ASIAN J. MATH.
Vol. 18, No. 5, pp. 779–816, November 2014

DIRAC LIE GROUPS*

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Abstract. A classical theorem of Drinfel'd states that the category of simply connected Poisson Lie groups H is isomorphic to the category of Manin triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h}$ is the Lie algebra of H. In this paper, we consider *Dirac Lie groups*, that is, Lie groups H endowed with a multiplicative Courant algebroid $\mathbb{A}$ and a Dirac structure $E \subseteq \mathbb{A}$ for which the multiplication is a Dirac morphism. It turns out that the simply connected Dirac Lie groups are classified by so-called *Dirac Manin triples*. We give an explicit construction of the Dirac Lie group structure defined by a Dirac Manin triple, and develop its basic properties.

Key words. Poisson Lie Groups, Multiplicative Dirac Structures, Multiplicative Courant algebroids, Lie groupoids, Lie bialgebras, Manin triples, Multiplicative Manin pairs, quasi-Poisson geometry, Group valued moment maps.

AMS subject classifications. 53D17 (Primary), 17B62, 53D20.

0. Introduction. Dirac structures were introduced by T. Courant [6] as a common framework for closed 2-forms and Poisson structures on manifolds. He showed that the integrability condition $d\omega = 0$ for 2-forms and $[\pi, \pi] = 0$ for bivector fields admits a common generalization to an integrability condition on Lagrangian subbundles $E \subseteq \mathbb{T}M = TM \oplus T^*M$ relative to a certain bracket on $\Gamma(\mathbb{T}M)$. Liu-Weinstein-Xu [21] generalized Courant's original set-up, replacing $\mathbb{T}M$ with a more general notion of a *Courant algebroid* $\mathbb{A} \to M$. The theory of Courant algebroids and Dirac structures was clarified and simplified in the work of Dorfman [7], Ševera [36, Letter no.7], Roytenberg [34], Uchino [39], and others. It has recently gained attention through the development of generalized complex geometry [11, 13], and it provides a unified setting for various types moment maps [1, 5].

A Poisson Lie group is a Lie group H, equipped with a Poisson structure such that the multiplication map is Poisson. To extend this definition to Dirac geometry, it is required that the Courant algebroid $\mathbb{A}$ itself has a multiplicative structure. As suggested by Mehta [27] and further explored in [20], we require that $\mathbb{A}$ carries a $\mathcal{VB}$-groupoid structure $\mathbb{A} \rightrightarrows \mathfrak{g}$ over the group $H \rightrightarrows \mathrm{pt}$, in such a way that the groupoid multiplication is a Courant morphism $\mathrm{Mult}_{\mathbb{A}} \colon \mathbb{A} \times \mathbb{A} \dashrightarrow \mathbb{A}$. (For the standard Courant algebroid $\mathbb{A} = \mathbb{T}H$ this structure is automatic, with $\mathfrak{g} = \mathfrak{h}^*$.) One then has a notion of a *multiplicative* Dirac structure $E \subseteq \mathbb{A}$. In the case of $\mathbb{T}H$ these were classified in the work of Ortiz [30] and Jotz [15], independently. While [15, 30] refer to multiplicative Dirac structures as Dirac Lie group structures, we will reserve this latter term for the case that the multiplication map is a Dirac morphism (i.e a morphism of Manin pairs as in [5]). For $\mathbb{A} = \mathbb{T}H$, only the Poisson Lie group structures are Dirac Lie group structures in our sense, but many more examples are obtained by considering more general Courant algebroids. These include the well known Cartan-Dirac structure (cf. [1] and references therein), and the examples in Section 5 of [16]. One of the goals of this paper is to develop the theory of Dirac Lie groups in this setting. The super-geometric interpretation of Dirac Lie group structures was previously studied

*Received January 23, 2013; accepted for publication July 5, 2013.

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in [20] under the name *MP Lie groups*.

By Drinfel'd's theorem [8], the category of simply connected Poisson Lie groups H is canonically equivalent to the category of Manin triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$. That is, $\mathfrak{d}$ is a Lie algebra with a vector space decomposition into two Lie subalgebras $\mathfrak{g}, \mathfrak{h}$, and equipped with a non-degenerate invariant symmetric bilinear form ('metric') for which $\mathfrak{g}, \mathfrak{h}$ are Lagrangian. According to a recent result of Michal Siran [38], the non-simply connected Poisson Lie groups are similarly classified by H-equivariant Manin triples.

We will show that Dirac Lie groups H are classified by H-equivariant *Dirac Manin triples* $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$. These consist of a Lie algebra $\mathfrak{d}$ with a vector space direct sum decomposition into two Lie subalgebras $\mathfrak{g}, \mathfrak{h}$, together with an invariant symmetric bilinear form β on the dual space $\mathfrak{d}^*$ such that $\mathfrak{g}$ is β-coisotropic, i.e. the restriction of β to $\mathrm{ann}(\mathfrak{g})$ is zero. Here β may be degenerate or even zero, and there is no compatibility requirement between β and $\mathfrak{h}$. We will prove:

THEOREM 0.1. *The category of Dirac Lie groups and the category of equivariant Dirac Manin triples are canonically equivalent.*

Theorem 0.1 may be deduced from the classification of *MP Lie groups* in [20], but we will give a direct proof, not using super geometry.

The functor from Dirac Manin triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ to Dirac Lie groups is constructed as follows. As a first step, we use a reduction procedure to construct a new Dirac Manin triple $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$, with a Lie algebra homomorphism $f\colon \mathfrak{q} \to \mathfrak{d}$ taking $\mathfrak{r}$ to $\mathfrak{h}$ and restricting to the identity on $\mathfrak{g}$. The new Dirac Manin triple is such that γ is non-degenerate and $\mathfrak{g}$ is Lagrangian in $\mathfrak{q}$. The corresponding Dirac Lie group $(\mathbb{A}, E)$ is described using a 'left trivialization'

$$\mathbb{A} = H \times \mathfrak{q}, \quad E = H \times \mathfrak{g},$$

where $H \times \mathfrak{q}$ is an *action Courant algebroid* [19]. An explicit description of the groupoid structure in terms of this trivialization is given in Theorem 5.2. It is rather cumbersome, however, to verify the compatibility properties from these explicit formulas. Therefore, we show that one can also obtain $(\mathbb{A}, E)$ by co-isotropic reduction of the direct product of the multiplicative Manin pairs $(\mathbb{T}H, TH)$ and $(\overline{\mathfrak{q}} \oplus \mathfrak{q}, \mathfrak{g} \oplus \mathfrak{g})$ (where $\overline{\mathfrak{q}} \oplus \mathfrak{q} \rightrightarrows \mathfrak{q}$ has the pair groupoid structure).

Of particular interest are the Dirac Lie group structures $(\mathbb{A}, E)$ over H for which the underlying Courant algebroid is *exact*, in the sense of Ševera. We prove that this is the case if and only if β is non-degenerate and $\mathfrak{g}$ is Lagrangian. In this case we construct a *canonical* isomorphism with the Courant algebroid $\mathbb{T}H_\eta$, with twisting 3-form the Cartan 3-form over H. We hence obtain another concrete description of the Dirac Lie group structure, in terms of differential forms and vector fields.

The organization of the paper is as follows. In Section 1 we summarize the basic theory of Courant algebroids, Dirac structures, and their morphisms. In Section 2 we introduce and motivate our definition of Dirac Lie groups. Next, in Sections 3 and 4 we classify Dirac Lie groups and their morphisms in terms of Lie theoretic data. In Section 5 we summarize the structural formulas for Dirac Lie groups obtained in the previous two sections, and use them to describe some examples explicitly. Following this, in Section 6 we relate Dirac Lie groups to the theory of quasi-Poisson geometry [2] and the language of quasi-Lie bialgebroids [35, 17, 32]. Finally, in Section 7 we study those Dirac Lie groups for which the underlying Courant algebroid is exact.

Acknowledgments. We thank Henrique Bursztyn and Pavol Ševera for discussions and helpful comments. David Li-Bland was supported by an NSERC CGS-D

Grant; Eckhard Meinrenken was supported by an NSERC Discovery Grant.

1. Preliminaries. We begin with a quick review of Courant algebroids and Dirac structures. A more slow-paced overview and further references can be found in our paper [19].

1.1. Basic definitions. A *Courant algebroid* over a manifold M is a vector bundle $\mathbb{A} \to M$, together with a bundle map $\mathfrak{a}\colon \mathbb{A} \to TM$ called the *anchor*, a bundle metric[1] $\langle\cdot,\cdot\rangle$, and a bilinear bracket $[\![\cdot,\cdot]\!]$ on its space of sections $\Gamma(\mathbb{A})$. These are required to satisfy the following axioms, for all sections $x_1, x_2, x_3 \in \Gamma(\mathbb{A})$:

c1) $[\![x_1, [\![x_2, x_3]\!]]\!] = [\![[\![x_1, x_2]\!], x_3]\!] + [\![x_2, [\![x_1, x_3]\!]]\!]$,
c2) $\mathsf{a}(x_1)\langle x_2, x_3\rangle = \langle [\![x_1, x_2]\!],\, x_3\rangle + \langle x_2,\, [\![x_1, x_3]\!]\rangle$,
c3) $[\![x_1, x_2]\!] + [\![x_2, x_1]\!] = \mathsf{a}^*(\mathrm{d}\langle x_1, x_2\rangle)$.

Here $\mathsf{a}^*\colon T^*M \to \mathbb{A}^* \cong \mathbb{A}$ is the dual map to a. The axioms c1)-c3) imply various other properties, in particular

c4) $[\![x_1, f x_2]\!] = f[\![x_1, x_2]\!] + \mathsf{a}(x_1)(f)x_2$,
c5) $[\![f x_1, x_2]\!] = f[\![x_1, x_2]\!] - \mathsf{a}(x_2)(f)x_1 + \langle x_1, x_2\rangle \mathsf{a}^*(\mathrm{d}f)$
c6) $\mathsf{a}([\![x_1, x_2]\!]) = [\mathsf{a}(x_1), \mathsf{a}(x_2)]$,

for sections $x_1, x_2 \in \Gamma(\mathbb{A})$ and functions $f \in C^\infty(M)$. We will refer to the bracket $[\![\cdot,\cdot]\!]$ as the *Courant bracket* (it is also know as the *Dorfman bracket*). Following Ševera [36], the Courant algebroid is called *exact* if the sequence

$$0 \to T^*M \to \mathbb{A} \to TM \to 0$$

is exact. In particular, the bundle metric of $\mathbb{A}$ is of split signature, and T^*M is a Lagrangian subbundle. Any choice of a Lagrangian splitting $\mathsf{l}\colon TM \to \mathbb{A}$ gives an isomorphism $\mathbb{A} \xrightarrow{\cong} TM \oplus T^*M$, with inverse map $v + \alpha \mapsto \mathsf{l}(v) + \mathsf{a}^*(\alpha)$. Under this identification, the anchor map a becomes projection to the first summand, the bilinear form is $\langle v_1 + \alpha_1, v_2 + \alpha_2\rangle = \langle \alpha_2, v_1\rangle + \langle \alpha_1, v_2\rangle$, and the Courant bracket reads

$$[\![v_1 + \alpha_1, v_2 + \alpha_2]\!] = [v_1, v_2] + \mathcal{L}_{v_1}\alpha_2 - \iota(v_2)\mathrm{d}\alpha_1 + \iota(v_1)\iota(v_2)\eta,$$

for vector fields $v_i \in \mathfrak{X}(M)$ and 1-forms $\alpha_i \in \Omega^1(M)$. Here $\eta \in \Omega^3(M)$ is the closed 3-form obtained as

$$\iota(v_1)\iota(v_2)\iota(v_3)\eta = \langle [\![\mathsf{l}(v_1), \mathsf{l}(v_2)]\!], \mathsf{l}(v_3)\rangle. \tag{1}$$

Conversely, given a closed 3-form η, these formulas define a Courant algebroid structure on $TM \oplus T^*M$. We will denote this Courant algebroid by $\mathbb{T}M_\eta$, or simply $\mathbb{T}M$ if $\eta = 0$. The set of Lagrangian splittings is an affine space, with $\Omega^2(M)$ as its space of motions, and a change of the splitting by a 2-form ω changes η to $\eta + \mathrm{d}\omega$.

Another important class of examples of Courant algebroids is as follows. Suppose $\mathfrak{g}$ is a Lie algebra equipped with an invariant metric. Given a Lie algebra action $\varrho\colon \mathfrak{g} \to \mathfrak{X}(M)$ on a manifold M, let $\mathbb{A} = M \times \mathfrak{g}$ with anchor map $\mathsf{a}(m, \xi) = \varrho(\xi)_m$, and with the bundle metric coming from the metric on $\mathfrak{g}$. As shown in [19], the Lie bracket on constant sections $\mathfrak{g} \subseteq C^\infty(M, \mathfrak{g}) = \Gamma(\mathbb{A})$ extends to a Courant bracket *if and only if* the stabilizers $\mathfrak{g}_m \subseteq \mathfrak{g}$ are coisotropic, i.e. $\mathfrak{g}_m \supseteq \mathfrak{g}_m^\perp$. Explicitly, for $\xi_1, \xi_2 \in \Gamma(\mathbb{A}) = C^\infty(M, \mathfrak{g})$ the Courant bracket reads (see [19, § 4])

$$[\![\xi_1, \xi_2]\!] = [\xi_1, \xi_2] + \mathcal{L}_{\varrho(\xi_1)}\xi_2 - \mathcal{L}_{\varrho(\xi_2)}\xi_1 + \varrho^*\langle \mathrm{d}\xi_1, \xi_2\rangle. \tag{2}$$

[1] In this paper, we take 'metric' to mean a non-degenerate symmetric bilinear form.

Here $\varrho^*\colon T^*M \to M \times \mathfrak{g}$ is the dual map to the action map $\varrho\colon M \times \mathfrak{g} \to TM$, using the metric to identify $\mathfrak{g}^* \cong \mathfrak{g}$. We refer to $M \times \mathfrak{g}$ with bracket (2) as an *action Courant algebroid*.

For any Courant algebroid $\mathbb{A}$, we denote by $\overline{\mathbb{A}}$ the Courant algebroid with the same bracket and anchor, but with the opposite bilinear form.

1.2. Involutive subbundles. Let $\mathbb{A} \to M$ be a Courant algebroid. A subbundle $E \subseteq \mathbb{A}$ along a submanifold $S \subseteq M$ is called *involutive* if it has the property

$$x_1|_S,\ x_2|_S \in \Gamma(E) \Rightarrow [\![x_1, x_2]\!]|_S \in \Gamma(E).$$

We stress that this property need not define a bracket on $\Gamma(E)$, in general. Using the properties c4 and c5 of Courant algebroids, one finds that if $E \to S$ is an involutive sub-bundle, with $0 < \operatorname{rank}(E) < \operatorname{rank}(\mathbb{A})$, then

$$\mathsf{a}(E) \subseteq TS, \ \ \mathsf{a}(E^\perp) \subseteq TS.$$

Note that the second property is not preserved under intersections of bundles, and indeed a sub-bundle given as the intersection of involutive sub-bundles need not be involutive (unless these subbundles are defined over the same submanifold). An involutive Lagrangian sub-bundle $E \subseteq \mathbb{A}$ along $S \subseteq M$ is called a *Dirac structure along* S. For instance, if $\mathbb{A} = \mathbb{T}M$ is the standard Courant algebroid, then $T^*M|_S$ and $TS \oplus \operatorname{ann}(TS)$ are Dirac structures along S.

A Dirac structure along $S = M$ is simply called a Dirac structure. These were introduced by Courant [6] and Liu-Weinstein-Xu [21]; the notion of a Dirac structure along a submanifold goes back to Ševera [36] and was developed in [4, 5, 32].

1.3. Courant relations. A *smooth relation* $S\colon M_0 \dashrightarrow M_1$ between manifolds is an immersed submanifold $S \subseteq M_1 \times M_0$. We will write $m_0 \sim_S m_1$ if $(m_1, m_0) \in S$. Given smooth relations $S\colon M_0 \dashrightarrow M_1$ and $S'\colon M_1 \dashrightarrow M_2$, the set-theoretic composition $S' \circ S$ is the image of

$$S' \diamond S = (S' \times S) \cap (M_2 \times (M_1)_\Delta \times M_0) \tag{3}$$

under projection to $M_2 \times M_0$. We say that the two relations *compose cleanly* if (3) is a clean intersection in the sense of Bott (i.e. it is smooth, and the intersection of the tangent bundles is the tangent bundle of the intersection), and the map from $S' \diamond S$ to $M_2 \times M_0$ has constant rank. In this case, the composition $S' \circ S\colon M_0 \dashrightarrow M_2$ is a well-defined smooth relation. See Appendix A for more information on the composition of smooth relations. For background on clean intersections of manifolds, see e.g. [14, page 490].

Specializing to vector bundles, Lie algebroids and Courant algebroids, we define

Definition 1.1.

a. A *vector bundle relation* ($\mathcal{VB}$*-relation*) $R\colon V_0 \dashrightarrow V_1$ between vector bundles $V_i \to M_i$ is a subbundle $R \subseteq V_1 \times V_0$ along a submanifold $S \subseteq M_1 \times M_0$.
b. A *Lie algebroid relation* ($\mathcal{LA}$*-relation*) $R\colon E_0 \dashrightarrow E_1$ between Lie algebroids $E_i \to M_i$ is a Lie subalgebroid $R \subseteq E_1 \times E_0$ along a submanifold $S \subseteq M_1 \times M_0$.
c. A *Courant relation* ($\mathcal{CA}$*-relation*) $R\colon \mathbb{A}_0 \dashrightarrow \mathbb{A}_1$ between Courant algebroids $\mathbb{A}_i \to M_i$ is a Dirac structure $R \subseteq \mathbb{A}_1 \times \overline{\mathbb{A}_0}$ along a submanifold $S \subseteq M_1 \times M_0$.

We will sometimes depict $\mathcal{VB}$-relations by diagrams as follows:

$$\begin{tikzcd} V_0 \arrow[r, dashed, "R"] \arrow[d] & V_1 \arrow[d] \\ M_0 \arrow[r, dashed, "S"'] & M_1 \end{tikzcd}$$

Letting $p_{M_i}\colon S \to M_i,\ (m_1, m_0) \mapsto m_i$, we define the kernel $\ker(R) \subseteq p_{M_0}^* V_0$ and the range $\operatorname{ran}(R) \subseteq p_{M_1}^* V_1$ of a $\mathcal{VB}$-relation to be the kernel and the range of the bundle map $R \to p_{M_1}^* V_1,\ (v_1, v_0) \mapsto v_1$.

Given sections $\sigma_i \in \Gamma(V_i)$, we will write $\sigma_0 \sim_R \sigma_1$ if (σ_1, σ_0) restricts to a section of R. Given a relation $S\colon M_0 \to M_1$ and functions $f_i \in C^\infty(M_i)$, we write $f_0 \sim_S f_1$ if $f_0(m_0) = f_1(m_1)$ for all $(m_1, m_0) \in S$. The following is clear from the definitions:

Proposition 1.2. *Suppose $\mathbb{A}_0, \mathbb{A}_1$ are Courant algebroids and $R\colon \mathbb{A}_0 \to \mathbb{A}_1$ is a $\mathcal{VB}$-relation with underlying relation $S\colon M_0 \dashrightarrow M_1$. Suppose $\sigma_0 \sim_R \sigma_1$ and $\tau_0 \sim_R \tau_1$. Then*

a. If R is Lagrangian, $\langle \sigma_0, \tau_0 \rangle \sim_S \langle \sigma_1, \tau_1 \rangle$.
b. If R is involutive, $[\![\sigma_0, \tau_0]\!] \sim_R [\![\sigma_1, \tau_1]\!]$.

The composition $R' \circ R$ of two $\mathcal{VB}$-relations is called *clean* if it is clean as a composition of submanifolds. It is then automatic that $R' \diamond R$ and $R' \circ R$ are smooth subbundles along $S' \diamond S$ and $S' \circ S$, respectively, where S', S are the base manifolds of R', R. Conversely, if the base manifolds compose cleanly, and the pointwise fibers of $R' \diamond R,\ R' \circ R$ have constant rank, then the subbundles compose cleanly.

Remark 1.3. Here it is convenient to work with the characterization of vector bundles and their morphisms in terms of scalar multiplication, due to Grabowski and Rotkiewicz [10]. Specifically, a smooth submanifold of a vector bundle is a vector subbundle if and only if it is invariant under scalar multiplication [10, Theorem 2.3], and a smooth map between vector bundles is a vector bundle homomorphism if and only if it intertwines scalar multiplication [10, Theorem 2.4].

The following proposition shows that the clean composition of $\mathcal{CA}$-relations is again a $\mathcal{CA}$-relation. There is a parallel statement for $\mathcal{LA}$-relations, with a similar proof.

Proposition 1.4. *Suppose $\mathbb{A}_i \to M_i$ are Courant algebroids, and that $R\colon \mathbb{A}_0 \dashrightarrow \mathbb{A}_1$ and $R'\colon \mathbb{A}_1 \dashrightarrow \mathbb{A}_2$ are $\mathcal{VB}$-relations with clean composition.*

a. If R, R' are involutive then so is $R' \circ R$.
b. If R, R' are Lagrangian then so is $R' \circ R$.

In particular, if R, R' are Courant relations then so is $R' \circ R$.

