma_{\theta/2}$, contained in M_h^*. Therefore $a' \in \gamma_{\theta/2}$ and this implies that, for λ sufficiently close to 1, a_3 lies in the hyperbolic angular sector $T_\theta = \{(r \sin u, r \cos u) \in \mathbb{H}^2, r \in [0,1), u \in [0,\theta]\}$.

Let a_4 be the orthogonal projection of a_3 over γ_θ. As λ goes to 1, a_3 goes to a' and a_4 goes to the projection a'_θ of a'. We recall that a_1 is the orthogonal projection of a_3 on γ_0 so a_1 goes to the projection a'_0 of a' on γ_0. Since $h > h_n$ and M_h^* (for $\lambda = 1$) is invariant by the rotation of angle π about $\gamma_{\theta/2}$, we deduce that the angle between $\overline{a' a'_0}$ and $\overline{a' a'_\theta}$ is strictly smaller than $\pi/2$. Thus the angle between $\overline{a_3 a_1}$ and $\overline{a_3 a_4}$ is strictly less than $\pi/2$, for λ close to 1.

Let us observe what happens when λ is close to 0. By construction, a_3 is at distance $h/2$ from the geodesic γ_0 (i.e. a_3 lies on $d(h/2)$, the equidistant curve of γ_0 at distance $h/2$). Besides the distance from $\mathbf{0}$ to a_3 is less than the sum of the lengths of c_2^* and c_3^*. So this distance is less than $\ln\left(\frac{1+\lambda}{1-\lambda}\right) + h/2$. So for λ small, the distance between $\mathbf{0}$ and a_3 is close to $h/2$. This implies that a_3 lies outside the angular sector T_θ when λ is close to zero.

By continuity, there is a largest λ, denoted by λ_0, such that $a_3 \in \gamma_\theta$. In particular, a_3 is contained in T_θ for any $\lambda > \lambda_0$. For $\lambda > \lambda_0$ close to λ_0, $a_3 \in T_\theta$ is close to γ_θ. So the angle between $\overline{a_3 a_1}$ and $\overline{a_3 a_4}$ is bigger than $\pi/2$. A continuity argument says that there exists $\lambda_1 \in (\lambda_0, 1)$ such that $a_3 \in T_\theta$ and the angle between $\overline{a_3a_1}$ and $\overline{a_3a_4}$ is equal to $\pi/2$ (see the proof of Claim A.1 for a similar argument). This value λ_1 is the one we look for; so from now on, we fix $\lambda = \lambda_1$.

The domain Ω is included in the convex polygonal domain of vertices $\mathbf{0}$, a_1, a_3 and a_4. We denote by $\widetilde{\Omega}$ the domain obtained from Ω by reflection with respect to the geodesics γ_0 and γ_θ successively. The boundary of $\widetilde{\Omega}$ has $2n$ vertices which are the images of a_3 and is composed of n geodesic arcs corresponding to $\overline{a_1a_3}$ and n concave arcs corresponding to c_3^*. This domain is included in the convex polygonal domain $\mathcal{P}$, which is constructed by the same symmetries from the geodesic polygon of vertices $\mathbf{0}, a_1, a_3, a_4$ (this polygon corresponds to the polygon $\mathcal{P}_{a_3}$ in Appendix A). $\mathcal{P}$ has $2n$ vertices coming from a_3, all of them with interior angle $\pi/2$; and its edges have lengths h and b, alternatively, where b is twice the length of the geodesic arc $\overline{a_3a_4}$. Such a polygon $\mathcal{P}$ is then the fundamental piece of a colorable tiling of $\mathbb{H}^2$ (see Proposition A.2).

Let us now extend M^*_{h,λ_1} by successive reflection symmetries with respect to the planes $\gamma_0 \times \mathbb{R}$ and $\gamma_\theta \times \mathbb{R}$. We get a minimal surface $\widetilde{M}$ which is a vertical graph over $\widetilde{\Omega}$ with value 0 along the concave arcs and $+\infty$ on the geodesic arcs. Moreover, this surface is in $\{t \geq 0\}$ and has all the symmetries of the polygonal domain $\mathcal{P}$. By reflection symmetry with respect to the horizontal slice $\{t = 0\}$, we get an embedded minimal surface whose boundary consists of $2n$ vertical geodesic lines passing through the vertices of $\mathcal{P}$. Such a surface is topologically a sphere minus n points.

From Proposition A.2, $\mathcal{P}$ is the fundamental piece of a hyperbolic colorable tiling. Thus we can extend the surface by successive reflection symmetries along the vertical geodesics contained in its boundary, getting a properly embedded minimal surface M which is invariant by the group of symmetries generated by the rotation around the vertices of the tiling. Moreover the surface projects only on tiles in black of $\mathcal{P}$. This proves Proposition 6.2.

