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SMALL FOUR-MANIFOLDS WITHOUT NON-SINGULAR SOLUTIONS OF NORMALIZED RICCI FLOWS*

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Abstract. It is known [6] that connected sums $X\#K3\#(\Sigma_g \times \Sigma_h)\#\ell_1(S^1 \times S^3)\#\ell_2\overline{\mathbb{C}P^2}$ satisfy the Gromov-Hitchin-Thorpe type inequality, but can not admit non-singular solutions of the normalized Ricci flow for any initial metric, where $\Sigma_g \times \Sigma_h$ is the product of two Riemann surfaces of odd genus, $\ell_1, \ell_2 > 0$ are sufficiently large positive integers, $g, h > 3$ are also sufficiently large positive odd integers, and X is a certain irreducible symplectic 4-manifold. These exmples are closely related with a conjecture of Fang, Zhang and Zhang [10]. In the current article, we point out that there still exist 4-manifolds with the same property even if $\ell_1 = \ell_2 = 0$ and $g = h = 3$. The topology of these new examples are smaller than that of previously known examples.

Key words. Four-manifold, Ricci flow, non-singular solution.

AMS subject classifications. 53C44, 57R57, 53C25.

1. Introduction.

Let X be a closed oriented Riemannian manifold of dimension $n \geq 3$. The normalized Ricci flow on X is the following evolution equation:

$$\frac{\partial}{\partial t}g = -2Ric_g + \frac{2}{n}\Big(\frac{\int_X s_g d\mu_g}{vol_g}\Big)g, \tag{1}$$

where Ric_g is the Ricci curvature of the evolving Riemannian metric g, s_g is the scalar curvature of the evolving Riemannian metric g, $vol_g := \int_X d\mu_g$ and $d\mu_g$ is the volume measure with respect to g. In [19], Hamilton introduced a nice class of solutions of (1), which is so called non-singular. Recall that a solution $\{g(t)\}$, $t \in [0, T)$, to (1) is called non-singular if $T = \infty$ and the Riemannian curvature tensor $Rm_{g(t)}$ of $g(t)$ satisfies $\sup_{X\times[0,T)} |Rm_{g(t)}| < \infty$. In particular, Hamilton [19] classified non-singular solutions to the normalized Ricci flow on a closed 3-manifold. After this pioneering work of Hamilton in dimension 3, Fang, Y.G. Zhang and Z.Z. Zhang [10] studied the properties of non-singular solutions to the normalized Ricci flow in higher dimensions. In the beautiful article [10], among other things, it was proved that the existence of the non-singular solution of the normalized Ricci flow forces a constraint on the Euler characteritic $\chi(X)$ and signature $\tau(X)$ of a given 4-manifold X. Based on this result, they proposed the following conjecture:

CONJECTURE 1 (Conjecture 1.8 in [10]). *Let X be a closed oriented smooth Riemannian 4-manifold with $||X|| \neq 0$ and $\overline{\lambda}(X) < 0$, where $||X||$ is the Gromov's simplicial volume of X and $\overline{\lambda}(X)$ is the Perelman's $\overline{\lambda}$ invariant. If there is a non-singular solution to the normalized Ricci flow on X, then the Gromov-Hitchin-Thorpe type inequality holds:*

$$2\chi(X) - 3|\tau(X)| \geq \frac{1}{1296\pi^2}||X||. \tag{2}$$

Here, the Perelman's $\overline{\lambda}$ invariant [23, 24] of X is a differential topological invariant defined by $\overline{\lambda}(X) = \sup_{g\in\mathcal{R}_X} \lambda_g(vol_g)^{2/n}$, where $\mathcal{R}_X$ is the space of all Riemmannian

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metrics on X, λ_g is the lowest eigenvalue of the elliptic operator $4\Delta_g + s_g$, and $\Delta = d^*d = -\nabla \cdot \nabla$ is the positive-spectrum Laplace-Beltrami operator associated with g. See also [20].