Proof. (a) Let $p\colon M_2 \times M_1 \times M_1 \times M_0 \to M_2 \times M_0$ be the projection, and let

$$Q\colon \mathbb{A}_2 \times \overline{\mathbb{A}_1} \times \mathbb{A}_1 \times \overline{\mathbb{A}_0} \dashrightarrow \mathbb{A}_2 \times \mathbb{A}_0$$

be the relation defined by $(\mathbb{A}_2)_\Delta \times (\mathbb{A}_1)_\Delta \times (\mathbb{A}_0)_\Delta$. Under this relation, $\tilde{\sigma} \sim_Q \sigma$ if and only if the restriction of $\tilde{\sigma} - p^*\sigma$ to $M_2 \times (M_1)_\Delta \times M_0$ takes values in $0 \times (\mathbb{A}_1)_\Delta \times 0$. Since Q is involutive, we have

$$\tilde{\sigma} \sim_Q \sigma,\ \tilde{\tau} \sim_Q \tau \Rightarrow [\![\tilde{\sigma}, \tilde{\tau}]\!] \sim_Q [\![\sigma, \tau]\!].$$

Suppose R, R' are involutive. Let σ be a section of $\mathbb{A}_2 \times \overline{\mathbb{A}_0}$ whose restriction to $S' \circ S$ takes values in $R' \circ R$. Since $R' \diamond R \to R' \circ R$ is a surjective vector bundle homomorphism covering a submersion $S' \diamond S \to S' \circ S$, the restriction $\sigma|_{S' \circ S}$ admits a lift to a section $\tilde{\sigma}_{S' \diamond S}$ of $R' \diamond R$. By definition, $\tilde{\sigma}_{S' \diamond S} - p^*\sigma|_{S' \diamond S}$ takes values in $0 \times (\mathbb{A}_1)_\Delta \times 0$. Since the bundles $R' \times R \to S' \times S$ and $\mathbb{A}_2 \times (\mathbb{A}_1)_\Delta \times \mathbb{A}_0 \to M_2 \times (M_1)_\Delta \times M_0$ intersect cleanly, we may choose $\tilde{\sigma} \in \Gamma(\mathbb{A}_2 \times \overline{\mathbb{A}_1} \times \mathbb{A}_1 \times \overline{\mathbb{A}_0})$ such that

(i) $\tilde{\sigma}|_{S' \diamond S} = \tilde{\sigma}_{S' \diamond S}$,

(ii) $\tilde{\sigma}|_{S' \times S}$ takes values in $R' \times R$,

(iii) $(\tilde{\sigma} - p^*\sigma)|_{M_2 \times (M_1)_\Delta \times M_0}$ takes values in $0 \times (\mathbb{A}_1)_\Delta \times 0$, i.e. $\tilde{\sigma} \sim_Q \sigma$.

Note that (iii) implies that $\tilde{\sigma}|_{M_2 \times (M_1)_\Delta \times M_0}$ takes values in $\mathbb{A}_2 \times (\mathbb{A}_1)_\Delta \times \overline{\mathbb{A}_0}$.

Given another section τ of $\mathbb{A}_2 \times \overline{\mathbb{A}_0}$ whose restriction $\tau|_{S' \circ S}$ takes values in $R' \circ R$, let $\tilde{\tau}$ be constructed similarly. Since $R' \times R$ and $\mathbb{A}_2 \times (\mathbb{A}_1)_\Delta \times \overline{\mathbb{A}_0}$ are involutive, the restriction of $[\![\tilde{\sigma}, \tilde{\tau}]\!]$ to $S' \times S$ takes values in $R' \times R$, while the restriction to $M_2 \times (M_1)_\Delta \times M_0$ takes values in $\mathbb{A}_2 \times (\mathbb{A}_1)_\Delta \times \overline{\mathbb{A}_0}$. Hence $[\![\tilde{\sigma}, \tilde{\tau}]\!]|_{S' \diamond S}$ takes values in $R' \diamond R$. Since $[\![\tilde{\sigma}, \tilde{\tau}]\!] \sim_Q [\![\sigma, \tau]\!]$, this shows that $[\![\sigma, \tau]\!]|_{S' \circ S}$ takes values in $R' \circ R$.

Part (b) follows from the well-known statement that the composition of Lagrangian relations of vector spaces is again Lagrangian (Lemma A.1). $\square$

A *Courant morphism* [36] is a Courant relation $R\colon \mathbb{A}_0 \dashrightarrow \mathbb{A}_1$ such that the underlying relation $S\colon M_0 \dashrightarrow M_1$ is the graph of a map $\Phi\colon M_0 \to M_1$. (In contrast with vector bundle morphisms or Lie algebroid morphisms, one does not require that R be a graph.) As a special case of Proposition 1.4, the composition of Courant morphisms is again a Courant morphism.

EXAMPLE 1.5. Any smooth map $\Phi\colon M_0 \to M_1$ has a *standard lift* to a Courant morphism $R_\Phi\colon \mathbb{T}M_0 \dashrightarrow \mathbb{T}M_1$, given by

$$v_0 + \alpha_0 \sim_{R_\Phi} v_1 + \alpha_1 \quad \Leftrightarrow \quad v_1 = T\Phi(v_0) \text{ and } \alpha_0 = T\Phi^*(\alpha_1). \tag{4}$$

More generally, suppose $\eta_i \in \Omega^3(M_i)$ are closed 3-forms, and $\omega \in \Omega^2(M_0)$ with $\eta_0 = \Phi^*\eta_1 + \mathrm{d}\omega$. Then there is a Courant morphism $R_{\Phi,\omega}\colon (\mathbb{T}M_0)_{\eta_0} \dashrightarrow (\mathbb{T}M_1)_{\eta_1}$ given by [5]

$$v_0 + \alpha_0 \sim_{R_{\Phi,\omega}} v_1 + \alpha_1 \quad \Leftrightarrow \quad v_1 = T\Phi(v_0) \text{ and } \alpha_0 + \iota(v_0)\omega = T\Phi^*(\alpha_1).$$

1.4. Manin pairs. A *Manin pair* $(\mathbb{A}, E)$ is a Courant algebroid $\mathbb{A} \to M$ together with a Dirac structure $E \subseteq \mathbb{A}$. If $M = \mathrm{pt}$, this reduces to the usual notion of a Manin pair of Lie algebras. A *morphism of Manin pairs* [5]

$$R\colon (\mathbb{A}, E) \dashrightarrow (\mathbb{A}', E'),$$

with underlying map $\Phi\colon M \to M'$, is a morphism of Courant algebroids with the property that for all $m \in M$, any element of $E'_{\Phi(m)}$ is R-related to a unique element of E_m. Equivalently, in terms of composition of relations,

$$\Phi^* E' = R \circ E, \qquad \ker(R) \cap E = 0.$$

One obtains a bundle map $\Phi^*E' \to E$, associating to each $x' \in E'_{\Phi(m)}$ the unique $x \in E_m$ to which it is R-related. This bundle map is a comorphism of Lie algebroids [24], thus in particular the map $\Phi^*\colon \Gamma(E') \to \Gamma(E)$ preserves Lie brackets.

EXAMPLE 1.6. For any Manin pair $(\mathbb{A}, E)$ over M, there is a morphism of Manin pairs

$$R\colon (\mathbb{T}M, TM) \dashrightarrow (\mathbb{A}, E)$$

where $v + \alpha \sim_R x$ if and only if $v = \mathsf{a}(x)$ and $x - \mathsf{a}^*(\alpha) \in E$.

EXAMPLE 1.7. Suppose M, M' are Poisson manifolds with bivector fields π, π'. Let $\Phi\colon M \to M'$ be a smooth map. Then the standard lift $R_\Phi\colon \mathbb{T}M \dashrightarrow \mathbb{T}M'$ (cf. (4)) defines a morphism of Manin pairs $R_\Phi\colon (\mathbb{T}M, \mathrm{Gr}_\pi) \dashrightarrow (\mathbb{T}M', \mathrm{Gr}_{\pi'})$ if and only if Φ is a Poisson map.

2. Dirac Lie groups. The definition of Dirac Lie group structures (Definition 2.5 below) requires that the ambient Courant algebroid itself be multiplicative, in the sense that it has the structure of a $\mathcal{CA}$-groupoid.

2.1. $\mathcal{CA}$-groupoids. For any groupoid $H \rightrightarrows H^{(0)}$, the space $H^{(k)} \subset H^k$ of k-arrows is the manifold of k-tuples $(g_1, \ldots, g_k)$ such that each composition $g_i g_{i+1}$ is defined. Let $\mathrm{Mult}_H\colon H^{(2)} \to H,\ (X, Y) \to X \circ Y$ denote the groupoid multiplication, and

$$\mathrm{gr}(\mathrm{Mult}_H) = \{(X \circ Y, X, Y)|\ \ (X, Y) \in H^{(2)}\} \subseteq H^3$$

its graph.

DEFINITION 2.1. Let $H \rightrightarrows H^{(0)}$ be a Lie groupoid.

a. A *$\mathcal{VB}$-groupoid* over H is a vector bundle $V \to H$, equipped with a groupoid structure such that $\mathrm{gr}(\mathrm{Mult}_V) \subseteq V^3$ is a vector subbundle along $\mathrm{gr}(\mathrm{Mult}_H)$.
b. An *$\mathcal{LA}$-groupoid* over H is a Lie algebroid $E \to H$, equipped with a groupoid structure such that $\mathrm{gr}(\mathrm{Mult}_E) \subseteq E^3$ is a Lie subalgebroid along $\mathrm{gr}(\mathrm{Mult}_H)$.
c. A *$\mathcal{CA}$-groupoid* over H is a Courant algebroid $\mathbb{A} \to H$, equipped with a groupoid structure such that $\mathrm{gr}(\mathrm{Mult}_\mathbb{A}) \subseteq \mathbb{A} \times \overline{\mathbb{A}} \times \overline{\mathbb{A}}$ is a Dirac structure along $\mathrm{gr}(\mathrm{Mult}_H)$.

In other words, we require that the groupoid multiplication is a $\mathcal{VB}$-relation, $\mathcal{LA}$-relation or $\mathcal{CA}$-relation, respectively. It is common to indicate a $\mathcal{VB}$-groupoid V by a diagram

$$\begin{tikzcd} V \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & V^{(0)} \arrow[d] \\ H \arrow[r, shift left] \arrow[r, shift right] & H^{(0)} \end{tikzcd}$$

REMARK 2.2.

a. The definition of $\mathcal{VB}$-groupoids given above is shorter than Pradines' original definition [33], which requires that all the groupoid structure maps of V are morphisms of vector bundles. The equivalence of the two definitions follows from Grabowski-Rotkiewicz's Remark 1.3. For instance, since $V^{(0)} \subseteq V$ is a smooth submanifold invariant under scalar multiplication, it is a vector subbundle. Similarly, since $s_V, t_V\colon V \to V^{(0)}$ are smooth maps intertwining scalar multiplication they are vector bundle morphisms.

b. $\mathcal{LA}$-groupoids are due to Mackenzie [22, 23]. The definition above implies that $V^{(0)}$ is a Lie subalgebroid along $H^{(0)}$, and that all the groupoid structure maps are morphisms of Lie algebroids.
c. The concept of a $\mathcal{CA}$-groupoid (also called Courant groupoid) was suggested by Mehta [27, Example 3.8] and Ortiz [31, Section 7.3], and developed in detail in [20].

A *relation of $\mathcal{CA}$-groupoids* $R\colon \mathbb{A}_0 \dashrightarrow \mathbb{A}_1$ is a $\mathcal{CA}$-relation such that $R \subseteq \mathbb{A}_1 \times \overline{\mathbb{A}_0}$ is a Lie subgroupoid. If the underlying relation $S\colon H_0 \dashrightarrow H_1$ is the graph of a groupoid homomorphism, then R is called a *morphism of $\mathcal{CA}$-groupoids.* Relations and morphisms of $\mathcal{VB}, \mathcal{LA}$-groupoids are defined similarly.

PROPOSITION 2.3. *Let $\mathbb{A} \to H$ be a $\mathcal{CA}$-groupoid. Then the set of units $\mathbb{A}^{(0)} \subseteq \mathbb{A}$ is a Dirac structure along $H^{(0)} \subseteq H$. Furthermore, the groupoid inversion defines a morphism of Courant algebroids* $\mathrm{Inv}_{\mathbb{A}}\colon \mathbb{A} \dashrightarrow \overline{\mathbb{A}}$ *over* $\mathrm{Inv}_H\colon H \to H$.

Proof. Define a relation $D\colon \overline{\mathbb{A}} \times \mathbb{A} \dashrightarrow \mathbb{A}$, where $(x_1, x_2) \sim_D x$ if and only if $x = x_1^{-1} \circ x_2$. Since

$$D = \{(x_1^{-1} \circ x_2, x_1, x_2)|\ t(x_1) = t(x_2)\} \subseteq \mathbb{A} \times \mathbb{A} \times \overline{\mathbb{A}}.$$

is obtained from $\mathrm{gr}(\mathrm{Mult}_{\mathbb{A}})$ by a cyclic permutation of components (and an overall sign change of the metric), it is a Dirac structure along the graph of the relation $H \times H \dashrightarrow H,\ (g_1, g_2) \sim g_1^{-1} g_2$. On the other hand, we may think of the diagonal in $\overline{A} \times \mathbb{A}$ as a Courant relation $\mathbb{A}_\Delta\colon 0 \dashrightarrow \overline{\mathbb{A}} \times \mathbb{A}$, with underlying relation $H_\Delta : \mathrm{pt} \dashrightarrow H \times H$. Observe $\mathbb{A}^{(0)} = D \circ \mathbb{A}_\Delta$, where the composition is clean. Hence $\mathbb{A}^{(0)}$ is a Dirac structure along $H^{(0)}$. Similarly, the graph of the groupoid inversion $\mathrm{gr}(\mathrm{Inv}_{\mathbb{A}}) \subseteq \overline{\mathbb{A}} \times \overline{\mathbb{A}}$ is a clean composition of Courant relations $\mathrm{gr}(\mathrm{Inv}_{\mathbb{A}}) = \mathbb{A}^{(0)} \circ \mathrm{gr}(\mathrm{Mult}_{\mathbb{A}})$. ∎

Note that D and $\mathbb{A}_\Delta$ are relations of Courant groupoids, if we take $\overline{\mathbb{A}} \times \mathbb{A}$ with the pair groupoid structure.

2.2. Multiplicative Manin pairs and Dirac Lie group structures. DEFINITION 2.4. *[20, 31, 5, 27]*A *multiplicative Manin pair* is a Manin pair $(\mathbb{A}, E)$, where $\mathbb{A} \rightrightarrows \mathbb{A}^{(0)}$ is a $\mathcal{CA}$-groupoid over $H \rightrightarrows H^{(0)}$, and $E \rightrightarrows E^{(0)}$ is a $\mathcal{VB}$-subgroupoid of $\mathbb{A}$. A *morphism of multiplicative Manin pairs* $R\colon (\mathbb{A}_0, E_0) \dashrightarrow (\mathbb{A}_1, E_1)$ is a morphism of Manin pairs which is also a morphism of $\mathcal{CA}$-groupoids $R\colon \mathbb{A}_0 \dashrightarrow \mathbb{A}_1$.

The involutivity condition implies that E inherits the structure of an $\mathcal{LA}$-groupoid.

As shown in Proposition 2.3, for any $\mathcal{CA}$-groupoid structure $\mathbb{A} \rightrightarrows \mathbb{A}^{(0)}$, the space $\mathbb{A}^{(0)}$ of units is a Dirac structure along $H^{(0)}$. In this paper, we are mainly concerned with the case that $H^{(0)} = \mathrm{pt}$, such that H is a group. In this case, the groupoid multiplication defines a Courant morphism $R = \mathrm{gr}(\mathrm{Mult}_{\mathbb{A}})$ covering the group multiplication,

$$\begin{tikzcd} \mathbb{A} \times \mathbb{A} \arrow[r, dashed, "R"] \arrow[d] & \mathbb{A} \arrow[d] \\ H \times H \arrow[r, "\mathrm{Mult}_H"'] & H \end{tikzcd}$$

Definition 2.5. A *Dirac Lie group structure* on a Lie group H is a multiplicative Manin pair $(\mathbb{A}, E)$ over H such that the multiplication morphism $\mathrm{gr}(\mathrm{Mult}_{\mathbb{A}})\colon (\mathbb{A}, E) \times (\mathbb{A}, E) \dashrightarrow (\mathbb{A}, E)$ is a morphism of Manin pairs.

Given Dirac Lie group structures $(\mathbb{A}, E), (\mathbb{A}', E')$ on Lie groups H, H', a morphism of multiplicative Manin pairs $(\mathbb{A}, E) \dashrightarrow (\mathbb{A}', E')$ is called a *morphism of Dirac Lie groups*.

Remark 2.6. In other words, we define the category of Dirac Lie groups to be the subcategory of *group like* objects in the category of Manin pairs. Meanwhile, the category of multiplicative Manin pairs is the subcategory of *groupoid like* objects in the category of Manin pairs. Our definition is more restrictive than that of Ortiz [31, 30] and Jotz [15], where Dirac Lie group structures are taken to be arbitrary multiplicative Manin pairs over H. Note that [15, 30] only explore the case $\mathbb{A} = \mathbb{T}H$. In an earlier paper, Milburn [28] gives a 'categorical' definition of what he calls *Dirac groups*, similar to the Ortiz-Jotz definition.

Proposition 2.7. *A multiplicative Manin pair* $(\mathbb{A}, E)$ *defines a Dirac Lie group structure if and only if* E *is a wide subgroupoid of* $\mathbb{A}$, *i.e* $\mathbb{A}^{(0)} = E^{(0)}$.

Proof. Suppose $(\mathbb{A}, E)$ is a multiplicative Manin pair over H, and that $\mathbb{A}^{(0)} = E^{(0)}$. We will show that the multiplication morphism is a morphism of Manin pairs.

By Proposition 2.3, $\mathfrak{g} = \mathbb{A}^{(0)}$ is Lagrangian, as is E_e. Since E contains the units, it follows that $E_e = \mathfrak{g}$. More generally, for any $h \in H$ the source and target map give isomorphisms $s_h, t_h\colon E_h \to \mathfrak{g}$. Hence if $h_1, h_2 \in H$ are given, then any $x \in E_{h_1 h_2}$ can be uniquely written as a product $x = x_1 \circ x_2$ with $x_i \in E_{h_i}$: x_1 is uniquely determined by $t(x_1) = t(x)$, and then $x_2 = x_1^{-1} \circ x$. This shows that $\mathrm{Mult}_{\mathbb{A}}$ gives a morphism of Manin pairs.

Conversely, suppose $E^{(0)}$ is a proper subspace of $A^{(0)}$. Then $\dim E^{(0)} < \mathrm{rank}(E) = \dim A^{(0)}$. In particular, $\ker(s|_E)$ is non-trivial. Since

$$\{(x^{-1}, x) \mid x \in \ker(s|_E)\} \subseteq \ker(\mathrm{Mult}_{\mathbb{A}}) \cap (E \times E),$$

this shows that $\mathrm{Mult}_{\mathbb{A}}$ does not define a morphism of Manin pairs. $\square$

2.3. Examples.

Example 2.8. For any Courant algebroid $\mathbb{A} \to M$, the direct product $\overline{\mathbb{A}} \times \mathbb{A} \to M \times M$, with groupoid structure that of a pair groupoid, defines a $\mathcal{CA}$-groupoid structure over the pair groupoid $M \times M \rightrightarrows M$:

$$\begin{tikzcd}
\overline{\mathbb{A}} \times \mathbb{A} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathbb{A} \arrow[d] \\
M \times M \arrow[r, shift left] \arrow[r, shift right] & M
\end{tikzcd}$$

If $(\mathbb{A}, E)$ is a Manin pair, then $(\overline{\mathbb{A}} \times \mathbb{A}, E \times E)$ becomes a multiplicative Manin pair. If $M = \mathrm{pt}$, so that $\mathbb{A} = \mathfrak{g}$ is a quadratic Lie algebra, the diagonal $\mathfrak{g}_\Delta \subseteq \overline{\mathfrak{g}} \oplus \mathfrak{g}$ defines a Dirac Lie group structure on $H = \{e\}$.

Example 2.9. The standard Courant algebroid over any Lie groupoid $H \rightrightarrows H^{(0)}$ is a $\mathcal{CA}$-groupoid $\mathbb{T}H \rightrightarrows TH^{(0)} \oplus A^*H$, where $AH \to H^{(0)}$ is the Lie algebroid of

H, and A^*H its dual. The $\mathcal{VB}$-groupoid structure is given as the direct sum of the tangent prolongation $TH \rightrightarrows TH^{(0)}$ and the cotangent groupoid $T^*H \rightrightarrows A^*H$. See [27, Example 3.8] and [20, Example 9]. Both $(\mathbb{T}H, T^*H)$ and $(\mathbb{T}H, TH)$ are multiplicative Manin pairs.

In particular, if H is a Lie group, the $\mathcal{VB}$-groupoid structure on $\mathbb{T}H$ is the direct product of the group $TH \rightrightarrows \mathrm{pt}$ with the symplectic groupoid $T^*H \rightrightarrows \mathfrak{h}^*$:

$$\begin{tikzcd} \mathbb{T}H \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathfrak{h}^* \arrow[d] \\ H \arrow[r, shift left] \arrow[r, shift right] & \mathrm{pt} \end{tikzcd}$$

If $(\mathbb{T}H, E)$ is a Dirac Lie group structure, then $E \cap TH = 0$ since the source and target maps $E \to \mathfrak{h}^*$ are surjective. Thus E is the graph of a bivector field $\pi \in \Gamma(\wedge^2 TH)$. The condition that E is a subgroupoid translates into the condition that π is multiplicative, i.e. a Poisson-Lie group structure. In fact the following was obtained by Ortiz [30] and Jotz [15], as part of a general classification of multiplicative Manin pairs for $\mathbb{A} = \mathbb{T}H$:

Proposition 2.10. *The Dirac Lie group structures for the standard Courant algebroid over a Lie group H are exactly those of the form $(\mathbb{T}H, \mathrm{Gr}_\pi)$ where π defines a Poisson-Lie group structure on H. If $(H, \pi), (H', \pi')$ are Poisson Lie groups and $\Phi\colon H \to H'$ is a Lie group homomorphism, then the standard lift of Φ is a Dirac Lie group morphism if and only if Φ is a Poisson Lie group morphism, i.e. $\pi \sim_\Phi \pi'$.*

As a special case, any Lie group has a 'trivial' Dirac Lie group structure $(\mathbb{T}H, T^*H)$. The Manin pair $(\mathbb{T}H, TH)$ is multiplicative, but is not a Dirac Lie group structure in our sense since TH is not a wide subgroupoid.

Example 2.11. For any multiplicative Manin pair $(\mathbb{A}, E)$, the morphism $(\mathbb{T}H, TH) \dashrightarrow (\mathbb{A}, E)$ (cf. Example 1.6) is a morphism of multiplicative Manin pairs.