Remark 6.3. If $n = 2$, the above contruction can be done without selecting the value of the parameter λ. Thus we get the surface $\widetilde{M}$ that can be extended by symmetry with respect to $\{t = 0\}$ to get a minimal surface whose boundary consists of 4 vertical geodesic lines. This surface is topologically an annulus. So this surface is a solution to the following Plateau problem: finding a minimal annulus bounded by four vertical geodesic lines. In this sense, it is very similar to the Karcher saddle [5] of $\mathbb{R}^3$. But in our situation it can't be extended by symmetry along its boundary into an embedded minimal surface of $\mathbb{H}^2 \times \mathbb{R}$.

Appendix A. Geodesic polygonal domains with right angles. In this appendix, we give some facts about the tilings of the hyperbolic plane that we consider in the paper.

Let $n \geq 3$ be an integer and define $\theta = \pi/n$. Let y_l be the point $(l\sin(\theta/2), l\cos(\theta/2))$ in $\mathbb{H}^2$, for $0 < l < 1$. Rotating y_l around $\mathbf{0}$ by $k\theta$ ($k = 1, \cdots, 2n-1$), we get the $2n$ vertices of a regular convex geodesic polygon in $\mathbb{H}^2$. We denote by h the length of one of its $2n$ edges. h is an increasing function of l. When l varies from 0 to 1, the interior angle of the polygon at y_l decreases from $\pi - \theta$ to 0. Thus there is one value of l such that this angle is $\pi/2$. We denote by h_n the associated value of h.

Let y be in T_θ. Considering the successive image of y by the reflections with respect to $\gamma_{k\theta}$ (($k = 1, \cdots, 2n$), we construct the $2n$ vertices of a convex polygon whose edges have alternative lengths a_y and b_y, where $a_y/2$ is the distance from y to

γ_0 and $b_y/2$ the one to γ_θ. We denote by $\mathcal{P}_y$ this polygon and by α_y the interior angle of $\mathcal{P}_y$ at the vertex y (the angle is the same at every vertex).

Claim A.1. *For any $a \geq h_n$, there is $y \in T_\theta$ such that $a_y = a$ and $\alpha_y = \pi/2$.*

Proof. Let $d(a/2)$ be the equidistant curve to γ_0 at distance $a/2$ in $\{x \geq 0\}$. Let y be on the part of $d(a/2)$ between $\gamma_{\theta/2}$ and γ_θ. Then $a_y = a$. If $y \in \gamma_{\theta/2}$, $\mathcal{P}_y$ is a regular convex polygon ($a_y = b_y$) and $\alpha_y \leq \pi/2$, since $a \geq h_n$. For y close to γ_θ, $\alpha_y > \pi/2$. By continuity, there is y such that $\alpha_y = \pi/2$. □

Proposition A.2. *Let $y \in T_\theta$ be such that $\alpha_y = \pi/2$. Then $\mathcal{P}_y$ is the fundamental piece of a tiling of $\mathbb{H}^2$. This tiling is given by considering the successive images of $\mathcal{P}_y$ by reflection with respect to its edges. Moreover, this tiling is colorable i.e. we can associate to any tile a color (black or white) such that two tiles having a common edge do not have the same color.*

For such a tiling, every vertex lies in four tiles: two are black and two are white. Two tiles of the same color with a common vertex are exchanged by the symmetry around this vertex. Proposition A.2 is a consequence of Poincaré's polyhedron Theorem [7].

Appendix B. Some interesting minimal surfaces. In this appendix, we recall some known minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ that we used in the paper.

Let us consider the half-space model for the hyperbolic plane : $\mathbb{H}^2 = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^+\}$ with the hyperbolic metric $g = \frac{1}{x_2^2}(dx_1^2 + dx_2^2)$.

On $\{x_1 > 0\}$, the function $v(x_1, x_2) = \log(\frac{\sqrt{x_1^2+x_2^2}+x_2}{x_1})$ is a solution to (1). Its graph is then a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. On the boundary of $\{x_1 > 0\}$, v takes the value $+\infty$ on the geodesic line $\{x_1 = 0\}$ and takes the value 0 on the asymptotic boundary of $\{x_1 > 0\}$. This solution was discovered independently by U. Abresch and R. Sa Earp. This surface is used in Subsection 4.1

On the entire $\mathbb{H}^2$, another solution to (1) is given by $u_a(x_1, x_2) = a\log(x_1^2 + x_2^2)$. This solution is invariant by the Z-flow, for Z normal to $\{x_1 = 0\}$. In fact the graph of u_a is a minimal surface foliated by horizontal geodesics in $\mathbb{H}^2 \times \mathbb{R}$ normal to $\{x_1 = 0\} \times \mathbb{R}$. Adding a constant c to u_a, we create a foliation of $\mathbb{H}^2 \times \mathbb{R}$ by such surfaces. When a varies in $\mathbb{R}$, we get a family of minimal surfaces which are similar to planes in $\mathbb{R}^3$. Moreover, for any non vertical tangent plane at $(0, 1, 0)$ which is tangent to Z, one surface in this family is tangent to this tangent plane. In order to have the complete family, we can add the vertical minimal plane $\{x_1^2 + x_2^2 = 1\} \times \mathbb{R}$. These surfaces are the P surfaces used in the proof of Claim 5.1.