To the best of our knowledge, Conjecture 1 still remains open. However, in a joint work with Baykur [6], the present author has shown that the converse of Conjecture 1 dose not hold in general. In fact, for $\ell_1, \ell_2 > 0$ which are sufficiently large positive integers, and for $g, h > 3$ which are also sufficiently large positive odd integers, it was proved in [6] that a connected sum of type

$$M := X\#K3\#(\Sigma_g \times \Sigma_h)\#\ell_1(S^1 \times S^3)\#\ell_2\overline{\mathbb{C}P^2}$$

has the following properties, where X is a certain irreducible symplectic 4-manifold, $K3$ is the $K3$ surface, $\Sigma_g \times \Sigma_h$ is the product of two Riemann surfaces Σ_g, Σ_h of odd genus and $\overline{\mathbb{C}P^2}$ is the complex projective plane with the reversed orientation:

1. M has $||M|| \neq 0$ and satisfies the strict case of the inequality (2):

$$2\chi(M) - 3|\tau(M)| > \frac{1}{1296\pi^2}||M||.$$

2. M admits infinitely many distinct smooth structures for which Perelman's $\bar{\lambda}$ invariant is negative and there is no non-singular solution to the normalized Ricci flow for any initial metric.

In what follows, we call these properties 1 and 2 ∞-property $\mathcal{R}$ for simplicity. Notice that the existence of 4-manifolds with ∞-property $\mathcal{R}$ particularly implies that the converse of Conjecture 1 does not hold in general. Namely, this tells us that the existence and non-existence of non-singular solutions are not controled by the topological information like (2). Moreover, these also provide us new examples of 4-manifold without Einstein metrics because Einstein metric is an example of non-singular solution.

However, in the construction of these examples, we took sufficiently large integers ℓ_1, ℓ_2, g, h. Therefore, it is a natural question to ask whether there still exists a 4-manifold with ∞-property $\mathcal{R}$ for small ℓ_1, ℓ_2, g, h. The main purpose of the current article is to give a positive answer to this question. Namely, we shall prove that there still exist 4-manifolds with ∞-property $\mathcal{R}$ even if $\ell_1 = \ell_2 = 0$ and $g = h = 3$. In what follows, N_p denotes a 4-manifold with fundamental group $\mathbb{Z}_p$, p odd, which is obtained from the product $L(p,1) \times S^1$ of Lens space $L(p,1)$ and S^1 by performing a 0-surgery along $\{pt\} \times S^1$. The main result of the current article is as follows:

Theorem A. *For any positive integer* $0 \leq n \leq 7$*, there exists an irreducible symplectic 4-manifold* X_n *which is homeomoprhic to*

1. $4\mathbb{C}P^2\#(13+n)\overline{\mathbb{C}P^2}\#(S^1 \times S^3)$ *or*
2. $3\mathbb{C}P^2\#(12+n)\overline{\mathbb{C}P^2}\#N_p$ *or*
3. $3\mathbb{C}P^2\#(12+n)\overline{\mathbb{C}P^2}$

and a connected sum $X_n\#K3\#(\Sigma_3 \times \Sigma_3)$ *has* ∞*-property* $\mathcal{R}$.

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2. Preliminaries. In the following, for any closed 4-manifold X, $b^+(X)$ (resp. $b^-(X)$) denotes the dimension of a maximal linear subspace of $H^2(X, \mathbb{R})$ on which the cup product pairing is positive (resp. negative) definite. Notice that $b_2(X) = b^+(X) + b^-(X)$ and $\tau(X) = b^+(X) - b^-(X)$.

2.1. Non-vanishing theorem of BF_X. Let X be a closed smooth Riemannian 4-manifold X with $b^+(X) > 1$. Recall that a spin^c-structure $\mathfrak{s}$ on X induces a pair of spinor bundles $S^{\pm}$ which are Hermitian vector bundles of rank 2. A Riemannian metric on X and a unitary connection A on the determinant line bundle $\mathcal{L}$ induce the twisted Dirac operator $\mathcal{D}_A : \Gamma(S^+) \longrightarrow \Gamma(S^-)$. The Seiberg-Witten monopole equations [27] over X are the following non-linear partial differential equations for a unitary connection A of $\mathcal{L}$ and a spinor $\phi \in \Gamma(S^+)$:

$$\mathcal{D}_A\phi = 0, \ F_A^+ = iq(\phi),$$

here F_A^+ is the self-dual part of the curvature of A and $q : S^+ \to \wedge^+$ is a certain natural real-quadratic map, where $\wedge^+$ is the bundle of self-dual 2-forms. In what follows, we denote the first Chern class of the complex line bundle $\mathcal{L}$ associated with $\mathfrak{s}$ by $c_1(\mathfrak{s})$.