Example 2.12. [1, § 3.4] Let G be a Lie group whose Lie algebra $\mathfrak{g}$ carries an invariant metric B. Then there is a $\mathcal{CA}$-groupoid structure on G,

$$\begin{tikzcd} G \times (\overline{\mathfrak{g}} \oplus \mathfrak{g}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathfrak{g} \arrow[d] \\ G \arrow[r, shift left] \arrow[r, shift right] & \mathrm{pt} \end{tikzcd}$$

Here the $\mathcal{VB}$-groupoid structure is the direct product of the group $G \rightrightarrows \mathrm{pt}$ with the pair groupoid $\overline{\mathfrak{g}} \oplus \mathfrak{g} \rightrightarrows \mathfrak{g}$. As a Courant algebroid, $\mathbb{A} = G \times (\overline{\mathfrak{g}} \oplus \mathfrak{g})$ is the action Courant algebroid for the following action of $\overline{\mathfrak{g}} \oplus \mathfrak{g}$ on G

$$\varrho(\zeta_1, \zeta_2) = \zeta_2^L - \zeta_1^R$$

where ζ^L, ζ^R are the left-,right-invariant vector fields defined by $\zeta \in \mathfrak{g}$. Since the action ϱ is transitive, the Courant algebroid $\mathbb{A}$ is exact. In fact there is an explicit isomorphism of Courant algebroids $\kappa\colon G \times (\overline{\mathfrak{g}} \oplus \mathfrak{g}) \to \mathbb{T}G_\eta$, where $\eta = \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]) \in \Omega^3(G)$ is the Cartan 3-form:

$$\kappa(\zeta_1, \zeta_2) = \Big(\zeta_2^L - \zeta_1^R,\ \tfrac{1}{2} B(\theta^L, \zeta_2) + \tfrac{1}{2} B(\theta^R, \zeta_1)\Big). \tag{5}$$

Here, $\theta^L,\ \theta^R \in \Omega^1(G,\mathfrak{g})$ are the left-,right-invariant Maurer-Cartan forms, defined by the property

$$\iota(\zeta^L)\theta^L = \zeta = \iota(\zeta^R)\theta^R,\quad \zeta \in \mathfrak{g}.$$

The subbundle $E = G \times \mathfrak{g}_\Delta$ defines a Dirac Lie group structure on G. This is the *Cartan-Dirac structure* on G, found independently by Alekseev, Ševera and Strobl. Its multiplicative properties were noted in [1].

2.4. Constructions with $\mathcal{CA}$-groupoids. In this section we will collect some further properties and constructions for $\mathcal{CA}$-groupoids. While we are mainly interested in the case $H^{(0)} = \mathrm{pt}$, the general proofs are more conceptual and in any case not harder.

2.4.1. Basic properties. Given a Lie groupoid $H \rightrightarrows H^{(0)}$, let $TH \rightrightarrows TH^{(0)}$ be its tangent prolongation and $T^*H \rightrightarrows A^*H$ the cotangent groupoid.

PROPOSITION 2.13. *For any $\mathcal{CA}$-groupoid $\mathbb{A} \rightrightarrows \mathbb{A}^{(0)}$ over $H \rightrightarrows H^{(0)}$, the anchor map defines a morphism of $\mathcal{VB}$-groupoids $\mathsf{a}\colon \mathbb{A} \to TH$, while a^* defines a morphism of $\mathcal{VB}$-groupoids $\mathsf{a}^*\colon T^*H \to \mathbb{A}$.*

Proof. By definition of a $\mathcal{CA}$-groupoid, the image of $\mathrm{gr}(\mathrm{Mult}_{\mathbb{A}})$ under the anchor map lies in $T\,\mathrm{gr}(\mathrm{Mult}_H) = \mathrm{gr}(\mathrm{Mult}_{TH})$. Hence, the graph of a is a $\mathcal{VB}$-subgroupoid of $TH \times \mathbb{A}$, proving that a is a $\mathcal{VB}$-groupoid homomorphism. Dualizing, $\mathsf{a}^*\colon T^*H \to \mathbb{A}^*$ is a $\mathcal{VB}$-groupoid homomorphism. But the isomorphism $\mathbb{A}^* \cong \mathbb{A}$ given by the metric is an isomorphism of $\mathcal{VB}$-groupoids. ☐

COROLLARY 2.14. *For any $\mathcal{CA}$-groupoid $\mathbb{A}$ over H, the diagonal morphism $\mathbb{T}H \dashrightarrow \overline{\mathbb{A}} \times \mathbb{A}$ given by*

$$v + \alpha \sim (x,y) \Leftrightarrow v = \mathsf{a}(x),\quad y - x = \mathsf{a}^*(\alpha)$$

is a morphism of $\mathcal{CA}$-groupoids.

Proof. As shown in [19, Proposition 1.6], the diagonal morphism is a morphism of Courant algebroids. By Proposition 2.13, it is also a morphism of $\mathcal{VB}$-groupoids. ☐

2.4.2. Reduction and Pull-backs.

PROPOSITION 2.15 (Coisotropic reduction). *Let $\mathbb{A} \rightrightarrows \mathbb{A}^{(0)}$ be a $\mathcal{CA}$-groupoid over $H \rightrightarrows H^{(0)}$, and let $C \subseteq \mathbb{A}$ be a $\mathcal{VB}$-subgroupoid along a subgroupoid $K \subseteq H$. Assume that*

a. C is co-isotropic,
b. C is involutive,
c. $\mathsf{a}(C) \subseteq TK,\ \mathsf{a}(C^\perp) = 0$.

Then the quotient $\mathbb{A}_C = C/C^\perp$ defines a $\mathcal{CA}$-groupoid structure on K, in such a way that the inclusion map $K \to H$ lifts to a morphism of $\mathcal{CA}$-groupoids, $C/C^\perp \dashrightarrow \mathbb{A}$. If $E \subseteq \mathbb{A}$ defines a multiplicative Manin pair $(\mathbb{A}, E)$, and E is transverse to C then

$$(\mathbb{A}_C, E_C) = (C/C^\perp,\ (E \cap C)/(E \cap C^\perp))$$

is again a multiplicative Manin pair.

Here transversality means $E|_K + C = \mathbb{A}|_K$, or equivalently $E \cap C^\perp = 0$.

Proof. Since C is a co-isotropic $\mathcal{VB}$-subgroupoid of $\mathbb{A}$, $C^\perp$ is a $\mathcal{VB}$-subgroupoid of C, and $\mathbb{A}_C = C/C^\perp$ inherits a $\mathcal{VB}$-groupoid structure (see Corollary C.5 from Appendix C for details). By [19, Proposition 2.1], the Courant bracket on $\mathbb{A}$ descends to a Courant bracket on the quotient $\mathbb{A}_C$, in such a way that

$$S = \{(x, [x]) | \; x \in C\} \subseteq \mathbb{A} \times \overline{\mathbb{A}_C} \tag{6}$$

is a Courant morphism $S\colon \mathbb{A}_C \dashrightarrow \mathbb{A}$. Here $[x] \in \mathbb{B}$ denotes the image of $x \in C$. The graph of the groupoid multiplication of $\mathbb{A}_C$ is a transverse composition of Courant relations,

$$\mathrm{gr}(\mathrm{Mult}_{\mathbb{A}_C}) = \mathrm{gr}(\mathrm{Mult}_{\mathbb{A}}) \circ (S \times S \times S),$$

hence it is itself a Courant relation. Thus $\mathbb{A}_C$ has a $\mathcal{CA}$-groupoid structure. Since S is a Dirac structure along the graph of the inclusion, and also a subgroupoid, it defines a $\mathcal{CA}$-groupoid morphism.

If $E \subseteq \mathbb{A}$ is a multiplicative Dirac structure transverse to C, then $E_C = E \circ S$ is a transverse composition, and is a multiplicative Dirac structure in $\mathbb{A}_C$. ∎

PROPOSITION 2.16 (Pull-backs). *Let $\mathbb{A} \rightrightarrows \mathbb{A}^{(0)}$ be a $\mathcal{CA}$-groupoid over $H \rightrightarrows H^{(0)}$, and $\Phi\colon K \to H$ a homomorphism of Lie groupoids. Suppose that Φ is transverse to the anchor map $\mathsf{a}\colon \mathbb{A} \to TH$. Then the pull-back Courant algebroid $\Phi^!\mathbb{A} \rightrightarrows \Phi^*\mathbb{A}^{(0)}$ inherits the structure of a $\mathcal{CA}$-groupoid over $K \rightrightarrows K^{(0)}$.*

Proof. By definition (see [19, Proposition 2.7]), the pull-back Courant algebroid is a reduction $\Phi^!\mathbb{A} = (\mathbb{A} \times \mathbb{T}K)_C$ relative to the coisotropic subbundle C along $\mathrm{gr}(\Phi) \cong K$,

$$C = \mathbb{A} \times_{TH} \mathbb{T}K \subseteq \mathbb{A} \times \mathbb{T}K,$$

the fiber product relative to the maps $\mathsf{a}_{\mathbb{A}}\colon \mathbb{A} \to TH$ and $d\Phi \circ \mathsf{a}_{\mathbb{T}K}\colon \mathbb{T}K \to TH$. Proposition C.1 shows that C is a Lie groupoid. Its space of units $C^{(0)} = \mathbb{A}^{(0)} \times_{TH^{(0)}} A^*K$ is a smooth subbundle of $\mathbb{A}^{(0)} \times A^*K$ along $\mathrm{gr}(\Phi|_{K^{(0)}}) \cong K^{(0)}$. Corollary C.5 from Appendix C shows that $C^\perp \subseteq C$ is a subgroupoid. Hence $C/C^\perp$ inherits a $\mathcal{CA}$-groupoid structure. ∎

3. Classification of Dirac Lie group structures. In this Section we will give the general classification and construction of Dirac Lie group structures over Lie groups H. The classification will be given in terms of *H-equivariant* Dirac Manin triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$.

3.1. Vacant $\mathcal{LA}$-groupoids. Following Mackenzie [22], a $\mathcal{VB}$-groupoid $V \to H$ will be called *vacant* if it has the property $V^{(0)} = V|_{H^{(0)}}$.

LEMMA 3.1. *For any Dirac Lie group structure $(\mathbb{A}, E)$ over a group H, the sub-bundle E is a vacant $\mathcal{LA}$-groupoid.*

Proof. The Lie algebroid bracket is induced from the Courant bracket on $\mathbb{A}$. Since $E^{(0)} \cong \mathbb{A}^{(0)}$ is a Lagrangian subspace of $\mathbb{A}_e$, it must must coincide with E_e. ∎

As shown by Mackenzie [22], vacant $\mathcal{LA}$-groupoids over groups are characterized in terms of Lie-theoretic data. We will review his theory from a mildly different perspective; further details are given in Appendix B.

DEFINITION 3.2. Let H be a Lie group with Lie algebra $\mathfrak{h}$. A *Lie algebra triple* $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Lie algebra $\mathfrak{d}$ with a vector space decomposition $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ into two Lie subalgebras $\mathfrak{g}, \mathfrak{h}$. Given an action of H on $\mathfrak{d}$ by automorphisms, which integrates the adjoint action of $\mathfrak{h} \subseteq \mathfrak{d}$ and restricts to the adjoint action of H on $\mathfrak{h}$, we refer to $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ as an *H-equivariant Lie algebra triple.*

We will simply write $h \mapsto \mathrm{Ad}_h$ for the action of H on $\mathfrak{d}$. Part (b) of the following Proposition associates a vacant $\mathcal{LA}$-groupoid $E \to H$ to any H-equivariant Lie algebra triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$. It is realized as a $\mathcal{LA}$-subgroupoid of the direct product of $TH \rightrightarrows 0$ with the pair groupoid $\mathfrak{g} \oplus \mathfrak{g} \rightrightarrows \mathfrak{g}$.

Proposition 3.3. *Let $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ be an H-equivariant Lie algebra triple.*

a. The subset

$$V = \{(v, X, X') |\ v \in T_h H,\ X, X' \in \mathfrak{d},\ \mathrm{Ad}_h X' - X = \iota(v)\theta^R_h\},$$

is an $\mathcal{LA}$-subgroupoid of $TH \times (\mathfrak{d} \oplus \mathfrak{d}) \rightrightarrows \mathfrak{d}$, of rank equal to $\dim \mathfrak{g} + 2 \dim \mathfrak{h}$. *Its object space is $V^{(0)} = \mathfrak{d}$.*

b. The subset

$$E = \{(v, \xi, \xi') |\ v \in T_h H,\ \xi, \xi' \in \mathfrak{g},\ \mathrm{Ad}_h \xi' - \xi = \iota(v)\theta^R_h\}$$

is a vacant $\mathcal{LA}$-subgroupoid of $TH \times (\mathfrak{g} \oplus \mathfrak{g}) \rightrightarrows \mathfrak{g}$, of rank equal to $\dim \mathfrak{g}$. *Its object space is $E^{(0)} = \mathfrak{g}$. The source map $(v, \xi, \xi') \mapsto \xi'$ is a trivialization of E, and defines a morphism of Lie algebroids $E \to \mathfrak{g}$.*

Proof. (a) The $\mathcal{VB}$-groupoid $TH \times (\mathfrak{d} \oplus \mathfrak{d}) \rightrightarrows \mathfrak{d}$ may be regarded as a direct sum of two $\mathcal{VB}$-subgroupoids $TH \rightrightarrows 0$ and $H \times (\mathfrak{d} \oplus \mathfrak{d}) \rightrightarrows \mathfrak{d}$. Right trivialization $TH \cong \mathfrak{h} \rtimes H$ gives a fiberwise injective group isomorphism

$$TH \to \mathfrak{d} \rtimes H,\quad v \mapsto (\iota_v \theta^R_h, h) \tag{7}$$

where h is the base point of v, and the semi-direct product is relative to Ad. On the other hand, the map

$$H \times (\mathfrak{d} \oplus \mathfrak{d}) \to \mathfrak{d} \rtimes H,\quad (h, X, X') \mapsto \mathrm{Ad}_h(X') - X \tag{8}$$

is a fiberwise surjective $\mathcal{VB}$-groupoid homomorphism. The fibered product of the two maps (7), (8) is equal to V, which is hence a $\mathcal{VB}$-subgroupoid of rank $\dim \mathfrak{h} + 2 \dim \mathfrak{d} - \dim \mathfrak{d} = \dim \mathfrak{g} + 2 \dim \mathfrak{h}$.

Let $H \times H$ act on H by $(h_1, h_2).h = h_1 h h_2^{-1}$, on TH by the tangent lift of this action, and on $\mathfrak{d} \oplus \mathfrak{d}$ by $(h_1, h_2).(X, X') = (\mathrm{Ad}_{h_1} X, \mathrm{Ad}_{h_2} X')$. We obtain a diagonal action on $TH \times (\mathfrak{d} \oplus \mathfrak{d})$ by $\mathcal{LA}$-groupoid automorphisms. The maps (7), (8) are equivariant relative to the action $(h_1, h_2).(Y, h) = (\mathrm{Ad}_{h_1} Y, h_1 h h_2^{-1})$ on $\mathfrak{d} \rtimes H$, hence V is $H \times H$-invariant. To verify that V is a $\mathcal{LA}$-subgroupoid, it hence suffices to check near the group unit. In particular, we may assume that H is connected and simply connected. Let D be a connected Lie group with Lie algebra $\mathfrak{d}$, and with the action of $\mathfrak{d} \oplus \mathfrak{d}$ by $(X, X') \mapsto X'^L - X^R$. The corresponding action Lie algebroid embeds as a Lie subalgebroid

$$\{(v, X, X') |\ v = X'^L|_d - X^R|_d\} \subseteq TD \times (\mathfrak{d} \oplus \mathfrak{d}) \tag{9}$$

(where $d \in D$ is the base point of $v \in TD$). On a neighborhood of $e \in H$, the group homomorphism $H \to D$ exponentiating $\mathfrak{h} \to \mathfrak{d}$ is an embedding, and V is simply the intersection of (9) with $TH \times (\mathfrak{d} \oplus \mathfrak{d})$. In particular, it is a Lie subalgebroid of $TH \times (\mathfrak{d} \oplus \mathfrak{d})$.

(b) The same argument as for V shows that E is a subbundle of rank $\dim \mathfrak{g}$. Since E is the intersection of V with the $\mathcal{LA}$-subgroupoid $TH \times (\mathfrak{g} \oplus \mathfrak{g})$, it is itself an $\mathcal{LA}$-subgroupoid. Since E has trivial intersection with the subbundle of elements of the form $(v, \xi, 0)$, the source map $TH \times (\mathfrak{g} \oplus \mathfrak{g}) \to \mathfrak{g}$, $(v, \xi, \xi') \mapsto \xi'$ defines a trivialization of E. Furthermore, since this projection is a Lie algebroid homomorphism, the same is true for its restriction to E. ☐

PROPOSITION 3.4. *There is a 1-1 correspondence between*
(i) Vacant $\mathcal{LA}$-groupoids $E \rightrightarrows \mathfrak{g}$ over groups $H \rightrightarrows \mathrm{pt}$, and
(ii) H-equivariant Lie algebra triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$.

The proof of Proposition 3.4 is elementary but tedious, so we summarize the construction here. More details can be found in Appendix B. The direction $(ii) \Rightarrow (i)$ is part (b) of Proposition 3.3. In the opposite direction $(i) \Rightarrow (ii)$, let $\mathfrak{h}$ be the Lie algebra of H, and put $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ as a vector space. One finds that $\mathfrak{d}$ has a unique action Ad of H, extending the adjoint action on $\mathfrak{h} \subseteq \mathfrak{d}$ and such that

$$\iota(\mathsf{a}(z))\theta_h^R = \mathrm{Ad}_h\, s(z) - t(z) \tag{10}$$

for all $z \in E_h$. Furthermore, $\mathfrak{d}$ has a unique Lie bracket such that $\mathfrak{g}, \mathfrak{h}$ are Lie subalgebras and such that the differential of $\mathrm{Ad}\colon H \to\in \mathrm{Aut}(\mathfrak{d})$ gives the adjoint action $\mathrm{ad}\colon \mathfrak{h} \to \mathrm{End}(\mathfrak{d})$.

3.2. Dirac Manin triples. If V is a vector space with an element $\beta \in S^2V$, we denote by $\beta^\sharp\colon V^* \to V$ the map $\beta^\sharp(\mu) = \beta(\mu, \cdot)$. A subspace $U \subseteq V$ is called β-coisotropic if $\beta^\sharp(\mathrm{ann}(U)) \subseteq U$.

DEFINITION 3.5. A *Dirac Manin triple* $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ is a triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ of Lie algebras, together with an element $\beta \in (S^2\mathfrak{d})^\mathfrak{d}$ such that $\mathfrak{g}$ is β-coisotropic.

If $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is an H-equivariant triple, and β is H-invariant, we call $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ an *H-equivariant Dirac Manin triple.*

If H is simply connected, then the H-equivariance conditions are automatic. If β is non-degenerate, and both $\mathfrak{g}$ and $\mathfrak{h}$ are Lagrangian Lie subalgebras, then the Dirac Manin triple is an ordinary Manin triple.

We will now associate to any Dirac Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ a new Dirac Manin triple $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$, where γ is non-degenerate and $\mathfrak{g}$ is Lagrangian in $\mathfrak{q}$. Let $\mathfrak{d}^*_\beta$ be the Lie algebra, equal to $\mathfrak{d}^*$ as a vector space, with the Lie bracket

$$\langle [\mu_1, \mu_2], \xi\rangle = \langle \mu_2, [\xi, \beta^\sharp(\mu_1)]\rangle, \quad \mu_1, \mu_2 \in \mathfrak{d}^*_\beta,\ \xi \in \mathfrak{g}.$$

The element β, viewed as a bilinear form on $\mathfrak{d}^*_\beta$, is invariant under the bracket. The co-adjoint action of $\mathfrak{d}$ is by derivations of the bracket, hence we may form the semi-direct product

$$\widehat{\mathfrak{d}} = \mathfrak{d} \ltimes \mathfrak{d}^*_\beta.$$

The bilinear form

$$\widehat{\beta}((\xi_1, \mu_1), (\xi_2, \mu_2)) = \beta(\mu_1, \mu_2) + \langle \mu_1, \xi_2\rangle + \langle \mu_2, \xi_1\rangle$$

on $\widehat{\mathfrak{d}}$ is invariant and non-degenerate. Note that $\mathfrak{d} \subseteq \mathfrak{d} \ltimes \mathfrak{d}^*_\beta$ is a Lagrangian Lie subalgebra, and $\mathfrak{d}^*_\beta$ is a Lie algebra ideal. This defines a new Dirac Manin triple $(\mathfrak{d} \ltimes \mathfrak{d}^*_\beta, \mathfrak{d}, \mathfrak{d}^*_\beta)_{\widehat{\beta}}$.

REMARK 3.6. As observed by Drinfel'd [9], there is a 1-1 correspondence between (i) Manin pairs $(\widehat{\mathfrak{d}}, \mathfrak{d})$ with a Lie algebra ideal complementary to $\mathfrak{d}$, and (ii) Lie algebras $\mathfrak{d}$ with invariant elements $\beta \in S^2\mathfrak{d}$. One may interpret this as a classification of $\mathcal{CA}$-groupoids over $H = \mathrm{pt}$. Here $\widehat{\mathfrak{d}} \equiv \mathfrak{d} \ltimes \mathfrak{d}^*_\beta \rightrightarrows \mathfrak{d}$ is the action Lie groupoid for the translation action of $\mathfrak{d}^*_\beta$ on $\mathfrak{d}$ via the map $\beta^\sharp \colon \mathfrak{d}^*_\beta \to \mathfrak{d}$.