If we look for solutions of (1) of the form $u(x_1, x_2) = f(x_1/x_2)$, we obtain solutions which are invariant by a translation along the geodesic $\{x_1 = 0\}$. The above solution v is one such solution. In fact, for any $h > \pi$, there is $d_h > 0$ and a function f_h which is defined on $[d_h, +\infty)$ such that $u_h(x_1, x_2) = f_h(x_1/x_2)$ is a solution to (1) (see [18, 9]). This function f_h is a decreasing function with $f_h(d_h) = h/2$ and $\lim_{+\infty} f_h = 0$ and $\lim_{d_h} f_h' = -\infty$. The function u_h is then defined on the set of points at distance larger than d_h from $\{x_1 = 0\}$ and has boundary value $h/2$ on the equidistant and 0 on the asymptotic boundary. When $h \to +\infty$, u_h converge to the above solution v. The graph of u_h is a minimal surface inside $\{0 < t \leq h/2\}$ which is foliated by horizontal equidistant lines to $\{x_1 = 0\} \times \mathbb{R}$ and is vertical along its boundary. Then this graph can be extended by symmetry with respect to $\{t = h/2\}$ to a complete minimal surface S_h which is a vertical bigraph, included in $\{0 < t < h\}$, foliated by horizontal equidistant lines to $\{x_1 = 0\} \times \mathbb{R}$. Moreover, the supremum of the vertical gap on S_h is h. The surfaces S_h are used in Subsections 5.1 and 5.3 as barriers in our construction.

Appendix C. Alexandrov reflection. In Subsection 5.1, we construct a minimal surface Σ_∞ as the limit of surfaces Σ_R. These surfaces Σ_R are minimal disks bounded by a Jordan curve Γ_R. We claim that the Alexandrov reflection technique can be applied with respect to horizontal slices to prove that Σ_R is a vertical bigraph with respect to $\{t = h/2\}$. Since there are two vertical arcs in Γ_R, we need to explain how the classical Alexandrov reflection technique works along these vertical edges.

In order to lighten the notation, we put $\Sigma = \Sigma_R$ and $\Gamma = \Gamma_R$. For $l \in [0, h]$, we define Π_l the horizontal slice $\{t = l\}$. We denote by P_l and Q_l the points in the vertical edges of Γ at height l (since the arguments work the same for both points in the sequel, we will assume that there is only one). Let $\Sigma_l = (\Sigma \cap \pi_l) \setminus \{P_l, Q_l\}$. We also define Σ_l^+ (resp. Σ_l^-) the part of Σ above (resp. below) Π_l minus its boundary. Finally we denote by Σ_l^{+*} and Σ_l^{-*} the symmetric of Σ_l^+ and Σ_l^- by Π_l.

The main step of the Alexandrov reflection technique is to prove that, for any $l \in (h/2, h]$, $\Sigma_l^- \cap \Sigma_l^{+*} = \emptyset$ and Σ is never vertical along Σ_l.

The property is true for $l = h$ since $\Sigma_h^{+*} = \emptyset$ and Σ is inside the convex hull of its boundary.

We notice that for any $l \in (h/2, h)$, if $\Sigma_l^- \cap \Sigma_l^{+*} = \emptyset$ is proved, then Σ is never vertical along Σ_l follows easily.

Now we consider $l_0 \in (h/2, h]$ such that the property is satisfied for any $l \geq l_0$. Let us assume that there exists a sequence of $l_k < l_0$ with $l_k \to l_0$ and, for any k, there is $p_k \in \Sigma_{l_k}^- \cap \Sigma_{l_k}^{+*}$.

Since $\Sigma_{l_0}^- \cap \Sigma_{l_0}^{+*} = \emptyset$, the limit p_∞ of p_k is either in Σ_{l_0} or in the vertical edge. Since Σ is not vertical along Σ_{l_0}, $p_\infty \notin \Sigma_{l_0}$. So p_∞ is in the vertical edge. Since $\Sigma_{l_0}^- \cap \Sigma_{l_0}^{+*} = \emptyset$, the tangent space to $\Sigma_{l_0}^-$ and $\Sigma_{l_0}^{+*}$ are different for any point in the the vertical edge except at P_{l_0} so the only possible limit is $p_\infty = P_{l_0}$.