An element $\mathfrak{a} \in H^2(X, \mathbb{Z})/\text{torsion} \subset H^2(X, \mathbb{R})$ is called monopole class [21, 22] of X if there exists a spin^c-structure $\mathfrak{s}$ with $c_1^{\mathbb{R}}(\mathfrak{s}) = \mathfrak{a}$ which has the property that the corresponding Seiberg-Witten monopole equations have a solution for every Riemannian metric on X. Here $c_1^{\mathbb{R}}(\mathfrak{s})$ is the image of $c_1(\mathfrak{s})$ in $H^2(X, \mathbb{R})$. It is known [22, 16] that the set of all monopole classes of X is finite.

There are several ways to detect the existence of monopole classes. For any closed oriented smooth 4-manifold X with $b^+(X) > 1$, one can define the Seiberg-Witten invariant [27] for any spin^c-structure $\mathfrak{s}$ by integrating a cohomology class on the moduli space of solutions of the Seiberg-Witten monopole equations associated with $\mathfrak{s}$:

$$SW_X : Spin(X) \longrightarrow \mathbb{Z},$$

where $Spin(X)$ is the set of all spin^c-structures on X. We call the first Chern class $c_1(\mathfrak{s})$ Seiberg-Witten basic class of X if $SW_X(\mathfrak{s}) \neq 0$ for a spin^c-structure $\mathfrak{s}$. In particular, Seiberg-Witten basic classes are monopole classes. Moreover, there is a sophisticated refinement of the idea of the construction of the Seiberg-Witten invariant, which is due to Bauer and Furuta [3, 4, 5]. The invariant is called the stable cohomotopy Seiberg-Witten invariant and denote it by BF_X. This invariant detects the presence of a monopole class by element of a certain complicated stable cohomotopy group $\pi^0_{\mathbb{T},\mathcal{U}}(\text{Pic}(X); \text{ind } l)$, where see [5] for the definition of the stable cohomotopy group:

$$BF_X(\mathfrak{s}) \in \pi^0_{\mathbb{T},\mathcal{U}}(\text{Pic}(X); \text{ind } l).$$

It is known [16] that the non-triviality of the stable cohomotopy Seiberg-Witten invariants implies the existence of monopole classes.

To state a non-vanishing theorem of the stable cohomotopy Seiberg-Witten invariants, we need to fix some notations. For any spin^c-structure $\mathfrak{s}$ on X, we introduce the following quantity:

$$\mathfrak{S}^{ij}(\mathfrak{s}) := \frac{1}{2} < c_1(\mathfrak{s}) \cup \mathfrak{e}_i \cup \mathfrak{e}_j, [X] >,$$

where $\mathfrak{e}_1, \mathfrak{e}_2, \cdots, \mathfrak{e}_s$ is a set of generators of $H^1(X, \mathbb{Z})$ and $s = b_1(X)$. Here $[X]$ is the fundamental class of X_i and $< \cdot, \cdot >$ is the pairing between cohomology and homology.

Definition 2 ([6]). *A closed oriented smooth 4-manifold X with $b^+(X) \geq 2$ is called BF-admissible if the following holds:*

1. *There is a spin^c-structure $\mathfrak{s}$ with $SW_X(\mathfrak{s}) \equiv 1 \pmod 2$ and $c_1^2(\mathfrak{s}) = 2\chi(X) + 3\tau(X)$*
2. $b^+(X) - b_1(X) \equiv 3 \pmod 4$.
3. $\mathfrak{S}^{ij}(\mathfrak{s}) \equiv 0 \pmod 2$ *for any* i, j.

Then, we have

THEOREM 3 ([18]). *For $i = 1, 2, 3$, let X_i be BF-admissible, closed oriented smooth 4-manifolds. Then a connected sum $\#_{i=1}^{j} X_i$ has a non-trivial stable cohomotopy Seiberg-Witten invariant, where $j = 2, 3$.*

We shall use Theorem 3 to prove Theorem A.