The Lie subalgebra $\mathfrak{c} = \mathfrak{g} \ltimes \mathfrak{d}^*_\beta$ is coisotropic, since it contains the Lagrangian Lie subalgebra $\mathfrak{g} \ltimes \mathrm{ann}(\mathfrak{g})$. Hence $\mathfrak{c}^\perp$ is an ideal in $\mathfrak{c}$, and the quotient

$$\mathfrak{q} = \mathfrak{c}/\mathfrak{c}^\perp$$

is a Lie algebra with a non-degenerate invariant metric induced from that on $\mathfrak{d} \ltimes \mathfrak{d}^*_\beta$. Let $\gamma \in (S^2\mathfrak{q})^\mathfrak{q}$ be given by the dual metric on $\mathfrak{q}^*$. The inclusion $\mathfrak{g} \hookrightarrow \mathfrak{d} \ltimes \mathfrak{d}^*_\beta$ descends to an inclusion $\mathfrak{g} \hookrightarrow \mathfrak{q}$ as a Lagrangian Lie subalgebra, thus $(\mathfrak{q}, \mathfrak{g})$ is a Manin pair.

Since $\mathfrak{d}^*_\beta$ is an ideal complementary to $\mathfrak{d}$, the same is true of $(\mathfrak{d}^*_\beta)^\perp$. Let $\widehat{f} \colon \mathfrak{d} \ltimes \mathfrak{d}^*_\beta \to \mathfrak{d}$ be the projection with kernel $(\mathfrak{d}^*_\beta)^\perp$. Explicitly, $\widehat{f}(\xi, \mu) = \xi + \beta^\sharp(\mu)$. This is a Lie algebra homomorphism, and since $\mathfrak{c}^\perp \subseteq (\mathfrak{d}^*_\beta)^\perp$, it descends to a Lie algebra homomorphism

$$f \colon \mathfrak{q} \to \mathfrak{d},$$

with the important properties $f(\xi) = \xi$ for $\xi \in \mathfrak{g}$ and $\beta^\sharp = f \circ f^*$. Finally, $\mathfrak{r} = f^{-1}(\mathfrak{h})$ is a Lie algebra complement to $\mathfrak{g}$. We have thus defined a Dirac Manin triple

$$(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma,$$

where γ is non-degenerate and $\mathfrak{g}$ is Lagrangian. We denote by $p_\mathfrak{r} \in \mathrm{End}(\mathfrak{q})$ the projection to $\mathfrak{r}$ along $\mathfrak{g}$ and by $p_\mathfrak{h} \in \mathrm{End}(\mathfrak{d})$ the projection to $\mathfrak{h}$ along $\mathfrak{g}$; thus $f \circ p_\mathfrak{r} = p_\mathfrak{h} \circ f$.

EXAMPLES 3.7. We describe the triple $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ associated to $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ in some extreme cases.

(i) If $\beta = 0$ one obtains (independent of $\mathfrak{h}$)

$$(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma = (\mathfrak{g} \ltimes \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)_\gamma$$

with γ the bilinear form given by the pairing. The map $f \colon \mathfrak{q} \to \mathfrak{d}$ is projection to $\mathfrak{g} \subseteq \mathfrak{d}$.

(ii) If β is non-degenerate, defining a non-degenerate metric on $\mathfrak{d}$, one finds

$$(\mathfrak{q}, \mathfrak{g}, \rangle)_\gamma = (\mathfrak{d} \oplus \overline{\mathfrak{g}/\mathfrak{g}^\perp}, \mathfrak{g}_\Delta, \mathfrak{h} \oplus 0)_\gamma$$

where $\mathfrak{g}/\mathfrak{g}^\perp$ is the quotient Lie algebra with metric induced from that on $\mathfrak{d}$, and $\overline{\mathfrak{g}/\mathfrak{g}^\perp}$ is the same Lie algebra with the opposite metric. $\mathfrak{g}_\Delta$ is embedded 'diagonally' as $\xi \mapsto (\xi, [\xi])$ (where $[\xi]$ is the image in $\mathfrak{g}/\mathfrak{g}^\perp$), and the homomorphism f is projection to the first summand.

(iii) In particular, if β is non-degenerate and $\mathfrak{g}$ is *Lagrangian*, we obtain $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma = (\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$, with f the identity map.

3.3. From Dirac Manin triples to Dirac Lie group structures. Let $(\mathfrak{d},\mathfrak{g},\mathfrak{h})_\beta$ be an H-equivariant Dirac Manin triple, and let $(\mathfrak{q},\mathfrak{g},\mathfrak{r})_\gamma$ and $f\colon \mathfrak{q}\to\mathfrak{d}$ be as in Section 3.2. We will obtain a Dirac Lie group structure $(\mathbb{A},E)$ on H by reduction from the direct product of the multiplicative Manin pairs

$$(\mathbb{T}H,TH)\times(\overline{\mathfrak{q}}\oplus\mathfrak{q},\mathfrak{g}\oplus\mathfrak{g}),$$

where $\mathbb{T}H\rightrightarrows\mathfrak{h}^*$ is the standard $\mathcal{CA}$-groupoid structure, and $\overline{\mathfrak{q}}\oplus\mathfrak{q}\rightrightarrows\mathfrak{q}$ is the pair groupoid.

Proposition 3.8. *The subset $C\subseteq\mathbb{T}H\times(\overline{\mathfrak{q}}\oplus\mathfrak{q})$ given as*

$$C=\{(v+\alpha,\zeta,\zeta')|\ \mathrm{Ad}_h f(\zeta')-f(\zeta)=\iota(v)\theta^R_h\} \tag{11}$$

(where $h\in H$ is the base point of $v+\alpha\in\mathbb{T}H$) is a coisotropic, involutive $\mathcal{VB}$-subgroupoid, with $\mathsf{a}(C^\perp)=0$. The reduction of $(\mathbb{T}H,TH)\times(\overline{\mathfrak{q}}\oplus\mathfrak{q},\mathfrak{g}\oplus\mathfrak{g})$ relative to C is a Dirac Lie group structure $(\mathbb{A},E)$.

Proof. An argument similar to that given in the proof of Proposition 3.3 shows that C is a $\mathcal{VB}$-subgroupoid of rank $\dim\mathfrak{q}+\dim\mathfrak{d}$. Furthermore, C is the pre-image of the $\mathcal{LA}$-subgroupoid V from Proposition 3.3 under the $\mathcal{VB}$-groupoid homomorphism

$$\mathbb{T}H\times(\overline{\mathfrak{q}}\oplus\mathfrak{q})\to TH\times(\mathfrak{d}\oplus\mathfrak{d}),\ (v+\alpha,\zeta,\zeta')\mapsto(v,f(\zeta),f(\zeta')). \tag{12}$$

Since (12) preserves brackets, and since V is a Lie subalgebroid, it follows that C is involutive. The orthogonal bundle $C^\perp$ has rank equal to $\dim\mathfrak{d}$, and is spanned by the sections

$$\psi(\mu)=\big(-\langle\mu,\theta^R\rangle,\ f^*(\mu),\ f^*(\mathrm{Ad}_{h^{-1}}\mu)\big),\quad \mu\in\mathfrak{d}^*.$$

Indeed, the pairing with $(v+\alpha,\zeta,\zeta')\in\Gamma(C)$ is $\langle\mu,-\iota(v)\theta^R_h-f(\zeta)+\mathrm{Ad}_h f(\zeta')\rangle=0$ as required. The property $C^\perp\subseteq C$ follows by checking the definition of C on the sections $\psi(\mu)$,

$$\mathrm{Ad}_h\big(f(f^*(\mathrm{Ad}_{h^{-1}}\mu))\big)-f(f^*(\mu))=0,$$

using the H-equivariance of $f\circ f^*=\beta^\sharp$. The object space of $\mathbb{T}H\times(\overline{\mathfrak{q}}\oplus\mathfrak{q})$ is $\mathfrak{h}^*\times\mathfrak{q}$, embedded as the space of units $T^*_eH\times\mathfrak{q}_\Delta$. This is contained in C, hence $C^{(0)}=\mathfrak{h}^*\times\mathfrak{q}$. On the other hand, $(C^\perp)^{(0)}\cong\mathfrak{d}^*$, embedded in $C^{(0)}$ by the map $\mathfrak{d}^*\to\mathfrak{h}^*\times\mathfrak{q}$, $\mu\mapsto(p^*_{\mathfrak{h}}(\mu),f^*(\mu))$. We next show that $TH\times(\mathfrak{g}\oplus\mathfrak{g})$ is transverse to C, or equivalently that $TH\times(\mathfrak{g}\oplus\mathfrak{g})\cap C^\perp$ is trivial. Indeed, vanishing of the T^*H-component of $\psi(\mu)$ is equivalent to $\mu\in\mathrm{ann}(\mathfrak{h})$, but then the last two components are contained in $f^*(\mathrm{ann}(\mathfrak{h}))=\mathfrak{r}^\perp$. Coisotropic reduction by C (cf. Proposition 2.15) gives the multiplicative Manin pair $(\mathbb{A},E)=((\mathbb{T}H\times(\overline{\mathfrak{q}}\oplus\mathfrak{q}))_C,\ (TH\times(\mathfrak{g}\oplus\mathfrak{g}))_C)$. We have $\mathbb{A}^{(0)}=C^{(0)}/(C^\perp)^{(0)}=(\mathfrak{h}^*\times\mathfrak{q})/\mathfrak{d}^*\cong\mathfrak{g}$ (the last identification is obtained by taking $0\times\mathfrak{g}\hookrightarrow\mathfrak{h}^*\times\mathfrak{q}$ as a complement to $\mathfrak{d}^*$), and also $E^{(0)}=\mathfrak{g}$ since

$$(TH\times(\mathfrak{g}\oplus\mathfrak{g}))^{(0)}=\{0\}\times\mathfrak{g}\subseteq\mathfrak{h}^*\times\mathfrak{q}.$$

Since $\mathbb{A}^{(0)}=E^{(0)}$, it follows that $(\mathbb{A},E)$ is a Dirac Lie group structure on H. ∎

Remark 3.9. Similar to the Cartan Courant algebroid from Example 2.12, $D\times(\overline{\mathfrak{q}}\oplus\mathfrak{q})$ is an action Courant algebroid for the action $\varrho(\zeta,\zeta')=f(\zeta')^L-f(\zeta)^R$. If the inclusion of $\mathfrak{h}$ into $\mathfrak{d}$ lifts to a morphism of Lie groups $i\colon H\to D$, then the Dirac Lie group $(\mathbb{A}, E)$ is the pullback $i^!\big(D \times (\overline{\mathfrak{q}} \oplus \mathfrak{q}, \mathfrak{g} \oplus \mathfrak{g})\big)$. In fact, by Section 2.4.2 we have $i^!\big(D \times (\overline{\mathfrak{q}} \oplus \mathfrak{q})\big) = C/C^\perp$ where

$$C = \big\{(d, \zeta, \zeta'; v + \alpha) \in \big(D \times (\overline{\mathfrak{q}} \oplus \mathfrak{q})\big) \times \mathbb{T}H \mid d = i(h),\ \varrho(\zeta, \zeta')|_d = (T_h i)(v)\}$$

(with h the base point of $v + \alpha$). The construction in Proposition 3.8 generalizes this to cases where the inclusion $i : \mathfrak{h} \to \mathfrak{d}$ fails to integrate to a morphism of Lie groups.

3.4. From Dirac Lie group structures to Dirac Manin triples. In this Section we will show that any Dirac Lie group structure $(\mathbb{A}, E)$ on H arises by the reduction procedure from the last section, from a unique H-equivariant Dirac Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$.

3.4.1. Definition of $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$. As remarked in Section 3.1, the Dirac structure E is a vacant $\mathcal{LA}$-groupoid over H. Hence it corresponds to a unique H-equivariant Lie algebra triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{q} := \mathbb{A}_e$, and let $f\colon \mathfrak{q} \to \mathfrak{d}$ be the linear map given as the sum of the target and anchor map at the group unit $e \in H$,

$$f(\zeta) = t_e(\zeta) + \mathsf{a}_e(\zeta), \quad \zeta \in \mathfrak{q}. \tag{13}$$

Let $\gamma \in S^2\mathfrak{q}$ be dual to the given metric on $\mathbb{A}_e$. Write $\mathfrak{q} = \mathfrak{g} \oplus \mathfrak{r}$, with $\mathfrak{r}$ be the kernel of $t_e\colon \mathbb{A}_e \to \mathfrak{g}$, and $\mathfrak{g}$ embedded as E_e. Thus $f(\tau) = \mathsf{a}_e(\tau)$ for $\tau \in \mathfrak{r}$ and $f(\xi) = \xi$ for $\xi \in \mathfrak{g}$.

Define $\beta \in S^2\mathfrak{d}$ by

$$\beta^\sharp = f \circ f^*\colon \mathfrak{d}^* \to \mathfrak{d}.$$

This defines $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$, but we will need to show that β is H-invariant and that this triple gives $(\mathbb{A}, E)$. We will also show that $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ is the triple associated to $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$. (Among other things, we will have to show that $\mathfrak{q}$ is a Lie algebra and that f is a Lie algebra homomorphism.) As before, we denote by $p_\mathfrak{r} \in \mathrm{End}(\mathfrak{q})$ the projection to $\mathfrak{r}$ along $\mathfrak{g}$ and by $p_\mathfrak{h} \in \mathrm{End}(\mathfrak{d})$ the projection to $\mathfrak{h}$ along $\mathfrak{g}$. Thus $t_e = 1 - p_\mathfrak{r}$ and $p_\mathfrak{h} \circ f = f \circ p_\mathfrak{r}$.

3.4.2. Trivialization of $\mathbb{A}$. Since $t, s\colon E \to \mathfrak{g}$ are fiberwise isomorphisms, we have $\mathbb{A} = E \oplus \ker(t) = E \oplus \ker(s)$ as vector bundles. Let

$$j\colon \mathbb{A} \to E$$

be the projection along $\ker(t)$. The trivialization $E = H \times \mathfrak{g}$ given by the source map $s\colon E \to \mathfrak{g}$ extends to a trivialization $\mathbb{A} = H \times \mathfrak{q}$, as follows.

Proposition 3.10.

a. *The map*

$$\mathbb{A} \to \mathfrak{q},\ x \mapsto j(x)^{-1} \circ x$$

defines a trivialization, $\mathbb{A} \cong H \times \mathfrak{q}$, compatible with the metric.

b. *The constant sections of $\mathbb{A} \cong H \times \mathfrak{q}$ form a Lie algebra under Courant bracket. Thus $\mathfrak{q}$ inherits a Lie algebra structure.*

c. *The subspace $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{q}$, and the trivialization of $\mathbb{A}$ restricts to the given trivialization $E \cong H \times \mathfrak{g}$.*

d. *The subspace $\mathfrak{r}$ is a Lie subalgebra of $\mathfrak{q}$, and the trivialization of $\mathbb{A}$ restricts to a trivialization* $\ker(t) \cong H \times \mathfrak{r}$.

e. Restriction of the anchor map to constant sections defines an action $\mathfrak{q} \to \mathfrak{X}(H)$ with coisotropic stabilizers, so that $\mathbb{A}$ is the corresponding action Courant algebroid (cf. Equation (2)).

Proof. For each $h \in H$, the map $\mathbb{A}_h \to \mathfrak{q},\ x \mapsto j(x)^{-1} \circ x$ has inverse $\mathfrak{q} \to \mathbb{A}_h,\ \zeta \mapsto y \circ \zeta$, where $y \in E_h$ is the unique element such that $s(y) = t(\zeta)$. It is clear that the resulting trivialization extends that of E. The trivialization is compatible with the metric, since $\langle j(x)^{-1} \circ x,\, j(x)^{-1} \circ x\rangle = \langle j(x)^{-1},\, j(x)^{-1}\rangle + \langle x,\, x\rangle = \langle x,\, x\rangle$.

By definition, a section $\sigma \in \Gamma(\mathbb{A})$ is 'constant' relative to the trivialization of $\mathbb{A}$ if and only if $\sigma_{h_1h_2} \circ \sigma_{h_2}^{-1} \in E$, for all h_1, h_2. This can be rephrased in terms of morphisms: Let $P_E \colon \mathbb{A} \times \mathbb{A} \dashrightarrow \mathbb{A}$ be the Courant morphism, with underlying map $H \times H \to H$ projection to the second factor, where $(x_1, x_2) \sim_{P_E} x$ if and only if $x_1 \in E$ and $x = x_2$. Thus $P_E \subseteq \mathbb{A} \times \overline{\mathbb{A}} \times \overline{\mathbb{A}}$ is obtained from $\mathbb{A}_\Delta \times E$ by interchanging the last two components. We note that $\sigma \in \Gamma(\mathbb{A})$ is constant if and only if and only if there is a section $\hat{\sigma} \in \Gamma(\mathbb{A} \times \mathbb{A})$ such that

$$\hat{\sigma} \sim_{\mathrm{Mult}_{\mathbb{A}}} \sigma, \quad \hat{\sigma} \sim_{P_E} \sigma.$$

Note that $\hat{\sigma}$ is uniquely determined by the constant section σ: Its value at h_1, h_2 is $\hat{\sigma}_{h_1,h_2} = (\sigma_{h_1h_2} \circ \sigma_{h_2}^{-1}, \sigma_{h_2}) \in E_{h_1} \times \mathbb{A}_{h_2}$. Given another constant section σ', we have

$$[\![\hat{\sigma}, \hat{\sigma}']\!] \sim_{\mathrm{Mult}_{\mathbb{A}}} [\![\sigma, \sigma']\!], \quad [\![\hat{\sigma}, \hat{\sigma}']\!] \sim_{P_E} [\![\sigma, \sigma']\!],$$

since Courant morphism preserve Courant brackets. Hence $[\![\sigma, \sigma']\!]$ is constant. It follows that the space of constant sections is closed under Courant bracket. Furthermore, if σ, σ' are constant, then $[\![\sigma, \sigma']\!] + [\![\sigma', \sigma]\!] = \mathsf{a}^*\mathrm{d}\langle\sigma, \sigma'\rangle = 0$ since $\langle\sigma, \sigma'\rangle$ is constant. Hence the resulting bracket on $\mathfrak{q}$ is skew-symmetric, and hence is a Lie bracket.

It is obvious that the trivialization of $\mathbb{A}$ restricts to the given trivialization of E. Since E is involutive, the constant sections with values in E form a Lie subalgebra, thus $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{q}$. On the other hand, $t(x) = 0 \Leftrightarrow t(j(x)) = 0 \Leftrightarrow s(j(x)) = 0 \Leftrightarrow t(j(x)^{-1} \circ x) = 0$, shows that the trivialization takes $\ker(t)$ to $\mathfrak{r}$. The $\mathfrak{r}$-valued constant sections σ are exactly those for which $\widehat{\sigma} = 0 \times \sigma$. Since this property is preserved under Courant bracket, it follows that $\mathfrak{r}$ is a Lie subalgebra of $\mathfrak{q}$.

Since the anchor map takes Courant brackets to Lie brackets, we obtain a $\mathfrak{q}$-action on H. By construction, the Courant bracket on $\mathbb{A}$ extends the Lie bracket on constant sections, and the anchor map extends the action map. As shown in [19] (cf. also Section 1.1), this implies that the action of $\mathfrak{q}$ has coisotropic stabilizers. $\square$

The first part of the Proposition may be phrased as the statement that the trivializing map $\mathbb{A} \to \mathfrak{q}$ defines a morphism of Manin pairs

$$T \colon (\mathbb{A}, E) \dashrightarrow (\mathfrak{q}, \mathfrak{g}), \tag{14}$$

where $x \sim_T \zeta$ if and only if $\zeta = j(x)^{-1} \circ x$.

3.4.3. Construction of the coisotropic subgroupoid $C \subseteq \mathbb{T}H \times (\overline{\mathfrak{q}} \oplus \mathfrak{q})$. In the following discussion, whenever we write a composition of groupoid elements we take it to be implicit that the elements are composable.

PROPOSITION 3.11. *Let $(\mathbb{A}, E)$ be a Dirac Lie group structure on H, and define a Lie algebra structure on $\mathfrak{q} = \mathbb{A}_e$ as above. Then the subset $C \subseteq \mathbb{T}H \times (\overline{\mathfrak{q}} \oplus \mathfrak{q})$ given as*

$$C = \{(v + \alpha, \zeta, \zeta') |\ \exists x \in \mathbb{A} \colon v = \mathsf{a}(x),\ \zeta \circ x \circ \zeta'^{-1} \in E\} \tag{15}$$

is an involutive co-isotropic $\mathcal{VB}$-subgroupoid satisfying $\mathsf{a}(C^\perp) = 0$. *There is a canonical isomorphism of $\mathcal{CA}$-groupoids* $\mathbb{A} \to C/C^\perp$*, taking E to the reduction of* $TH \times (\mathfrak{g} \oplus \mathfrak{g})$.

Proof. Recall the definition of the division morphism $D\colon \overline{\mathbb{A}} \times \mathbb{A} \dashrightarrow \mathbb{A}$ from the proof of Proposition 2.3 where $(x_1, x_2) \sim_D x$ if and only if $x_1^{-1} \circ x_2 = x$. Together with the trivialization $T\colon \mathbb{A} \dashrightarrow \mathfrak{q}$, we obtain a morphism $K = (T \times T) \circ D^\top\colon \mathbb{A} \dashrightarrow \overline{\mathfrak{q}} \oplus \overline{\mathfrak{q}}$. Under this morphism, $x \sim_K (\zeta_1, \zeta_2)$ if and only if $\zeta_1 \circ x \circ \zeta_2^{-1} \in E$.

Let $R\colon \mathbb{A} \dashrightarrow \mathbb{T}H \times \mathbb{A}$ be the morphism, with underlying map the diagonal inclusion, defined by the property that $x \sim_R (v + \alpha, y)$ if and only if $v = \mathsf{a}(x)$ and $y - x = \mathsf{a}^*(y)$. Thus $R \subset \mathbb{T}H \times \mathbb{A} \times \overline{\mathbb{A}}$ is obtained from the diagonal morphism cf. Corollary 2.14) by permutation of the components and a sign change of the metric. The composition of R with $\mathbb{T}H_\Delta \times K\colon \mathbb{T}H \times \mathbb{A} \dashrightarrow \mathbb{T}H \times (\overline{\mathfrak{q}} \oplus \mathfrak{q})$ is clean, and defines a morphism

$$Q = (\mathbb{T}H_\Delta \times K) \circ R\colon \mathbb{A} \dashrightarrow \mathbb{T}H \times (\overline{\mathfrak{q}} \oplus \mathfrak{q})$$

with underlying map $H \to H$ the identity map. Explicitly,

$$y \sim_Q (v + \alpha, \zeta_1, \zeta_2) \Leftrightarrow \exists x \in \mathbb{A}\colon \zeta_1 \circ x \circ \zeta_2^{-1} \in E,\ v = \mathsf{a}(x),\ y - x = \mathsf{a}^*(\alpha). \tag{16}$$

Since $R,\ K,\ \mathbb{T}H_\Delta$ are all $\mathcal{CA}$-groupoid morphism, the same is true of Q.