Let us first consider the case $l_0 < h$, and let (x, y, z) be an orthogonal coordinate system at P_{l_0} such that (x, y) are euclidean coordinates in the vertical plane tangent to Σ at P_{l_0}, where ∂_x is a vertical down pointing vector field and ∂_y is a horizontal vector field. Σ is then locally the graph of a function $z = w(x, y)$ over $\{y \geq 0\}$. w vanishes on $\{y = 0\}$ and has vanishing differential at the origin. We notice that $\{z = 0\}$ is a minimal surface thus from the proof of Theorem 5.3 in [1], w can be written in the following way:

$$w(x, y) = p(x, y) + q(x, y),$$

where p is a homogeneous harmonic polynomial of degree d and q satisfies

$$|q(X)| + |X||\nabla q(X)| + \cdots + |X|^d|\nabla^d q(X)| \leq C|X|^{d+1}.$$

Since $\Sigma_{l_0}^- \cap \Sigma_{l_0}^{+*} = \emptyset$, then $w(x, y) - w(-x, y)$ has a sign for any $|(x, y)| < \varepsilon$ with $x \neq 0$ and $y \neq 0$. We assume that the coordinate z is chosen such that this sign is $+$. Thus $0 \leq w(x, y) - w(-x, y) = p(x, y) - p(-x, y) + q(x, y) - q(-x, y)$ for x and y non negative close to 0, and it does not vanish for positive values of x and y. Thus the degree of $p(x, y) - p(-x, y)$ has to be 2, and $p(x, y) = \alpha xy$ with $\alpha > 0$.

When $l_0 = h$, we also get that Σ is the graph of function w over $[0, \varepsilon]^2$ with $w(x, y) = \alpha xy + q(x, y)$, for the same choice of coordinate system, with α and q satifying the same hypotheses as above.

Since $p_k \to P_{l_0}$, for k large enough we get $p_k = (x_k, y_k, w(x_k, y_k))$, with $(x_k, y_k) \in [-l_k, \varepsilon] \times [0, \epsilon]$. Since $p_k \in \Sigma_{l_k}^- \cap \Sigma_{l_k}^{+*}$, we have :

$$w(x_k, y_k) = w(2(l_0 - l_k) - x_k, y_k) \tag{3}$$

But if $(x, y) \in [\lambda, \varepsilon] \times [0, \varepsilon]$ we have:

$$w(x,y) - w(2\lambda - x, y) \geq 2\alpha(x-\lambda)y - 2 \sup_{u\in[-\varepsilon,\varepsilon]} |\partial_x q(u,y)|(x-\lambda).$$

Since $0 = w(x, 0) = q(x, 0)$, we get

$$|\partial_x q(u,y)| \leq \sup_{v\in[0,\varepsilon]} |\partial_y \partial_x q(u,v)| y \leq C\sqrt{u^2+\varepsilon^2}y.$$

Thus

$$\begin{aligned} w(x,y) - w(2\lambda - x, y) &\geq 2\alpha(x-\lambda)y - C2\sqrt{2}\varepsilon(x-\lambda)y \\ &\geq 2[\alpha - \sqrt{2}C\varepsilon](x-\lambda)y, \end{aligned}$$

which is positive if $x > \lambda$, $y > 0$ and ε is small enough. This contradicts (3) when k is large enough.

We then have proved that: for any $l \in (h/2, h]$, $\Sigma_l^- \cap \Sigma_l^{+*} = \emptyset$ and Σ is never vertical along Σ_l.

Therefore, we obtain that either $\Sigma_{h/2}^- = \Sigma_{h/2}^{+*}$ and it is a vertical bigraph with respect to $\{t = h/2\}$, or $\Sigma_{h/2}^-$ and $\Sigma_{h/2}^{+*}$ are two non intersecting minimal surfaces with the same boundary. In this second case, $\Sigma_{h/2}^-$ is clearly below $\Sigma_{h/2}^{+*}$ along the $\Gamma \cap \Pi_0$. By symmetry by $\Pi_{h/2}$ this implies that $\Sigma_{h/2}^+$ is below $\Sigma_{h/2}^{-*}$ along $\Gamma \cap \Pi_h$. But doing Alexandrov reflection technique as above with the slices Π_l, $l \in [0, h/2]$, we get that $\Sigma_{h/2}^+$ is above $\Sigma_{h/2}^{-*}$ along $\Gamma \cap \Pi_h$. Finally, we have proved $\Sigma_{h/2}^- = \Sigma_{h/2}^{+*}$.

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