2.2. Irreducible BF-admissible 4-manifolds. We need to find BF-admissible 4-manifolds to prove Theorem A. For this purpose, let us recall the following nice result on the existence of irreducible symplectic 4-manifolds, where notice that it is known [12] that any simply connected minimal symplectic 4-manifold is irreducible.

THEOREM 4 (Theorem A in [1]). *Let a and b integers satisfying $2a + 3b \geq 0$, and $a + b \equiv 0 \pmod 4$. If, in addtion, $b \leq -2$ is satisfied. Then there exists a simply connected minimal symplectic 4-manifold X with $(\chi(X), \tau(X)) = (a, b)$ and odd intersection form, except possibly for (a, b) equal to $(7, -3)$, $(11, -3)$, $(13, -5)$, or $(15, -7)$.*

Consider a symplectic 4-manifold with $(\chi(X), \tau(X)) = (a, b)$ in Theorem 4. Since $\chi(X) = 2 - 2b_1(X) + b_2(X) = 2 + b^+(X) + b^-(X) = a$ and $\tau(X) = b^+(X) - b^-(X) = b$ hold, we have $b^+(X) = \alpha - 1$ and $b^-(X) = \beta - 1$, where $\alpha := (a + b)/2$ and $\beta := (a - b)/2$. Since X has odd intersection form, the celebrated result of Freedman [11] tells us that X is homeomorphic to

$$(\alpha - 1)\mathbb{C}P^2 \# (\beta - 1)\overline{\mathbb{C}P^2}.$$

Suppose now that $b^+(X) = \alpha - 1 \equiv 3 \pmod 4$, i.e., $a + b \equiv 0 \pmod 8$. Then X satisfies the second condition in Definition 2, where notice that $b_1(X) = 0$. Since X is a symplectic 4-manifold with $b^+(X) > 1$, a famous result of Taubes [25] tells us that X satisfies the first condition in Definition 2. In fact, we can take a canonical spin^c-structure compatible with a symplectic structure. The third condition in Definition 2 is also satisfied since we have $b_1(X) = 0$. Hence we obtain the following existence result of BF-admissible 4-manifolds:

COROLLARY 5. *Let (a, b) be a pair of integers satisfying $2a + 3b \geq 0$, $a + b \equiv 0 \pmod 8$, and $b \leq -2$ is satisfied, except possibly for (a, b) equal to $(11, -3)$, $(13, -5)$, or $(15, -7)$. Set as $\alpha = (a + b)/2$ and $\beta = (a - b)/2$. Then, there exists a BF-admissible, irreducible symplectic 4-manifold which is homeomorphic to $(\alpha - 1)\mathbb{C}P^2 \# (\beta - 1)\overline{\mathbb{C}P^2}$.*

In the case of non-simply connected, we have a similar result as follows:

THEOREM 6 (Theorem B in [6]). *Let a and b are integers satisfying $2a + 3b \geq 0$, $a + b \equiv 0 \pmod 8$, and $b \leq -2$ is satisfied, except possibly for (a, b) equal to $(11, -3)$, $(13, -5)$, or $(15, -7)$. Set as $\alpha = (a + b)/2$ and $\beta = (a - b)/2$. Then, there exists a BF-admissible, irreducible symplectic 4-manifold with fundamental group $\mathbb{Z}$ which is homeomorphic to $\alpha\mathbb{C}P^2 \# \beta\overline{\mathbb{C}P^2} \# (S^1 \times S^3)$ and a BF-admissible, irreducible symplectic 4-manifold with fundamental group $\mathbb{Z}_p$ which is homeomorphic to $(\alpha - 1)\mathbb{C}P^2 \# (\beta - 1)\overline{\mathbb{C}P^2} \# N_p$.*

2.3. Obstruction to the non-singular solutions. We also use the following result on estimates on Perelman's $\bar{\lambda}$ invariant, which was proved in [6]:

THEOREM 7. *For $m = 1, 2, 3$, let X_m be BF-admissible 4-manifolds. And assume that $\sum_{i=1}^{j}(2\chi(X_i) + 3\tau(X_i)) > 0$, where $j = 2, 3$. Then, Perelman's $\bar{\lambda}$ invariant of a connected sum $Z := \#_{i=1}^{j} X_i$ satisfies*

$$\bar{\lambda}(Z) \leq -4\pi\sqrt{2\sum_{i=1}^{j} C(X_i)} < 0, \tag{3}$$

where $C(X_i) := 2\chi(X_i) + 3\tau(X_i)$.