We claim that $\ker(Q) = 0$. Indeed, suppose $y \sim_Q (0, 0, 0)$. The condition $x - y = \mathsf{a}^*(\alpha)$ with $\alpha = 0$ gives $x = y$, and the condition $\zeta_1 \circ x \circ \zeta_2^{-1} \in E$ with $\zeta_i = 0$ implies $x = 0$, as claimed. On the other hand, $\operatorname{ran}(Q) = C$. By Lemma 3.12 below, there is an isomorphism of Courant algebroids $\mathbb{A} \to C/C^\perp$.

Finally, we show that $E = (TH \times (\mathfrak{g} \oplus \mathfrak{g})) \circ Q$. Suppose (16) with $\alpha = 0$ and $\zeta_1, \zeta_2 \in \mathfrak{g}$. Then $x = y$, and $\zeta_1 \circ y \circ \zeta_2^{-1} \in E$. Since $\zeta_1, \zeta_2 \in \mathfrak{g} = E^{(0)}$ it follows that $y \in E$. Therefore $(TH \times (\mathfrak{g} \oplus \mathfrak{g})) \circ Q \subseteq E$, and the conclusion follows, since both sides are Lagrangian. ∎

LEMMA 3.12. *Let $R\colon \mathbb{A} \dashrightarrow \mathbb{A}'$ be a Courant morphism, with underlying map $\Phi\colon M \to M'$ a diffeomorphism. If* $\ker(R) = 0$*, then* $C = \operatorname{ran}(R)$ *is co-isotropic, with* $\mathsf{a}(C^\perp) = 0$*, and $\mathbb{A} \cong \mathbb{A}'_C$ as Courant algebroids. If $\mathbb{A}, \mathbb{A}'$ are $\mathcal{CA}$-groupoids, and $R\colon \mathbb{A} \dashrightarrow \mathbb{A}'$ is a $\mathcal{CA}$-groupoid morphism, then $\mathbb{A} \cong \mathbb{A}'_C$ is an isomorphism of $\mathcal{CA}$-groupoids.*

Proof. The inclusion $R \subseteq (\mathbb{A}' \times C)|_{\mathrm{gr}(\Phi)}$ shows $(0 \times C^\perp)|_{\mathrm{gr}(\Phi)} \subseteq R^\perp = R$, hence $C^\perp \subseteq C$ so that C is co-isotropic. Furthermore, since $\mathsf{a}(0, y') = (0, \mathsf{a}(y'))$ for $y' \in C^\perp$ is tangent to $\mathrm{gr}(\Phi)$, we see $\mathsf{a}(y') = 0$, hence $\mathsf{a}(C^\perp) = 0$. Let $P\colon \mathbb{A}' \dashrightarrow \mathbb{A}'_C$ be the Courant morphism defined by the reduction. Thus $y' \sim_P y''$ if and only if $y' \in C$, with y'' its image under the quotient map. We will show that $P \circ R\colon \mathbb{A} \dashrightarrow \mathbb{A}'_C$ is an isomorphism. Indeed, let $x \in \ker(P \circ R)$. Then $x \sim_R x',\ x' \sim_P 0$ for some $x' \in \mathbb{A}'$. By definition of P, we have $x' \in C^\perp$. Since $(0 \times C^\perp)|_{\mathrm{gr}(\Phi)} \subseteq R,\ x \sim_R x'$ implies $x \sim_R 0$, hence $x = 0$. The property $\ker(P \circ R) = 0,\ \operatorname{ran}(P \circ R) = \mathbb{A}'$ means that $P \circ R$ defines an isomorphism $\mathbb{A} \cong \mathbb{A}'_C$. If $R\colon \mathbb{A} \dashrightarrow \mathbb{A}'$ is a morphism of $\mathcal{CA}$-groupoids, then so is P and hence $P \circ R$. ∎

The co-isotropic subbundle C has an alternative description, similar to Proposition 3.8.

PROPOSITION 3.13. *The co-isotropic subbundle $C \subseteq \mathbb{T}H \times (\overline{\mathfrak{q}} \oplus \mathfrak{q})$ from Proposition 3.11 may be written,*

$$C = \{(v + \alpha, \zeta_1, \zeta_2)|\ \iota(v)\theta_h^R = \mathrm{Ad}_h\, f(\zeta_2) - f(\zeta_1)\}.$$

Proof. Given $(v+\alpha,\zeta_1,\zeta_2)\in \mathbb{T}H\times(\overline{\mathfrak{q}}\oplus\mathfrak{q})$, let $z\in E_h$ be the unique element with $s(z)=t(\zeta_2)$. By Equation (10), we have $\mathrm{Ad}_h\, t(\zeta_2)=\mathrm{Ad}_h\, s(z)=\iota(\mathsf{a}(z))\theta_h^R+t(z)$. Together with Equation (13) we obtain

$$\begin{aligned}\mathrm{Ad}_h\, f(\zeta_2)-f(\zeta_1)&=\mathrm{Ad}_h(t(\zeta_2)+\mathsf{a}(\zeta_2))-(t(\zeta_1)+\mathsf{a}(\zeta_1))\\&=\iota(\mathsf{a}(z))\theta_h^R+\mathrm{Ad}_h\,\mathsf{a}(\zeta_2)-\mathsf{a}(\zeta_1)+t(z)-t(\zeta_1).\end{aligned}$$

The first three terms lie in $\mathfrak{h}$, the last two in $\mathfrak{g}$. Hence the property $\iota(v)\theta_h^R=\mathrm{Ad}_h\, f(\zeta_2)-f(\zeta_1)$ is equivalent to the two conditions

$$\iota(v)\theta_h^R=\iota(\mathsf{a}(z))\theta_h^R+\mathrm{Ad}_h\,\mathsf{a}(\zeta_2)-\mathsf{a}(\zeta_1),\quad t(\zeta_1)=t(z) \tag{17}$$

Equation (17) is equivalent to the condition that $x:=\zeta_1^{-1}\circ z\circ\zeta_2$ is defined and $v=\mathsf{a}(x)$. ☐

REMARK 3.14. Define a bundle map $H\times\mathfrak{q}\to C,\ (h,\zeta)\mapsto(v,\xi,\zeta)$ where $\iota(v)\theta_h^R=p(\mathrm{Ad}_h\, f(\zeta))$ and $\xi=(1-p)\,\mathrm{Ad}_h\, f(\zeta)$. The sub-bundle given as its image is invariant under left groupoid multiplication by elements of $TH\times(\mathfrak{g}\oplus\mathfrak{g})$, and is a complement to $C^\perp$. Hence, its composition with the quotient map to $\mathbb{A}=C/C^\perp$ is the trivialization $H\times\mathfrak{q}\cong\mathbb{A}$ from Proposition 3.10.

3.4.4. Relation between $(\mathfrak{d},\mathfrak{g},\mathfrak{h})_\beta$ and $(\mathfrak{q},\mathfrak{g},\mathfrak{r})_\gamma$. We still have to show that β is H-invariant, and that $(\mathfrak{q},\mathfrak{g},\mathfrak{r})_\gamma$ is the Dirac Manin triple associated to $(\mathfrak{d},\mathfrak{g},\mathfrak{h})_\beta$ by the construction from Section 3.2.

PROPOSITION 3.15.
a. *The map $f\colon\mathfrak{q}\to\mathfrak{d}$ is a Lie algebra homomorphism.*
b. *The element $\beta\in S^2\mathfrak{d}$ defined by $\beta^\sharp=f\circ f^*$ is $\mathfrak{d}$-invariant as well as Ad_h-invariant.*
c. *$\mathfrak{g}$ is β-coisotropic.*
d. *The Lie subalgebra $\mathfrak{c}=\mathfrak{g}\ltimes\mathfrak{d}^*_\beta\subseteq\mathfrak{d}\ltimes\mathfrak{d}^*_\beta$ is coisotropic, and the map*

$$\mathfrak{c}\to\mathfrak{q},\ \ (\xi,\mu)\mapsto\xi+f^*(\mu)$$

descends to an isometric Lie algebra isomorphism $\mathfrak{c}/\mathfrak{c}^\perp\to\mathfrak{q}$.

Proof.
a. Since f restricts to a Lie algebra homomorphism $\mathfrak{r}\to\mathfrak{h}$, and is given by the identity map on $\mathfrak{g}$, we need only check that $f([\tau,\xi])=[f(\tau),\xi]$ for $\xi\in\mathfrak{g},\ \tau\in\mathfrak{r}$. Define sections of $C\subseteq\mathbb{T}H\times(\mathfrak{q}\oplus\mathfrak{q})$ by

$$(f(\tau)^L,0,\tau),\quad (p_\mathfrak{h}(\mathrm{Ad}_h\,\xi))^R,(1-p_\mathfrak{h})\,\mathrm{Ad}_h\,\xi,\xi).$$

Since C is involutive, their Courant bracket

$$\big(p_\mathfrak{h}(\mathrm{Ad}_h[f(\tau),\xi])^R,(1-p_\mathfrak{h})\,\mathrm{Ad}_h\,[f(\tau),\xi],[\tau,\xi]\big)$$

is again a section of C. Thus

$$\begin{aligned}p_\mathfrak{h}(\mathrm{Ad}_h[f(\tau),\xi])&=\mathrm{Ad}_h\, f([\tau,\xi])-(1-p_\mathfrak{h})\,\mathrm{Ad}_h\,[f(\tau),\xi]\\&=\mathrm{Ad}_h\, f([\tau,\xi])-\mathrm{Ad}_h\,[f(\tau),\xi]+p_\mathfrak{h}(\mathrm{Ad}_h[f(\tau),\xi]),\end{aligned}$$

giving $f([\tau,\xi])=[f(\tau),\xi]$ as desired.

b. By the same argument as in the proof of Proposition 3.8, the fiber of $C^\perp$ at $h \in H$ is spanned by the sections $\psi(\mu)$, $\mu \in \mathfrak{d}^*$. The property $C^\perp \subseteq C$ gives $0 = \mathrm{Ad}_h(f(f^*(\mu)) - f(f^*(\mu))$ as desired. This shows that β is invariant under the adjoint action of H. In particular it is $\mathfrak{h}$-invariant. Since f is a Lie algebra homomorphism, it is also equivariant under the adjoint action of $\mathfrak{g}$. Thus $\beta = f \circ f^*$ is $\mathfrak{g}$-invariant as well.

c. The dual map $f^* \colon \mathfrak{d}^* \to \mathfrak{q}$ takes $\mathrm{ann}(\mathfrak{g}) \subseteq \mathfrak{d}^*$ to $\mathfrak{g}^\perp \subseteq \mathfrak{q}$. Hence, for $\mu \in \mathrm{ann}(\mathfrak{g})$, $\beta(\mu, \mu) = \langle f^*(\mu), f^*(\mu) \rangle = 0$.

d. The map $\mathfrak{c} \to \mathfrak{q}$, $(\xi, \mu) \mapsto \xi + f^*(\mu)$ is surjective, since its image contains $\mathfrak{g}$ as well as the complement $f^*(\mathrm{ann}(\mathfrak{h})) = \mathfrak{r}^\perp$. The map clearly preserves the bilinear forms, hence its kernel must be $\mathfrak{c}^\perp$. Using the identity

$$[f^*(\mu_1), f^*(\mu_2)] = f^*([\beta^\sharp(\mu_1), \mu_2]),$$

(which is verified by pairing both sides with $\zeta \in \mathfrak{q}$), one finds that it is a Lie algebra homomorphism.

$\square$

It follows that $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ is an H-equivariant Dirac Manin triple, and that $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ and $f \colon \mathfrak{q} \to \mathfrak{d}$ result from the construction of Section 3.2. Propositions 3.11 and 3.13 define a canonical isomorphism between $(\mathbb{A}, E)$ and the Dirac Lie group constructed in Section 3.3.

4. Morphisms. To complete the proof of Theorem 0.1, it remains to show that the correspondence between Dirac Lie group structures and H-equivariant Dirac Manin triples respects morphisms. We will sketch the main aspects of this correspondence, leaving details to the reader.

4.1. Morphisms of Dirac Manin triples. Dirac Manin triples form a category relative to the following notion of morphism.

DEFINITION 4.1. A *morphism of Dirac Manin triples*

$$\mathfrak{k} \colon (\mathfrak{d}_0, \mathfrak{g}_0, \mathfrak{h}_0)_{\beta_0} \dashrightarrow (\mathfrak{d}_1, \mathfrak{g}_1, \mathfrak{h}_1)_{\beta_1}$$

is a $\beta_1 - \beta_0$-coisotropic Lie subalgebra $\mathfrak{k} \subseteq \mathfrak{d}_1 \times \mathfrak{d}_0$ such that

$$\mathfrak{g}_1 = \mathfrak{k} \circ \mathfrak{g}_0, \quad \mathfrak{h}_0 = \mathfrak{h}_1 \circ \mathfrak{k}.$$

If the Dirac Manin triples $(\mathfrak{d}_i, \mathfrak{g}_i, \mathfrak{h}_i)_{\beta_i}$ are H_i-equivariant, and given a group homomorphism $\Phi \colon H_0 \to H_1$, we speak of a *morphism of H_i-equivariant Dirac Manin triples* provided $\mathfrak{k}$ is invariant under the diagonal action of H_0 on $\mathfrak{d}_1 \times \mathfrak{d}_0$ and projects onto the graph of $T_e\Phi$.

See Appendix A for compositions of linear relations. The property $\mathfrak{h}_0 = \mathfrak{h}_1 \circ \mathfrak{k}$ implies $\ker(\mathfrak{k}) \subseteq \mathfrak{h}_0$, hence $\mathfrak{g}_0 \cap \ker(\mathfrak{k}) = 0$. Hence there is a linear map $\psi \colon \mathfrak{g}_1 \to \mathfrak{g}_0$, taking $\xi_1 \in \mathfrak{g}_1$ to the unique element $\xi_0 = \psi(\xi_1) \in \mathfrak{g}_0$ with $\xi_0 \sim_{\mathfrak{k}} \xi_1$. By a similar argument, there is a linear map $\phi \colon \mathfrak{h}_0 \to \mathfrak{h}_1$, taking $\nu_0 \in \mathfrak{h}_0$ to the unique element $\nu_1 = \phi(\nu_0) \in \mathfrak{h}_1$ with $\nu_0 \sim_{\mathfrak{k}} \nu_1$. Moreover a quick exercise in linear algebra shows that $\mathfrak{k}$ is the direct sum of the graphs of ψ, ϕ,

$$\mathfrak{k} = \mathrm{gr}(\psi)^\top \oplus \mathrm{gr}(\phi).$$

In particular

$$\dim \mathfrak{k} = \dim \mathfrak{g}_1 + \dim(\mathfrak{d}_0/\mathfrak{g}_0).$$

In the H_i-equivariant case, $\phi = T_e\Phi$.

Remark 4.2. Suppose that the β_i are non-degenerate and that the Lie subalgebras $\mathfrak{g}_i$ are Lagrangian. Then $\mathfrak{k} \subseteq \mathfrak{d}_1 \times \mathfrak{d}_0$ is $\beta_1 - \beta_0$-Lagrangian, for dimensional reasons. This implies that $\psi = \phi^*$, for the identification $\mathfrak{h}_i = \mathfrak{g}_i^*$ given by the pairing, and that ϕ preserves the induced bilinear forms on $\mathfrak{h}_i$. In particular, for ordinary Manin triples $(\mathfrak{d}_i, \mathfrak{g}_i, \mathfrak{h}_i)$, a morphism is given by a pair of Lie algebra homomorphisms $\psi\colon \mathfrak{g}_1 \to \mathfrak{g}_0$ and $\phi := \psi^*\colon \mathfrak{h}_0 \to \mathfrak{h}_1$.

The construction of $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ from $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ has the following functoriality property:

Proposition 4.3. *Let $\mathfrak{k}\colon (\mathfrak{d}_0, \mathfrak{g}_0, \mathfrak{h}_0)_{\beta_0} \dashrightarrow (\mathfrak{d}_1, \mathfrak{g}_1, \mathfrak{h}_1)_{\beta_1}$ be a morphism of Dirac Manin triples, given as the direct sum of the graphs of $\phi\colon \mathfrak{h}_0 \to \mathfrak{h}_1$ and $\psi\colon \mathfrak{g}_1 \to \mathfrak{g}_0$. Let $(\mathfrak{q}_i, \mathfrak{g}_i, \mathfrak{r}_i)_{\gamma_i}$ be the Dirac Manin triples associated to $(\mathfrak{d}_i, \mathfrak{g}_i, \mathfrak{h}_i)_{\beta_i}$ as in Section 3.2, with the corresponding maps $f_i\colon \mathfrak{q}_i \to \mathfrak{d}_i$. The one obtains a morphism of Dirac Manin triples*

$$\mathfrak{l}\colon (\mathfrak{q}_0, \mathfrak{g}_0, \mathfrak{r}_0)_{\gamma_0} \dashrightarrow (\mathfrak{q}_1, \mathfrak{g}_1, \mathfrak{r}_1)_{\gamma_1},$$

where $\mathfrak{l} \subseteq \mathfrak{q}_1 \times \mathfrak{q}_0$ is the direct sum of the graphs of ψ and $\kappa = \psi^*\colon \mathfrak{r}_0 = \mathfrak{g}_0^* \to \mathfrak{r}_1 = \mathfrak{g}_1^*$. One has $f(\mathfrak{l}) \subseteq \mathfrak{k}$ where $f = f_1 \times f_0$.

The proof is straightforward.

4.2. The Equivalence Theorem for morphisms. A morphism of Dirac Lie groups is defined to be a morphism of multiplicative Manin pairs.

Theorem 4.4 (Morphisms). *There is a 1-1 correspondence between morphisms of of H_i-equivariant Dirac Manin triples $\mathfrak{k}\colon (\mathfrak{d}_0, \mathfrak{g}_0, \mathfrak{h}_0)_{\beta_0} \dashrightarrow (\mathfrak{d}_1, \mathfrak{g}_1, \mathfrak{h}_1)_{\beta_1}$ and morphisms of the corresponding Dirac Lie groups, $L\colon (\mathbb{A}_0, E_0) \dashrightarrow (\mathbb{A}_1, E_1)$. This correspondence is compatible with the composition of morphisms.*

One direction of this result is rather simple: Given the morphism of H_i-equivariant Dirac Manin triples, with underlying group homomorphism $\Phi\colon H_0 \to H_1$, one takes

$$L = \mathrm{gr}(\Phi) \times \mathfrak{l} \subseteq (H_1 \times H_0) \times (\mathfrak{q}_1 \times \overline{\mathfrak{q}_0}) \cong \mathbb{A}_1 \times \overline{\mathbb{A}}_0$$

where $\mathfrak{l}$ is defined as above. The proof that this sets up a 1-1 correspondence is more cumbersome, and is omitted for the sake of brevity.

5. Explicit formulas.

5.1. The $\mathcal{CA}$-groupoid structure in terms of the trivialization. Let $(\mathbb{A}, E)$ be a Dirac Lie group structure on H. In this Section we will work out the formulas for the Courant algebroid structure and $\mathcal{VB}$-groupoid structure on $\mathbb{A}$ in terms of the trivialization $\mathbb{A} \xrightarrow{\cong} H \times \mathfrak{q}$, obtained in Proposition 3.10.

We will need some background from the theory of matched pairs. Given a Lie algebra triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$, one obtains actions of of $\mathfrak{h}$ on $\mathfrak{g} \cong \mathfrak{d}/\mathfrak{h}$ and $\mathfrak{g}$ on $\mathfrak{h} \cong \mathfrak{d}/\mathfrak{g}$, satisfying the compatibility conditions of a *matched pair* $\mathfrak{g} \bowtie \mathfrak{h}$ of Lie algebras. Moreover, letting $p_{\mathfrak{h}} \in \mathrm{End}(\mathfrak{d})$ be the projection to $\mathfrak{h}$ along $\mathfrak{g}$, the $\mathfrak{g}$-action on $\mathfrak{h}$ extends to a $\mathfrak{d}$-action on $\mathfrak{h}$ by

$$\nu \mapsto p_{\mathfrak{h}}([\xi, \nu]), \quad \xi \in \mathfrak{d}, \quad \nu \in \mathfrak{h}.$$

Similarly, an H-equivariant Lie algebra triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ defines a linear action $\bullet$ of the group H on $\mathfrak{g} = \mathfrak{d}/\mathfrak{h}$ and a Lie algebra action $\varrho\colon \mathfrak{g} \to \mathfrak{X}(H)$, satisfying the compatibility conditions of a *matched pair* $\mathfrak{g} \bowtie H$ *between a Lie group and a Lie algebra.* These actions are given by

$$h \bullet \xi = (1 - p_{\mathfrak{h}}) \operatorname{Ad}_h \xi. \quad h \in H, \ \xi \in \mathfrak{g} \tag{18}$$

and $\iota(\varrho(\xi))\theta_h^R = p_{\mathfrak{h}}(\operatorname{Ad}_h \xi)$ for $h \in H$, $\xi \in \mathfrak{g}$. Furthermore, the $\mathfrak{g}$-action on H combines with the $\mathfrak{h}$-action $\nu \to \nu^L$ to a Lie algebra action $\varrho\colon \mathfrak{d} \to \mathfrak{X}(H)$, by the same formula:

$$\iota(\varrho(\zeta))\theta_h^R = p_{\mathfrak{h}}(\operatorname{Ad}_h \zeta), \quad h \in H, \ \zeta \in \mathfrak{d}. \tag{19}$$

See Appendix B for more details. One has the following extension of Proposition 3.4.

Proposition 5.1. *There is a 1-1 correspondence between*
(i) Vacant $\mathcal{LA}$-groupoids $E \rightrightarrows \mathfrak{g}$ over groups $H \rightrightarrows \mathrm{pt}$,
(ii) H-equivariant Lie algebra triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$, and
(iii) Matched pairs $\mathfrak{g} \bowtie H$.
Furthermore, if $x \in E_h$ with $s(x) = \xi$ then $t(x) = h \bullet \xi$, $\mathsf{a}(x) = \varrho(\xi)_h$.

Here, the equivalence $(i) \Leftrightarrow (iii)$ was observed by Mackenzie [22, 23]. Again, further details are given in Appendix B.

Let $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ and $f\colon \mathfrak{q} \to \mathfrak{d}$ be as in Section 3.2. The action $\bullet$ of H on $\mathfrak{g}$ defines an action on the dual space $\mathfrak{g}^* \cong \mathfrak{r}^\perp \subseteq \mathfrak{q}$, which we again denote by $\bullet$. Recalling that f^* takes the Ad_h-invariant subspace $\mathfrak{g}^* \cong \operatorname{ann}(\mathfrak{h})$ isomorphically to $\mathfrak{r}^\perp$, this action is characterized by

$$h \bullet f^*(\mu) = f^*(\operatorname{Ad}_h \mu), \quad h \in H, \ \mu \in \operatorname{ann}(\mathfrak{h}). \tag{20}$$

The restriction of the metric on $\mathfrak{q}$ to the subspace $\mathfrak{r}^\perp$ is invariant under the H-action:

$$\langle h \bullet \nu, \ h \bullet \nu' \rangle = \langle \nu, \nu' \rangle, \quad \nu, \nu' \in \mathfrak{r}^\perp, \ h \in H. \tag{21}$$

This follows by writing $\nu = f^*(\mu), \nu' = f^*(\mu')$ and using the H-equivariance of $\beta^\sharp = f \circ f^*$. Write $p_{\mathfrak{r}} \in \operatorname{End}(\mathfrak{q})$ for the projection to $\mathfrak{r}$ along $\mathfrak{g}$. We are in a position to give explicit structural formulas for Dirac Lie groups.

Theorem 5.2. *The Dirac Lie group structure defined by the H-equivariant Dirac Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ is given by*

$$(\mathbb{A}, E) = (H \times \mathfrak{q}, \ H \times \mathfrak{g}).$$

Here $\mathbb{A}$ has the structure of an action Courant algebroid, for the action $\varrho^{\mathfrak{q}} = \varrho \circ f\colon \mathfrak{q} \to \mathfrak{X}(H)$,

$$\iota(\varrho^{\mathfrak{q}}(\zeta))\theta_h^R = p_{\mathfrak{h}}(\operatorname{Ad}_h f(\zeta)). \tag{22}$$

The $\mathcal{VB}$-groupoid structure has source and target maps $s, t\colon \mathbb{A} \to \mathfrak{g}$,

$$s(h, \zeta) = p_{\mathfrak{r}}^*(\zeta), \qquad t(h, \zeta) = h \bullet (1 - p_{\mathfrak{r}})(\zeta);$$

the inclusion of units is the map $\mathfrak{g} \to \mathbb{A}$, $\xi \mapsto (e, \xi)$, and the multiplication of composable elements is given by

$$(h_1, \zeta_1) \circ (h_2, \zeta_2) = (h_1 h_2, \zeta_2 + h_2^{-1} \bullet (1 - p_{\mathfrak{r}}^*)\zeta_1).$$

Proof. Let $\varrho^{\mathfrak{q}}\colon \mathfrak{q} \to \mathfrak{X}(H)$ be the Lie algebra action described in Proposition 3.10. By definition, $\varrho^{\mathfrak{q}}(\zeta) = \mathsf{a}(\sigma)$ where $\sigma \in \Gamma(\mathbb{A})$ is the constant section corresponding to ζ (i.e. $\sigma \sim_T \zeta$). For $\zeta = \tau \in \mathfrak{r}$, the constant section σ takes values in $\ker(t)$, hence $\sigma_h = 0_h \circ \sigma_e$ for all h. Since the anchor a is a groupoid homomorphism, it follows that $\varrho^{\mathfrak{q}}(\tau)_h = 0_h \circ \varrho^{\mathfrak{q}}(\tau)_0$. Equivalently, $\varrho^{\mathfrak{q}}(\tau)$ is left-invariant. Hence $\iota(\varrho^{\mathfrak{q}}(\tau))\theta^R_h = \mathrm{Ad}_h(\varrho^{\mathfrak{q}}(\tau)_e) = \mathrm{Ad}_h(f(\tau)) = p(\mathrm{Ad}_h(f(\tau)))$, i.e. $\varrho^q(\tau) = \varrho(f(\tau))$. On the other hand, Equation (18) shows $\iota(\varrho(\xi))\theta^R_h = p_{\mathfrak{h}}(\mathrm{Ad}_h(f(\xi)))$ for $\xi \in \mathfrak{g}$, hence $\varrho^{\mathfrak{q}}(\xi) = \varrho(f(\xi))$. This proves $\varrho^{\mathfrak{q}} = \varrho \circ f$.

We next consider the groupoid structure. Use the trivialization to write elements of $\mathbb{A}$ in the form $x = (h, \zeta)$. Recall that on the vacant $\mathcal{VB}$-subgroupoid E, the trivialization is given by the source map. Hence, by Proposition 5.1 we have

$$s(h,\xi) = \xi, \quad t(h,\xi) = h \bullet \xi \tag{23}$$

for $(h,\xi) \in E$, and

$$(h_1,\xi_1) \circ (h_2,\xi_2) = (h_1h_2,\xi_2) \tag{24}$$

for $(h_i,\xi_i) \in E$ with $h_2 \bullet \xi_2 = \xi_1$. Consider now a general element $(h,\zeta) \in \mathbb{A}$. By definition of the trivialization,

$$(h,\zeta) = j(h,\zeta) \circ (e,\zeta). \tag{25}$$

Since $s(j(h,\zeta)) = t(e,\zeta) = (1-p_{\mathfrak{r}})\zeta$ by definition of $p_{\mathfrak{r}}$, it follows that $j(h,\zeta) = (h,(1-p_{\mathfrak{r}})\zeta)$. We conclude that $t(h,\zeta) = t(j(h,\zeta)) = h\bullet(1-p_{\mathfrak{r}})\zeta$, and $s(h,\zeta) = s(e,\zeta) = p^*_{\mathfrak{r}}\zeta$.

To find the groupoid multiplication, consider first a product $(h_1,0)\circ(h_2,\nu)$ with $(h_2,\nu) \in \ker(t)$, i.e. $\nu \in \mathfrak{r}$. The product lies in $\ker(t)$, hence it is of the form $(h_1,0)\circ(h_2,\nu) = (h_1h_2,\nu')$ for some $\nu' \in \mathfrak{r}$. Taking inner products with the identity $(h_1, h_2 \bullet \xi) \circ (h_2,\xi) = (h_1h_2,\xi)$ for $\xi \in \mathfrak{g}$, we obtain $\langle 0, h_2 \bullet \xi\rangle + \langle \nu,\xi\rangle = \langle \nu',\xi\rangle$, hence $\nu' = \nu$. Thus

$$(h_1,0)\circ(h_2,\nu) = (h_1h_2,\nu), \quad \nu \in \mathfrak{r}.$$

Similarly, consider a product $(h_1,\tau)\circ(h_2,0) = (h_1h_2,\tau')$ with $(h_1,\tau) \in \ker(s)$, thus $\tau \in \mathfrak{r}^\perp$. Then $\tau' \in \mathfrak{r}^\perp$, and $\langle \tau, h_2\bullet\xi\rangle + \langle 0,\xi\rangle = \langle \tau',\xi\rangle$ for $\xi \in \mathfrak{g}$, proving $\tau' = (h_2)^{-1}\bullet\tau$.

For a general product $(h_1,\zeta_1)\circ(h_2,\zeta_2)$ of composable elements, write $\zeta_2 = (1-p_{\mathfrak{r}})\zeta_2 + p_{\mathfrak{r}}\zeta_2$ and $\zeta_1 = p^*_{\mathfrak{r}}\zeta_1 + (1-p^*_{\mathfrak{r}})\zeta_1$. We obtain

$$\begin{aligned}(h_1,\zeta_1)\circ(h_2,\zeta_2) &= (h_1,0)\circ(h_2,p_{\mathfrak{r}}\zeta_2) + (h_1,(1-p^*_{\mathfrak{r}})\zeta_1)\circ(h_2,0)\\ &\quad + (h_1,p^*_{\mathfrak{r}}\zeta_1)\circ(h_2,(1-p_{\mathfrak{r}})\zeta_2)\\ &= (h_1h_2, p_{\mathfrak{r}}\zeta_2 + h_2^{-1}\bullet(1-p^*_{\mathfrak{r}})\zeta_1 + (1-p_{\mathfrak{r}})\zeta_2)\\ &= (h_1h_2, \zeta_2 + h_2^{-1}\bullet(1-p^*_{\mathfrak{r}})\zeta_1).\end{aligned}$$

∎

5.2. Examples.

5.2.1. The standard Dirac Lie group structure. For any Lie group H, we have the H-equivariant Dirac Manin triple $(\mathfrak{h}\ltimes\mathfrak{h}^*, \mathfrak{h}^*, \mathfrak{h})_\beta$, with β the symmetric bilinear form given by the pairing. Since β is non-degenerate and $\mathfrak{g} = \mathfrak{h}^*$ is Lagrangian, we have (cf. Example 3.7) $\mathfrak{q} = \mathfrak{d}$, with f the identity. The projections $p_{\mathfrak{h}}$ and $(1-p^*_{\mathfrak{h}})$ coincide, and our formulas specialize to

$$s(h,\nu,\mu) = \mu, \quad t(h,\nu,\mu) = \mathrm{Ad}_h\,\mu,$$

$$(h_1, \nu_1, \mu_1) \circ (h_2, \nu_2, \mu_2) = (h_1 h_2, \nu_2 + \mathrm{Ad}_{h_2^{-1}} \nu_1, \mu_2).$$

The action of $\mathfrak{h} \ltimes \mathfrak{h}^*$ on H is given by the left-invariant vector fields, $\varrho(\nu, \mu) = \nu^L$. This is the standard Dirac Lie group structure $(\mathbb{A}, E) = (\mathbb{T}H, T^*H)$, written in left-trivialization.

5.2.2. The Cartan-Dirac structure. Given a Lie group G with an invariant metric on $\mathfrak{g}$, one can form the Dirac Manin triple $(\overline{\mathfrak{g}} \oplus \mathfrak{g}, \mathfrak{g}_\Delta, 0 \oplus \mathfrak{g})_\beta$ where β is given by the metric on $\overline{\mathfrak{g}} \oplus \mathfrak{g}$, and $\mathfrak{g}_\Delta$ is the diagonal. Again $\mathfrak{q} = \mathfrak{d}$, $f = \mathrm{id}$. For $h \in H = \{1\} \times G$ we have $\mathrm{Ad}_h(\xi, \xi') = (\xi, \mathrm{Ad}_h \xi')$. It follows that the action $\bullet$ on $\mathfrak{g}$ (hence also on $\mathfrak{r}^\perp$) is the trivial action:

$$h \bullet (\xi, \xi) = (1 - p_\mathfrak{h})(\xi, \mathrm{Ad}_h \xi) = (\xi, \xi).$$

The formulas for the groupoid structure simplify to

$$s(h, \xi, \xi') = \xi', \ t(h, \xi, \xi') = \xi, \quad (h_1, \xi_1, \xi_1') \circ (h_2, \xi_2, \xi_2') = (h_1 h_2, \xi_1, \xi_2').$$

From $\iota(\varrho(\xi, \xi'))\theta^R_h = p_\mathfrak{h} \mathrm{Ad}_h(\xi, \xi') = p_\mathfrak{h}(\xi, \mathrm{Ad}_h \xi') = \mathrm{Ad}_h \xi' - \xi$ we obtain

$$\varrho(\xi, \xi') = (\xi')^L - \xi^R.$$

The resulting Dirac Lie group structure $(\mathbb{A}, E) = (G \times (\overline{\mathfrak{g}} \oplus \mathfrak{g}),\ G \times \mathfrak{g}_\Delta)$ is the Cartan-Dirac structure from Example 2.12.

5.2.3. Dirac Lie group structures over $H = \mathrm{pt}$. If the group H is trivial, then the Dirac Manin triple is of the form $(\mathfrak{d}, \mathfrak{d}, 0)_\beta$. Dirac Lie group structures over pt are hence classified by Lie algebras $\mathfrak{d}$ with invariant elements $\beta \in S^2\mathfrak{d}$. (The same data also classify the $\mathcal{CA}$-groupoid structures $\mathbb{A}$ over $H = \mathrm{pt}$, since $\mathbb{A}$ extends uniquely to a Dirac Lie group structure by putting $E = \mathbb{A}^{(0)}$.) We find

$$(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma = (\mathfrak{d} \ltimes \mathfrak{d}^*_\beta, \mathfrak{d}, (\mathfrak{d}^*_\beta)^\perp)_{\widehat{\beta}}$$

with f the projection along $(\mathfrak{d}^*_\beta)^\perp$. The $\mathcal{VB}$-groupoid structure on $\mathfrak{d} \ltimes_\beta \mathfrak{d}^*_\beta \rightrightarrows \mathfrak{d}$ is given by $s(\xi, \mu) = \xi, \quad t(\xi, \mu) = \xi + \beta^\sharp(\mu)$, and the groupoid multiplication of composable elements is given by

$$(\xi_1, \mu_1) \circ (\xi_2, \mu_2) = (\xi_2, \mu_1 + \mu_2).$$

Note that $\mathfrak{d}$ is a subgroupoid, as required.

We can also classify the multiplicative Main pairs over $H = \mathrm{pt}$. Let us call a Lie algebra $\mathfrak{d}$ with an invariant element $\beta \in S^2\mathfrak{d}$ and a β-coisotropic Lie subalgebra $\mathfrak{g}$ a *Dirac-Manin pair.*

PROPOSITION 5.3. *There is a 1-1 correspondence between*
(i) Multiplicative Manin pairs $(\mathbb{A}, E)$ *over* $H = \mathrm{pt}$,
(ii) Dirac Manin pairs $(\mathfrak{d}, \mathfrak{g})_\beta$.

The correspondence is as follows: By Drinfel'd's observation, (see Remark 3.6), $\mathcal{CA}$-groupoids $\mathbb{A}$ over $H = \mathrm{pt}$ are of the form $\mathbb{A} = \mathfrak{d} \ltimes \mathfrak{d}^*_\beta$. Given a β-coisotropic Lie subalgebra $\mathfrak{g}$, one obtains a multiplicative Dirac structure $E = \mathfrak{g} \ltimes \mathrm{ann}(\mathfrak{g})$. The Lie algebra $\mathfrak{g}$ is recovered from E as the units, $\mathfrak{g} = E^{(0)}$.

6. The Lagrangian complement F. Let $(\mathbb{A}, E)$ be a Dirac Lie group structure on H. We will show that E has a distinguished Lagrangian complement. The splitting $\mathbb{A} = E \oplus F$ defines a bi-vector field π_H, and (H, π_H) is a quasi-Poisson $\mathfrak{g}$-space in the sense of Alekseev and Kosmann-Schwarzbach [2].

6.1. Quasi-Poisson $\mathfrak{g}$-manifolds. A *quasi-Manin triple* $(\mathfrak{q}, \mathfrak{g}, \mathfrak{n})_\gamma$ is Lie algebra $\mathfrak{q}$ with a non-degenerate element $\gamma \in (S^2\mathfrak{q})^\mathfrak{q}$, together with a Lagrangian Lie subalgebra $\mathfrak{g}$ and a Lagrangian subspace $\mathfrak{n}$ complementary to $\mathfrak{g}$. The quasi-Manin triple $(\mathfrak{q}, \mathfrak{g}, \mathfrak{n})_\gamma$ determines a trivector $\chi \in \wedge^3\mathfrak{g} \subseteq \wedge^3\mathfrak{q}$ by the equation

$$\chi(\zeta, \zeta', \zeta'') = \langle [\zeta, \zeta'], \zeta'' \rangle, \quad \zeta, \zeta', \zeta'' \in \mathfrak{n},$$

as well as a *cobracket*

$$\partial\colon \mathfrak{g} \to \wedge^2\mathfrak{g}, \quad \partial(\xi)(\zeta, \zeta') = \langle [\zeta, \zeta'], \xi \rangle, \quad \zeta, \zeta' \in \mathfrak{n}, \ \xi \in \mathfrak{g}.$$

Here $\mathfrak{g}$ is identified with $\mathfrak{n}^*$. Note that χ measures the failure of $\mathfrak{n}$ to define a Lie subalgebra of $\mathfrak{q}$. A *quasi-Poisson* $\mathfrak{g}$*-space* [2] for the quasi-Manin triple $(\mathfrak{q}, \mathfrak{g}, \mathfrak{n})_\gamma$ is a manifold M with an action $\varrho_M\colon \mathfrak{g} \to \mathfrak{X}(M)$ and a bivector field $\pi_M \in \mathfrak{X}^2(M)$ satisfying

$$\tfrac{1}{2}[\pi_M, \pi_M] = \varrho_M(\chi), \qquad \mathcal{L}_{\varrho_M(\xi)}\pi_M = \varrho_M(\partial\xi), \quad \xi \in \mathfrak{g}. \tag{26}$$

As shown in [5], this definition is equivalent to a morphism of Manin pairs,

$$K\colon (\mathbb{T}M, TM) \dashrightarrow (\mathfrak{q}, \mathfrak{g}). \tag{27}$$

Here K determines the $\mathfrak{g}$-action $\varrho_M\colon \mathfrak{g} \to \mathfrak{X}(M)$ by the condition $\varrho_M(\xi) \sim_K \xi, \ \xi \in \mathfrak{g}$, and the bivector field π_M on M is described in terms of its graph as $\mathrm{Gr}(\pi_M) = \mathfrak{n} \circ K \subseteq \mathbb{T}M$, the 'backward image' of $\mathfrak{n}$. More generally, any morphism of Manin pairs $R\colon (\mathbb{A}, E) \dashrightarrow (\mathfrak{q}, \mathfrak{g})$ determines a quasi-Poisson structure on M, by taking its composition with the morphism $(\mathbb{T}M, TM) \dashrightarrow (\mathbb{A}, E)$ from Example 1.6. Here the $\mathfrak{g}$-action is $\varrho_M(\xi) = \mathsf{a}(e(\xi))$ where $\mathfrak{g} \to \Gamma(E), \ \xi \mapsto e(\xi)$ is defined by the condition $e(\xi) \sim_R \xi$. The bi-vector field π_M is determined by the splitting $\mathbb{A} = E \oplus F$, and is locally given by the formula $\pi_M = \frac{1}{2}\mathsf{a}(e_i) \wedge \mathsf{a}(f^i)$ where e_i, f^j are sections of E, F with $\langle e_i, f^j \rangle = \delta^i_j$. (See e.g. [19, Theorem 3.16]).

6.2. Quasi-Poisson structures from Dirac Lie groups. Let $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ be an H-equivariant Dirac Manin triple, and let $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ be the Dirac Manin triple constructed from it. The following standard procedure turns the Lie algebra complement $\mathfrak{r}$ into a *Lagrangian* complement. As before we denote by $p_\mathfrak{r} \in \mathrm{End}(\mathfrak{q})$ the projection to $\mathfrak{r}$ along $\mathfrak{g}$, so that $1 - p_\mathfrak{r}$ and $p_\mathfrak{r}^*$ are the projections to $\mathfrak{g}$ along $\mathfrak{r}, \mathfrak{r}^\perp$, respectively. Their average $\frac{1}{2}((1-p_\mathfrak{r}) + p_\mathfrak{r}^*)$ is again a projection to $\mathfrak{g}$, and its kernel $\mathfrak{n}$ is the desired Lagrangian complement. Thus $\mathfrak{n}$ is the mid-point between $\mathfrak{r}, \mathfrak{r}^\perp$ in the affine space of complements to $\mathfrak{g}$. If ϵ_i is a basis of $\mathfrak{g}$, and ϕ^i a basis of $\mathfrak{r}^\perp$ with $\langle \epsilon_i, \phi^j \rangle = \delta^j_i$, the space $\mathfrak{n}$ has basis

$$\nu^i = \phi^i - \tfrac{1}{2}\sum_j \langle \phi^i, \phi^j \rangle \epsilon_j.$$

Note that the 'r-matrix'

$$\tfrac{1}{2}\sum_i \epsilon_i \wedge \nu^i = \tfrac{1}{2}\sum_i \epsilon_i \wedge \phi^i \in \wedge^2\mathfrak{q}$$

is independent of the choice of basis. Letting $\epsilon^i \in \mathfrak{g}^*$ be the dual basis, we have $\phi^i = f^*(\epsilon^i)$, and using $\langle f^*(\epsilon^i), f^*(\epsilon^j)\rangle = \beta(\epsilon^i, \epsilon^j)$ we obtain

$$\nu^i = f^*(\epsilon^i) - \tfrac{1}{2}\sum_j \beta(\epsilon^i, \epsilon^j)\epsilon_j. \tag{28}$$

The 'trivializing morphism' $T\colon (\mathbb{A}, E) \dashrightarrow (\mathfrak{q}, \mathfrak{g})$ gives H the structure of a quasi-Poisson space for the quasi-Manin triple $(\mathfrak{q}, \mathfrak{g}, \mathfrak{n})$. In terms of the trivialization $\mathbb{A} = H \times \mathfrak{q}$, the Lagrangian complement $F = \mathfrak{n} \circ T$ is simply the trivial bundle $H \times \mathfrak{n}$.

Proposition 6.1. *In the affine space of Lagrangian complements to E in $\mathbb{A}$, the sub-bundle F is the mid-point between* $\ker(s)$ *and* $\ker(t)$. *One has,*

$$F = \{x \in \mathbb{A}|\ h \bullet s(x) + t(x) = 0\} \tag{29}$$

where $h \in H$ indicates the base point of x.

Note that E is similarly given by a condition $h \bullet s(x) - t(x) = 0$.

Proof. The first claim follows since the trivialization of $\mathbb{A}$ restricts to isomorphisms $E \cong H \times \mathfrak{g}$, $\ker(t) \cong H \times \mathfrak{r}$ and $\ker(s) \cong H \times \mathfrak{r}^\perp$. The second part follows from the characterization of $\mathfrak{n}$ as the kernel of $\frac{1}{2}((1-p_\mathfrak{r}) + p_\mathfrak{r}^*)$, since $h \bullet s(h,\zeta) = h \bullet p_\mathfrak{r}^*(\zeta)$ and $t(h,\zeta) = h \bullet (1-p_\mathfrak{r})(\zeta)$ in the trivialization. ∎

Since a is given on constant sections of $\mathbb{A} = H \times \mathfrak{q}$ by the action map ϱ, we obtain:

Proposition 6.2. *For any Dirac Lie group structure $(\mathbb{A}, E)$ on H, with corresponding $\mathfrak{q}$-action $\varrho\colon \mathfrak{q} \to \mathfrak{X}(H)$, one obtains a quasi-Poisson structure on H, with bivector field*

$$\pi_H = \tfrac{1}{2}\sum_i \varrho(\epsilon_i) \wedge \varrho(f^*(\epsilon^i)).$$

and $\mathfrak{g}$-action $\varrho_H = \varrho|_\mathfrak{g}$.

6.3. Multiplicative properties. We next consider the multiplicative aspects of the quasi-Poisson structure. The composition of morphisms

$$(\mathbb{A}, E) \times (\mathbb{A}, E) \dashrightarrow (\mathbb{A}, E) \dashrightarrow (\mathfrak{q}, \mathfrak{g})$$

gives $H \times H$ the structure of a quasi-Poisson $\mathfrak{g}$-space $(H \times H, \pi_{H\times H})$, with the property that the underlying map $\mathrm{Mult}_H\colon H \times H \to H$ is a morphism of quasi-Poisson manifolds. The $\mathfrak{g}$-action $\varrho_{H\times H}$ is computed as follows. Using the trivialization $E = H \times \mathfrak{g}$, the equality $(h_1, \xi_1) \circ (h_2, \xi_2) = (h_1 h_2, \xi)$ holds if and only if $\xi_2 = \xi,\ \xi_1 = h_2 \bullet \xi$. Thus

$$\varrho_{H\times H}(\xi)_{(h_1,h_2)} = (\varrho_H(h_2 \bullet \xi)_{h_1}, \varrho_H(\xi)_{h_2}).$$

Proposition B.2 confirms that the multiplication in H is equivariant for this twisted action. (More generally this holds true for any matched pair between a Lie group and a Lie algebra.)

The bivector field $\pi_{H\times H}$ is determined by the splitting $(E \times E) \oplus F'$, where F' is the backward image $F' = \mathfrak{n} \circ (T \circ \mathrm{Mult}_\mathbb{A}) = F \circ \mathrm{Mult}_\mathbb{A} = \{(x_1, x_2) \in \mathbb{A} \times \mathbb{A}|\ x_1 \circ x_2 \in F\}$. Thus

$$F' = \{(x_1, x_2) \in \mathbb{A} \times \mathbb{A}|\ s(x_1) = t(x_2),\quad h_1 h_2 \bullet s(x_2) + t(x_1) = 0\}.$$

Since F' and $F \times F$ are both Lagrangian complements to $E \times E$, there is a unique section $\lambda \in \Gamma(\wedge^2(E \times E))$ with the property that

$$F' = (\mathrm{id} + \lambda^\sharp)(F \times F),$$

where $\lambda^\sharp \colon \mathbb{A} \times \mathbb{A} \to \mathbb{A} \times \mathbb{A}$ is the bundle map defined by λ, and

$$\pi_{H \times H} = \pi_H^{(1)} + \pi_H^{(2)} + \mathsf{a}(\lambda)$$

where $\pi_H^{(i)}$, $i = 1, 2$ is the bivector field π_H on the i-th factor of $H \times H$, and a is the anchor map for $\mathbb{A} \times \mathbb{A}$. (See [1, Proposition 1.18].) It remains to compute λ.

Proposition 6.3. *The section $\lambda \in \Gamma(\wedge^2(E \times E))$ is given in terms of the trivialization $E = H \times \mathfrak{g}$ as*

$$\lambda = -\tfrac{1}{2} \sum_{ij} \beta(\epsilon^i, \epsilon^j)\ (\epsilon_i, 0) \wedge (0, h_2^{-1} \bullet \epsilon_j).$$

Thus,

$$\mathsf{a}(\lambda) = -\tfrac{1}{2} \sum_{ij} \beta(\epsilon^i, \mathrm{Ad}_{h_2} \epsilon^j)\ \varrho_H(\epsilon_i)^{(1)} \wedge \varrho_H(\epsilon_j)^{(2)} \in \mathfrak{X}(H \times H),$$

where the superscripts (1), (2) *indicate the vector fields operating on the first resp. second H-factor.*

Proof. We will use the trivialization $\mathbb{A} = H \times \mathfrak{q}$, and omit base points to simplify notation. For all $(\tau_1, \tau_2) \in F \times F$ at a given base point h_1, h_2, there is a unique element $(\xi_1, \xi_2) \in E \times E$ such that $(\tau_1 + \xi_1, \tau_2 + \xi_2) \in F'$. Thus, $(\tau_1 + \xi_1) \circ (\tau_2 + \xi_2) \in F$, i.e.

$$s(\tau_1 + \xi_1) = t(\tau_2 + \xi_2), \quad t(\tau_1 + \xi_1) = -h_1 h_2 \bullet s(\tau_2 + \xi_2)$$

Using $t(\xi_i) = h_i \bullet s(\xi_i)$ and $t(\tau_i) = -h_i \bullet s(\tau_i)$, and solving for $\xi_i = s(\xi_i)$, we find

$$\xi_1 = -h_2 \bullet s(\tau_2), \quad \xi_2 = h_2^{-1} \bullet s(\tau_1).$$

This shows that λ is of the form $\lambda = \sum_i (\epsilon_i, 0) \wedge (0, s^i)$ for some $s^i \in \mathfrak{g}$ (depending on h_1, h_2). Taking $\tau_2 = 0$ and $\tau_1 = \nu^i$ the basis element of $\mathfrak{n}$, we find

$$(0, s^i) = \lambda^\sharp(\nu^i, 0) = (0, h_2^{-1} \bullet s(\nu^i)) = -\tfrac{1}{2} \sum_l \beta(\epsilon^i, \epsilon^j)(0, h_2^{-1} \bullet \epsilon_j).$$

∎

Example 6.4. Let us specialize the formulas to the Cartan-Dirac structure from Section 5.2.2, given by the G-invariant Dirac Manin triple $(\overline{\mathfrak{g}} \oplus \mathfrak{g}, \mathfrak{g}_\Delta, 0 \oplus \mathfrak{g})_\beta$. In this case $\mathfrak{q} = \mathfrak{d}$, $\mathfrak{r} = \mathfrak{h}$. We have $\mathfrak{r}^\perp = \mathfrak{g} \oplus 0$, and $\mathfrak{n} = \{(-\xi, \xi) | \xi \in \mathfrak{g}\}$ is the anti-diagonal. Letting e_i be a basis of $\mathfrak{g}$, with B-dual basis e^i, the corresponding basis of $\mathfrak{g}_\Delta$ is $\epsilon_i = (e_i, e_i)$, hence $f^*(\epsilon^i) = (-e^i, 0) \in \mathfrak{r}^\perp$, and the dual basis of $\mathfrak{n}$ is $\nu^i = \frac{1}{2}(-e^i, e^i)$. The resulting bivector field on G is

$$\pi_G = \tfrac{1}{2} \sum_i \varrho(e_i, e_i) \wedge \varrho(-e^i, 0) = \tfrac{1}{2} \sum_i ((e_i)^L - (e_i)^R) \wedge (e^i)^R = \tfrac{1}{2} \sum_i (e_i)^L \wedge (e^i)^R.$$

Since the action $\bullet$ is trivial, and $\beta(\epsilon^i, \epsilon^j) = -B(e^i, e^j)$, the section $\lambda \in \Gamma(\wedge^2(E \times E))$ is given by the formula $\lambda = \frac{1}{2} \sum_i (e_i, e_i)^{(1)} \wedge (e^i, e^i)^{(2)}$.

7. Exact Dirac Lie groups. A Dirac Lie group structure $(\mathbb{A}, E)$ on H is called *exact* if the underlying Courant algebroid $\mathbb{A}$ is exact (cf. Section 1). In this case, $\mathbb{A}$ has a distinguished isotropic splitting, giving an identification $\mathbb{A} \cong \mathbb{T}H_\eta$ for a suitable closed 3-form $\eta \in \Omega^3(H)$. Exact Dirac Lie group structures have the following characterization in terms of the associated Dirac Manin triples.

PROPOSITION 7.1. *Let $(\mathbb{A}, E)$ be a Dirac Lie group structure on H, with corresponding Dirac Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$. Then the Dirac Lie group structure is exact if and only if β is non-degenerate and $\mathfrak{g}$ is Lagrangian with respect to β.*

Proof. Let $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ be the Dirac Manin triple constructed from $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$.

$\Rightarrow$. Suppose $\mathbb{A}$ is exact. Then $\mathsf{a}_e\colon \mathfrak{q} = \mathbb{A}_e \to \mathfrak{h} = T_eH$ is surjective, with kernel $\mathfrak{g}$. It follows that a_e restricts to an isomorphism $\mathfrak{r} \to \mathfrak{h}$; hence $f\colon \mathfrak{q} \to \mathfrak{d}$ is an isomorphism. Since $f(\gamma) = \beta$, we conclude that β is non-degenerate and $\mathfrak{g}$ is Lagrangian.

$\Leftarrow$. If β is non-degenerate and $\mathfrak{g}$ is Lagrangian, then (cf. Example 3.7) the map $f\colon \mathfrak{q} \to \mathfrak{d}$ gives an isomorphism $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma \cong (\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$. Hence the action $\varrho^{\mathfrak{q}} = \varrho \circ f$ of $\mathfrak{q}$ on H is transitive. Hence $\mathsf{a}\colon \mathbb{A} \to TH$ is surjective. By dimension count its kernel is $\mathsf{a}^*(T^*H)$. $\square$

For the remainder of this section, we assume $(\mathbb{A}, E)$ is an exact Dirac Lie group structure, so that β is non-degenerate and $\mathfrak{g}$ is Lagrangian in $\mathfrak{d}$. Let $\langle\cdot,\cdot\rangle$ denote the bilinear form on $\mathfrak{d}$ dual to β. Using the isomorphism $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma \cong (\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ we will write $\mathfrak{d}, \mathfrak{h}, \beta$ in place of $\mathfrak{q}, \mathfrak{r}, \gamma$ and we omit the letter f. We will write $p \in \mathrm{End}(\mathfrak{d})$ for the projection to $\mathfrak{h}$ along $\mathfrak{g}$.

As in [37], we consider the exact Courant algebroid $\mathbb{T}H_\eta$, where η is the closed bi-invariant *Cartan 3-form*

$$\eta = \tfrac{1}{12}\langle\theta^R, [\theta^R, \theta^R]\rangle.$$

$\mathbb{T}H_\eta$ is a multiplicative Courant algebroid [1], where the multiplication is defined by the twisted Courant morphism

$$R_{\mathrm{Mult}_H, \sigma}\colon \mathbb{T}H_\eta \times \mathbb{T}H_\eta \dashrightarrow \mathbb{T}H_\eta$$

over the graph of the multiplication $\mathrm{Mult}_H : H \times H \to H$. Here $\sigma \in \Omega^2(H \times H)$ is the 2-form

$$\sigma = -\tfrac{1}{2}\langle \mathrm{pr}_1^*\,\theta^L, \mathrm{pr}_2^*\,\theta^R\rangle, \tag{30}$$

with $\mathrm{pr}_1, \mathrm{pr}_2\colon H \times H \to H$ the two projections.

THEOREM 7.2. *Let $(\mathbb{A}, E)$ be an exact Dirac Lie group structure on H.*

a. The action of $H \times H$ on H, given by $(h_1, h_2).h = h_1 h h_2^{-1}$ lifts canonically to a compatible action on the multiplicative Courant algebroid $\mathbb{A}$.

b. There exists a canonical $H \times H$-equivariant Lagrangian splitting $\mathsf{l}\colon TH \to \mathbb{A}$ inducing an $H \times H$-equivariant isomorphism of multiplicative Courant algebroids $\mathbb{A} \xrightarrow{\cong} \mathbb{T}H_\eta$.

Proof. Regard $\mathbb{A}$ as the reduction $C/C^\perp$ of $\mathbb{T}H \times (\overline{\mathfrak{d}} \oplus \mathfrak{d})$, as in Section 3.3. The lift of the $H \times H$-action to $\mathbb{T}H$, given as the direct sum of the tangent and cotangent lifts, is by Courant algebroid automorphisms, and is compatible with the groupoid structure $\mathbb{T}H \rightrightarrows \mathfrak{h}^*$. Similarly the $H \times H$-action on $\overline{\mathfrak{d}} \oplus \mathfrak{d}$ is by Courant algebroid automorphisms, and is compatible with the pair groupoid structure $\overline{\mathfrak{d}} \oplus \mathfrak{d} \rightrightarrows \mathfrak{d}$. The sub-bundle $C \subset \mathbb{T}H \times (\overline{\mathfrak{d}} \oplus \mathfrak{d})$ given by (11) is invariant under this action, hence so is $C^\perp$, and we obtain an induced action on $\mathbb{A} = C/C^\perp$ by Courant algebroid automorphisms compatible with the groupoid structure. Let $\Pi\colon \mathbb{A} \to \mathbb{A}$ denote the projection to $\ker(t)$ along $\ker(\mathsf{a})$. Since $\ker(t)$ and $\ker(\mathsf{a})$ are $H \times H$-invariant sub-bundles, the projection Π is $H \times H$-equivariant. $1 - \Pi$, Π^* are projections to $\ker(\mathsf{a})$ with kernels $\ker(t)$, $\ker(s)$ respectively. The kernel of the projection $\frac{1}{2}((1-\Pi)+\Pi^*)$ is a Lagrangian sub-bundle complementary to $\ker(\mathsf{a})$, defining an $H \times H$-invariant isotropic splitting $\mathsf{l}\colon TH \to \mathbb{A}$ having this sub-bundle as its range. By computing the resulting 3-form using (1), one verifies that this splitting yields an isomorphism $\mathbb{A} \xrightarrow{\cong} \mathbb{T}H_\eta$ with the desired properties. ☐

Remark 7.3. At the group unit, Π coincides with the projection $p\colon \mathfrak{d} = \mathbb{A}_e \to \mathfrak{h} = \ker(t_e)$ along $\mathfrak{g} = \ker(\mathsf{a}_e)$, hence $\mathsf{l}(T_eH)$ is the subspace $\mathfrak{n} = \ker((1-p)+p^*)$.

Remark 7.4. Under the trivialization $\mathbb{A} = H \times \mathfrak{d}$, the action of $\{e\} \times H \subset H \times H$ is given by $(e, h_2).(h, \zeta) = (hh_2^{-1}, \mathrm{Ad}_{h_2}\zeta)$. On the other hand, the isomorphism $\mathbb{T}H_\eta \to \mathbb{A}$, $v + \alpha \mapsto \mathsf{l}(v) + \mathsf{a}^*(\alpha)$ is given by $v + \alpha \mapsto (h, \zeta)$, where

$$\zeta = \mathrm{Ad}_{h^{-1}}\big((1 - \tfrac{1}{2}p^*)\iota_v\theta^R_h + t(\alpha)\big).$$

On recovers v and α via $\iota_v\theta^R_h = p(\mathrm{Ad}_h\zeta)$ and $\alpha = \langle\theta^R_h, \frac{1}{2}(p^* + (1-p))(\mathrm{Ad}_h\zeta)\rangle$.

Appendix A. Composition of relations. For more details on the theory summarized in this section, with particular emphasis on the symplectic setting, see Guillemin-Sternberg [12].

A (linear) *relation* $R\colon V_1 \dashrightarrow V_2$ between vector spaces V_1, V_2 is a subspace $R \subseteq V_2 \times V_1$. Write $v_1 \sim_R v_2$ if $(v_2, v_1) \in R$. Any linear map $A\colon V_1 \to V_2$ defines a relation $\mathrm{gr}(A)$. In particular, the identity map of V defines the diagonal relation $\mathrm{gr}(\mathrm{id}_V) = V_\Delta \subseteq V \times V$.

The *transpose relation* $R^\top\colon V_2 \to V_1$ consists of all (v_1, v_2) such that $(v_2, v_1) \in R$. We define

$$\ker(R) = \{v_1 \in V_1 |\ v_1 \sim 0\}, \quad \mathrm{ran}(R) = \{v_2 \in V_2 |\ \exists v_1 \in V_1\colon (v_2, v_1) \in R\}$$

Given another relation $R'\colon V_2 \dashrightarrow V_3$, the composition $R' \circ R\colon V_1 \dashrightarrow V_3$ consists of all (v_3, v_1) such that $v_1 \sim_R v_2$ and $v_2 \sim_{R'} v_3$ for some $v_2 \in V_2$.

We let $\mathrm{ann}^\natural(R)\colon V_1^* \to V_2^*$ be the relation such that $\mu_1 \sim_{\mathrm{ann}^\natural(R)} \mu_2$ if $\langle\mu_1, v_1\rangle = \langle\mu_2, v_2\rangle$ whenever $v_1 \sim_R v_2$. Thus $(\mu_2, \mu_1) \in \mathrm{ann}^\natural(R) \Leftrightarrow (\mu_2, -\mu_1) \in \mathrm{ann}(R)$. Note $\mathrm{ann}^\natural(V_\Delta) = (V^*)_\Delta$, and more generally

$$\mathrm{ann}^\natural(\mathrm{gr}(A)) = \mathrm{gr}(A^*)^\top \tag{31}$$

for linear maps $A\colon V_1 \to V_2$. Suppose W_1, W_2 are vector spaces with non-degenerate symmetric bilinear forms. A relation $L\colon W_1 \dashrightarrow W_2$ is called *Lagrangian* if $L \subseteq W_2 \times \overline{W_1}$ is a Lagrangian subspace, where $\overline{W_1}$ indicates W_1 with the opposite bilinear form.

Lemma A.1. *If $L\colon W_1 \dashrightarrow W_2$ and $L'\colon W_2 \dashrightarrow W_3$ are Lagrangian relations, then $L' \circ L\colon W_1 \dashrightarrow W_3$ is a Lagrangian relation.*

The analogous result for symplectic vector spaces is proved in detail in [12]; this proof carries over to Lagrangian spaces for vector spaces with split bilinear form.

LEMMA A.2. *For any relations $R\colon V_1 \to V_2$ and $R'\colon V_2 \to V_3$, one has* $\mathrm{ann}^\natural(R' \circ R) = \mathrm{ann}^\natural(R') \circ \mathrm{ann}^\natural(R)$.

Proof. Let $W_i = V_i \oplus V_i^*$ with the metric given by the pairing $\langle (v,\alpha),(v',\alpha')\rangle = \langle \alpha, v'\rangle + \langle \alpha', v\rangle$. By Lemma A.1, the composition of Lagrangian relations

$$(R' \oplus \mathrm{ann}^\natural(R')) \circ (R \oplus \mathrm{ann}^\natural(R)) = (R' \circ R) \oplus (\mathrm{ann}^\natural(R') \circ \mathrm{ann}^\natural(R)).$$

is again a Lagrangian relation. This means that $\mathrm{ann}^\natural$ of the first summand is equal to the second summand. □

The composition $R' \circ R$ can be regarded as the image of

$$R' \diamond R := (R' \times R) \cap (V_3 \times (V_2)_\Delta \times V_1)$$

under the projection to $V_3 \times V_1$.

LEMMA A.3.

$$\begin{aligned}\dim(R' \diamond R) &= \dim R' + \dim R - \dim V_2 \\ &\qquad + \dim(\ker(\mathrm{ann}^\natural(R')) \cap \ker(\mathrm{ann}^\natural(R)^\top)), \\ \dim(R' \circ R) &= \dim(R' \diamond R) - \dim(\ker(R') \cap \ker(R^\top)).\end{aligned}$$

Proof. The codimension of $(R' \times R) + (V_3 \times (V_2)_\Delta \times V_1)$ equals the dimension of its annihilator. It is thus equal to the dimension of

$$(\mathrm{ann}^\natural(R') \times \mathrm{ann}^\natural(R)) \cap (0 \times (V_2^*)_\Delta \times 0) \cong \ker(\mathrm{ann}^\natural(R')) \cap \ker(\mathrm{ann}^\natural(R)^\top).$$

This gives the formula for $\dim(R' \diamond R)$. On the other hand, the projection $R' \diamond R \to R' \circ R$ has kernel the intersection $(R' \times R) \cap (0 \times (V_2)_\Delta \times 0) \cong \ker(R') \cap \ker(R^\top)$. □

The composition of linear relations R, R' is called *transverse* if

$$\ker(R') \cap \ker(R^\top) = 0, \qquad \ker(\mathrm{ann}^\natural(R')) \cap \ker(\mathrm{ann}^\natural(R)^\top) = 0.$$

The first condition is equivalent to the claim that for $(v_3, v_1) \in R' \circ R$, there is a unique $v_2 \in V_2$ such that $(v_3, v_2) \in R'$ and $(v_2, v_1) \in R$. The second condition is equivalent to the transversality of $R' \times R$ with $V_3 \times (V_2)_\Delta \times V_1$. For transverse compositions, $R' \circ R$ varies smoothly with R', R. Either of the two conditions in the transversality condition can be replaced with the dimension formula $\dim(R' \circ R) = \dim(R') + \dim(R) - \dim V_2$. For Lagrangian relations, the dimension formula is automatic.

More generally, consider (non-linear) relations between manifolds. Here, 'clean composition' hypotheses are needed. Recall that the intersection of submanifolds $S_1, S_2 \subseteq M$ is *clean* (in the sense of Bott) if $S_1 \cap S_2$ is a submanifold, and $T(S_1 \cap S_2) = TS_1 \cap TS_2$. Equivalently, the intersection is clean if at all points $x \in S_1 \cap S_2$, there are local coordinates in which both S_1, S_2 are given as subspaces [14, page 491]. We say that the composition $R' \circ R$ of submanifolds $R \subseteq M_2 \times M_1$ and $R' \subseteq M_3 \times M_2$ is *clean* if

$$R' \diamond R = (R' \times R) \cap (M_3 \times (M_2)_\Delta \times M_1) \tag{32}$$

is a clean intersection, and the map $R' \diamond R \to M_3 \times M_1$ (forgetting the M_2-component) has constant rank. Thus $R' \circ R$ is an (immersed) submanifold, and the map $R' \diamond R \to R' \circ R$ is a submersion.

The composition is called transverse if the composition of tangent spaces is transverse everywhere. In this case $R' \diamond R$ is a smooth submanifold of dimension $\dim R' + \dim R - \dim M_2$, and the map $R' \diamond R \to R' \circ R$ is a covering.

Appendix B. Matched pairs and $\mathcal{LA}$-groupoids. A $\mathcal{VB}$-groupoid V over H is *vacant* if $V^{(0)} = V|_{H^{(0)}}$. Equivalently, the source map is a fiberwise isomorphism. In [22, 23], Mackenzie interpreted a vacant $\mathcal{LA}$-groupoid $E \rightrightarrows \mathfrak{g}$ over a group $H \rightrightarrows \mathrm{pt}$ as a matched pair between a Lie algebra $\mathfrak{g}$ and a Lie group H.[2] In this Section we review and elaborate these results, proving Proposition 5.1, in particular.

LEMMA B.1. *Suppose $E \rightrightarrows \mathfrak{g}$ is an $\mathcal{LA}$-groupoid over $H \rightrightarrows \mathrm{pt}$. Then the map*

$$(\mathsf{a}, t, s)\colon E \to TH \times (\mathfrak{g} \oplus \mathfrak{g}),\ \ x \mapsto (\mathsf{a}(x), t(x), s(x))$$

is a homomorphism of $\mathcal{LA}$-groupoids. If E is vacant, then (a, t, s) is an embedding as a subbundle.

Proof. Since $\mathsf{a}\colon E \to TH,\ s\colon E \to \mathfrak{g},\ t\colon E \to \mathfrak{g}$ are morphisms of Lie algebroids, the map (a, t, s) is one also. The equality

$$\begin{aligned}(\mathsf{a}(x_1 \circ x_2), t(x_1 \circ x_2), s(x_1 \circ x_2)) &= (\mathsf{a}(x_1) \circ \mathsf{a}(x_2),\ t(x_1), s(x_2))\\ &= (\mathsf{a}(x_1), t(x_1), s(x_1)) \circ (\mathsf{a}(x_2), t(x_2), s(x_2))\end{aligned}$$

for $s(x_1) = t(x_2)$ shows that (a, t, s) is a $\mathcal{VB}$-groupoid homomorphism. If E is vacant, so that t is a fiberwise isomorphism, the map $E \to H \times \mathfrak{g}$ taking $x \in E_h$ to $(h, t(x))$ is an isomorphism. In particular, (a, t, s) is an embedding as a subbundle. ∎

PROPOSITION B.2. *Let $E \rightrightarrows \mathfrak{g}$ be a vacant $\mathcal{LA}$-groupoid. For any $\xi \in \mathfrak{g}$ and $h \in H$, let $h^{-1} \bullet \xi \in \mathfrak{g}$ and $\varrho(\xi)_h \in T_h H$ be defined by the condition that there exist $x \in E_h$ with*

$$(\mathsf{a}(x),\ t(x),\ s(x)) = (\varrho(\xi)_h,\ h \bullet \xi,\ \xi). \tag{33}$$

Then the map $(h, \xi) \mapsto h \bullet \xi$ defines an action of H on $\mathfrak{g}$, while $\varrho\colon \mathfrak{g} \to \mathfrak{X}(H)$ is an action of $\mathfrak{g}$ on H. These actions satisfy the compatibility conditions,

$$h \bullet [\xi_1, \xi_2] = [h \bullet \xi_1,\ h \bullet \xi_2] + \mathcal{L}_{\varrho(\xi_1)}(h \bullet \xi_2) - \mathcal{L}_{\varrho(\xi_2)}(h \bullet \xi_1) \tag{34}$$

and

$$\varrho(\xi)_{h_1 h_2} = (\mathrm{Mult}_H)_*(\varrho(h_2 \bullet \xi)_{h_1}, \varrho(\xi)_{h_2}). \tag{35}$$

Conversely, given a pair of actions of H on $\mathfrak{g}$ and of $\mathfrak{g}$ on H, satisfying (34) *and* (35)*, the span of the sections $\alpha(\xi)$ is a vacant $\mathcal{LA}$-subgroupoid.*

Proof. For $h \in H,\ \xi \in \mathfrak{g}$ let $\alpha(\xi)_h = (\varrho(\xi)_h,\ h \bullet \xi,\ \xi)$ be the right hand side of (33). Since the image of E under (a, t, s) is a subgroupoid, we have

$$\alpha(h_2 \bullet \xi)_{h_1} \circ \alpha(\xi)_{h_2} = \alpha(\xi)_{h_1 h_2}.$$

Applying a, s to this identity gives (35) and the action property $(h_1 h_2) \bullet \xi = h_1 \bullet h_2 \bullet \xi$. On the other hand, since E is a Lie subalgebroid, $[\alpha(\xi_1), \alpha(\xi_2)] = \alpha([\xi_1, \xi_2])$.

[2]Note that in [22], Mackenzie uses the terminology of an *interaction* rather than a matched pair.

Application of s, a gives (34) and the property $[\varrho(\xi_1), \varrho(\xi_2)] = \varrho([\xi_1, \xi_2])$. Conversely, given actions ϱ and $\bullet$ satisfying (34) and (35), let E be the subbundle of $TH \times (\mathfrak{g} \oplus \mathfrak{g})$ spanned by the sections $\alpha(\xi)$. The compatibility conditions guarantee that it is a $\mathcal{VB}$-subgroupoid and also a Lie subalgebroid. □

We note that (34) and (35) are exactly the compatibility conditions for a *matched pair* $\mathfrak{g} \bowtie H$ *between a Lie group and a Lie algebra* as given in [22]. Therefore Proposition B.2 can be interpreted as proving a 1-1 correspondence between such matched pairs and vacant $\mathcal{LA}$-groupoids over a Lie group.

By differentiating the action of H on $\mathfrak{g}$, we obtain a linear representation of $\mathfrak{h}$ on $\mathfrak{g}$ (still denoted $\bullet$). Similarly, since $\varrho(\xi)_e = 0$, we may linearize the action of $\mathfrak{g}$ on H to obtain a linear representation $\dot{\varrho}$ of $\mathfrak{g}$ on $\mathfrak{h}$. Concretely,

$$\dot{\varrho}(\xi)(\tau) = [\varrho(\xi), \widetilde{\tau}]|_e$$

where $\widetilde{\tau} \in \mathfrak{X}(H)$ with $\widetilde{\tau}|_e = \tau$. By linearizing (34) and (35), one obtains the following conditions, for all $\xi, \xi_1, \xi_2 \in \mathfrak{g}$ and $\tau, \tau_1, \tau_2 \in \mathfrak{h}$:

$$\tau \bullet [\xi_1, \xi_2] = [\tau \bullet \xi_1, \xi_2] - [\tau \bullet \xi_2, \xi_1] - \dot{\varrho}(\xi_1)(\tau) \bullet \xi_2 + \dot{\varrho}(\xi_2)(\tau) \bullet \xi_1, \tag{36}$$

$$\dot{\varrho}(\xi)([\tau_1, \tau_2]) = [\dot{\varrho}(\xi)(\tau_1), \tau_2] - [\dot{\varrho}(\xi)(\tau_2), \tau_1] - \varrho(\tau_1 \bullet \xi)(\tau_2) + \varrho(\tau_2 \bullet \xi)(\tau_1). \tag{37}$$

These are exactly the compatibility conditions for a *matched pair* of Lie algebras, as studied in [26, 18]. The conditions are equivalent to the statement that $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ has a Lie bracket, with $\mathfrak{g}, \mathfrak{h}$ as Lie subalgebras and such that

$$[\xi, \tau] = \dot{\varrho}_\xi(\tau) - \tau \bullet \xi, \quad \xi \in \mathfrak{g}, \ \tau \in \mathfrak{h}. \tag{38}$$

Let $p = p_{\mathfrak{h}} \colon \mathfrak{d} \to \mathfrak{h}$ be the projection to the second summand, and put $q = 1 - p$.

Proposition B.3.

a. *The adjoint action* $\mathrm{Ad}\colon H \to \mathrm{End}(\mathfrak{h})$ *admits a unique extension* $\mathrm{Ad}\colon H \to \mathrm{End}(\mathfrak{d})$ *with the property*

$$\mathrm{Ad}_h \xi = h \bullet \xi + \iota(\varrho(\xi))\theta^R_h \tag{39}$$

for all $h \in H$, $\xi \in \mathfrak{g}$. *Its derivative is the adjoint action of* $\mathfrak{h}$ *on* $\mathfrak{d}$.

b. *The action of* $\mathfrak{g}$ *on* H *combines with the* $\mathfrak{h}$*-action* $\nu \mapsto \nu^L$ *to an action of the Lie algebra* $\mathfrak{d}$. *We have*

$$\iota(\varrho(\zeta))\theta^R_h = p(\mathrm{Ad}_h \zeta), \quad \zeta \in \mathfrak{d}.$$

c. *The action* Ad_h *on* $\mathfrak{d}$ *is a Lie algebra automorphism of* $\mathfrak{d}$.

Proof. The proof of these facts involves some elementary but tedious computations which we omit it for brevity. □

Proposition B.3 shows that a vacant $\mathcal{LA}$-groupoid over $H \rightrightarrows \mathrm{pt}$ determines an H-equivariant Lie algebra triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$, as in Definition 3.2. The converse was established in Proposition 3.3.

Using the H-action on $\mathfrak{d}$ we can now characterize E directly in terms of the H-equivariant triple.

PROPOSITION B.4. *Suppose $E \to H$ is a vacant $\mathcal{LA}$-groupoid over a group H, and define the H-equivariant triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ as explained above. Then*

$$(\mathsf{a}, t, s)(E) = \{(v, \xi, \xi') |\ v \in T_h H,\ \xi, \xi' \in \mathfrak{g},\ \mathrm{Ad}_h \xi' - \xi = \iota(v)\theta_h^R\}$$

Proof. The condition $\mathrm{Ad}_h \xi' - \xi = \iota(v)\theta_h^R$ is just (39), proving the inclusion $\subseteq$. The opposite inclusion follows by dimension count: Given $\xi' \in \mathfrak{g}$, the elements ξ, v are determined as $\xi = q(\mathrm{Ad}_h \xi'),\ \iota(v)(\theta_h^R) = p(\mathrm{Ad}_h \xi)$. ∎

The correspondence between $\mathcal{LA}$-groupoids and H-equivariant triples is compatible with morphisms. Suppose $E_i \rightrightarrows \mathfrak{g}_i,\ i = 0, 1$ are vacant $\mathcal{LA}$-groupoids over groups H_i. A *morphism* (resp. *comorphism*) from E_0 to E_1 is a Lie group homomorphism $\Phi\colon H_0 \to H_1$, together with a vector bundle map $E_0 \to E_1$ (resp. $\Phi^* E_1 \to E_0$) whose graph is an $\mathcal{LA}$-subgroupoid of $E_1 \times E_0$ along the graph of Φ. If E_i are vacant, so that $E_i|_e \cong \mathfrak{g}_i$, such a morphism (resp. comorphism) defines a pair of Lie algebra homomorphisms $\mathrm{d}_e\Phi\colon \mathfrak{h}_0 \to \mathfrak{h}_1$ and $\mathfrak{g}_0 \to \mathfrak{g}_1$ (resp. $\mathfrak{g}_1 \to \mathfrak{g}_0$). Let $\mathfrak{d}_i = \mathfrak{g}_i \oplus \mathfrak{h}_i$.

PROPOSITION B.5.

a. *If $E_0 \to E_1$ is a morphism of $\mathcal{LA}$-groupoids, then the linear map $\mathfrak{d}_0 \to \mathfrak{d}_1$ given as the direct sum of the maps $\mathfrak{g}_0 \to \mathfrak{g}_1$ and $\mathfrak{h}_0 \to \mathfrak{h}_1$ is a Lie algebra homomorphism, equivariant relative to the underlying group homomorphism $\Phi\colon H_0 \to H_1$.*
b. *If $\Phi^* E_1 \to E_0$ is a comorphism of $\mathcal{LA}$-groupoids, then the subspace $\mathfrak{r} \subseteq \mathfrak{d}_1 \times \mathfrak{d}_0$, given as the direct sum of the graphs of the maps $\mathfrak{g}_1 \to \mathfrak{g}_0$ and $\mathfrak{h}_0 \to \mathfrak{h}_1$, is a Lie subalgebra, invariant under the action of H_0 (via its inclusion as $\mathrm{gr}(\Phi) \subseteq H_1 \times H_0$).*

Proof.

a. The statement is obvious if $E_0 \to E_1$ is an inclusion. The general case reduces to that of an inclusion, by letting $E_1' = E_1 \times E_0,\ H_1' = H_1 \times H_0$, $\Phi'(h_0) = (\Phi(h_0), h_0)$, and with the inclusion $E_0 \to E_1'$ the direct sum of the identity map with the map $E_0 \to E_1$.
b. Similar to (a), let $E_0' \hookrightarrow E_1' = E_1 \times E_0$ be the inclusion of the graph of the map $\Phi^* E_1 \to E_0$. By (a), applied to inclusions one obtains an $H_0 \cong \mathrm{gr}(\Phi)$-equivariant Lie algebra homomorphism $\mathfrak{d}_0' := \mathfrak{g}_1 \times \mathfrak{h}_0 \hookrightarrow \mathfrak{d}_1' := \mathfrak{d}_1 \times \mathfrak{d}_0$. Its range is $\mathfrak{r}$, which is hence an H_0-invariant Lie subalgebra.

∎

Conversely, given an $H_0 \cong \mathrm{gr}(\Phi)$-invariant Lie subalgebra $\mathfrak{r} \subseteq \mathfrak{d}_1 \times \mathfrak{d}_0$ given as the direct sum of the graphs of $\mathrm{d}_e\Phi$ and the graph of a Lie algebra homomorphism $\mathfrak{g}_0 \to \mathfrak{g}_1$ (resp. $\mathfrak{g}_1 \to \mathfrak{g}_0$), the resulting $\mathcal{LA}$-subgroupoid of $E_1 \times E_0$ defines a morphism (resp. comorphism) of $\mathcal{LA}$-groupoids.

Appendix C. Some constructions with $\mathcal{VB}$-groupoids. We will base our discussion on the following result

PROPOSITION C.1. *[29, § 5.3] Suppose H, G, K are Lie groupoids, and $\phi\colon G \to K$ and $\psi\colon H \to K$ are morphisms of Lie groupoids. If ϕ and ψ are transverse, then the fibered product $H \times_K G$ is a Lie groupoid, with $H^{(0)} \times_{K^{(0)}} G^{(0)}$ as its space of units.*

REMARK C.2. Note that, [29, § 5.3] makes the additional assumption that the restrictions $\phi^{(0)} = \phi|_{G^{(0)}}$ and $\psi^{(0)} = \psi|_{H^{(0)}}$ to the units are transverse. However, this property is automatic: Suppose $x \in G^{(0)}$, $y \in H^{(0)}$ are units with $w := \phi(x) = \psi(y)$. Then

$$T_xG = \ker(T_xs) \oplus T_xG^0, \quad T_yH = \ker(T_ys) \oplus T_yH^0, \quad T_wK = \ker(T_ws) \oplus T_wK^0.$$

Since the tangent maps to ϕ, ψ respect these decompositions, their transversality implies that of $\phi^{(0)}$, $\psi^{(0)}$.

Corollary C.3. *Suppose $\phi\colon V \to W$ is a fiberwise surjective homomorphism of $\mathcal{VB}$-groupoids over $G \to H$. Then* $\ker(\phi)$ *is a $\mathcal{VB}$-subgroupoid of V.*

Proof. Since ϕ is fiberwise surjective, it is transverse to the zero section $H \to W$. We may view $\ker(\phi)$ as the fibered product $V \times_W H$, where $H \to W$ is the inclusion of the zero section. By Proposition C.1, it is a Lie groupoid. By Definition 2.1 it is a $\mathcal{VB}$-groupoid. ◻

For the next result, we recall Pradines' observation [33] (see also [24, § 11.2]) that the dual of a $\mathcal{VB}$-groupoid $V \to H$ has a natural structure of $\mathcal{VB}$-groupoid,

$$\begin{tikzcd} V^* \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathrm{ann}(V^{(0)}) \arrow[d] \\ H \arrow[r, shift left] \arrow[r, shift right] & H^{(0)} \end{tikzcd}$$

where $\mathrm{ann}(V^{(0)})$ is the annihilator of $V^{(0)}$ in $V^*|_{H^{(0)}}$. The groupoid structure is given by $\langle \alpha_1 \circ \alpha_2,\ v_1 \circ v_2\rangle = \langle \alpha_1, v_1\rangle + \langle \alpha_2, v_2\rangle$, for composable elements $\alpha_1, \alpha_2 \in V^*$ and $v_1, v_2 \in V^*$, with α_i having the same base points as v_i.

Alternatively, one can define the groupoid multiplication in terms of its graph by

$$\mathrm{gr}(\mathrm{Mult}_{V^*}) = \mathrm{ann}^\natural(\mathrm{gr}(\mathrm{Mult}_V)) \tag{40}$$

(using the notation from Appendix A). Writing the groupoid axioms in terms of compositions of relations, it then follows from the vector bundle version of Lemma A.2, that the $\mathcal{VB}$-groupoid axioms of V imply those for V^*.

Suppose now that $\Phi :: V \to W$ is a morphism of $\mathcal{VB}$-groupoids, i.e.

$$\mathrm{gr}(\Phi) \circ \mathrm{Mult}_V \subseteq \mathrm{Mult}_W \circ \mathrm{gr}(\Phi \times \Phi).$$

By application of Lemma A.2 and (40) one obtains the corresponding equation for $\Phi^*\colon W^* \to V^*$ holds. Thus we have proven [24, Proposition 11.2.6], that the dual bundle map $\Phi^*\colon W^* \to V^*$ is again a morphism of $\mathcal{VB}$-groupoids.

Corollary C.4. *Suppose $C \subseteq V$ is a $\mathcal{VB}$-subgroupoid over groupoids $K \subseteq H$. Then* $\mathrm{ann}(C) \subseteq V^*$ *is a $\mathcal{VB}$-subgroupoid. Its space of objects is* $\mathrm{ann}(C) \cap \mathrm{ann}(V^{(0)})$.

Proof. Let $i\colon K \hookrightarrow H$ be inclusion. By Proposition C.1, the pull-back $i^*V^* \to K$ is a $\mathcal{VB}$-groupoid. It comes with a fiberwise surjective Lie groupoid homomorphism $i^*V^* \to C^*$, where the map on units is again fiberwise surjective. Its kernel is $\mathrm{ann}(C)$. ◻

A non-degenerate fiber metric $\langle \cdot, \cdot \rangle$ on a $\mathcal{VB}$-groupoid V is *multiplicative* if it satisfies

$$\langle v_1 \circ v_2, v_1' \circ v_2' \rangle = \langle v_1, v_1' \rangle + \langle v_2, v_2' \rangle \tag{41}$$

for composable elements v_1, v_2, resp. v_1', v_2', with v_i having the same base points as v_i'. Equivalently, the graph $\mathrm{gr}(\mathrm{Mult}_V) \subset V \times \overline{V} \times \overline{V}$ is an isotropic subbundle, where $\overline{V}$ denotes V with the opposite fiber metric.

The fiber metric $\langle\cdot,\cdot\rangle$ defines a map $\Psi : V \to V^*$, and (41) shows that

$$\Psi\big(\mathrm{gr}(\mathrm{Mult}_V)\big) \subseteq \mathrm{gr}(\mathrm{Mult}_{V^*}) = \mathrm{ann}^\natural\left(\mathrm{gr}(\mathrm{Mult}_V)\right). \tag{42}$$

Since, in addition, the fiber metric $\langle\cdot,\cdot\rangle$ is non-degenerate, Ψ defines an isomorphism of $\mathcal{VB}$-groupoids. This shows that (42) is an equality and

$$\Psi(V^{(0)}) = (V^*)^{(0)} = \mathrm{ann}(V^{(0)}).$$

Therefore both $\mathrm{gr}(\mathrm{Mult}_V)$ and $V^{(0)}$ are Lagrangian.

Corollary C.5. *Let $V \to H$ be a $\mathcal{VB}$-groupoid, equipped with a multiplicative non-degenerate fiber metric. Let $C \rightrightarrows C \cap V^{(0)}$ be a co-isotropic $\mathcal{VB}$-subgroupoid. Then $C^\perp \rightrightarrows C^\perp \cap V^{(0)}$ is a $\mathcal{VB}$-subgroupoid of C, and hence the quotient inherits a $\mathcal{VB}$-groupoid structure $C/C^\perp \rightrightarrows (C \cap V^{(0)})/(C^\perp \cap V^{(0)})$. Moreover, the natural non-degenerate fiber metric on $C/C^\perp$ is multiplicative.*

Proof. The identification $V^* \cong V$ identifies $\mathrm{ann}(C) \cong C^\perp \subseteq C$. By the previous Corollary, this is a $\mathcal{VB}$-subgroupoid of V. Hence $C^\perp \to C$ is an inclusion of $\mathcal{VB}$-groupoids. Therefore, the dual morphism,

$$C^* \to (C^\perp)^*, \tag{43}$$

is a surjective submersion of $\mathcal{VB}$-groupoids. Thus, by Corollary C.3, the kernel $(C/C^\perp)^* \cong C/C^\perp$ of (43) has a natural $\mathcal{VB}$-groupoid structure. Finally, it is clear that the restriction of the fiber metric $\langle\cdot,\cdot\rangle$ to $C/C^\perp$ satisfies (41), since it does so for V. □

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