We should notice that the first non-trivial bound for Perelman's $\bar{\lambda}$ invariant of 4-manifold was proved in an interesting article [9] by using Seiberg-Witten monopole equations.

We also have the following obstruction to the existence of non-singular solution to the normalized Ricci flow, which was also proved in [6]:

THEOREM 8. *For $i = 1, 2, 3$, let X_i be BF-admissible 4-manifolds. Assume also that $\sum_{i=1}^{j}(2\chi(X_i) + 3\tau(X_i)) > 0$ is satisfied, where $j = 2, 3$. Then, on a connected sum $Z := \#_{i=1}^{j} X_i$, there is no non-singular solution to the normalized Ricci flow for any initial metric if*

$$12(j-1) > \sum_{i=1}^{j}\Big(2\chi(X_i) + 3\tau(X_i)\Big). \tag{4}$$

3. Proof of Theorem A.

3.1. Case 1.

LEMMA 9. *For any pair (k, ℓ) of positive integers satisfying*

$$-7 \leq 5k - \ell < 8,\ 5\ell - k \geq -103, \tag{5}$$

the following inequalities are satisfied simultaneously:

$$5\ell - k + 88 + 8\Big(2 - \frac{12}{1296\pi^2}\Big) > 0, \tag{6}$$

$$5k - \ell + 8\Big(2 - \frac{12}{1296\pi^2}\Big) > 8, \tag{7}$$

$$5k - \ell < 8. \tag{8}$$

Proof. The inequality (7) is equivalent to

$$5k - \ell > -8\Big(1 - \frac{12}{1296\pi^2}\Big). \tag{9}$$

Since $\pi > 3$ holds, we get

$$1 - \frac{12}{1296\pi^2} > 1 - \frac{12}{1296\cdot 3^2}.$$

Therefore,

$$-8\Big(1-\frac{12}{1296\pi^2}\Big) < -8\Big(1-\frac{12}{12963^2}\Big) < -7. \tag{10}$$

This tells us that (9) always holds if $-7 \leq 5k-\ell$ is satisfied. Hence both (7) and (8) are satisfied under $-7 \leq 5k-\ell < 8$. Similarly, we also have

$$-88-8\Big(2-\frac{12}{1296\pi^2}\Big) = -96-8\Big(1-\frac{12}{1296\pi^2}\Big) < -103.$$

Therefore (6) holds if $5\ell - k \geq -103$ is satisfied. □

Let Y_0 be a Kummer surface with an elliptic fibration $Y_0 \to \mathbb{C}P^1$. Let Y_m be obtained from Y_0 by performing a logarithmic transformation of order $2m+1$ on a non-singular fiber of Y_0. Then, Y_m are simply connected spin manifolds with $b^+(Y_m)=3$ and $b^-(Y_m)=19$. By the Freedman classification [11], Y_m must be homeomorphic to a $K3$ surface. And Y_m is a Kähler surface with $b^+(Y_m)>1$ and hence a result of Witten [27] tells us that $\pm c_1(Y_m)$ are monopole classes of Y_m for each m. We have $c_1(Y_m) = 2m\mathfrak{f}$, where $\mathfrak{f}$ is Poincaré dual to the multiple fiber which is introduced by the logarithmic transformation. See also [2]. Notice also that Y_m is a BF-admissible 4-manifold.

On the other hand, let $X_{k,\ell}$ be any 4-manifold which is homeomorphic to $k\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}\#(S^1\times S^3)$. Then, we have $2\chi(X_{k,\ell})+3\tau(X_{k,\ell}) = 5k-\ell$ and $2\chi(X_{k,\ell})-3\tau(X_{k,\ell}) = 5\ell-k$. Consider the following connected sum

$$M^{k,\ell}_{g,h}(m) := X_{k,\ell}\#Y_m\#(\Sigma_g\times\Sigma_h). \tag{11}$$

Then we also have

$$2\chi(M^{k,\ell}_{g,h}(m))+3\tau(M^{k,\ell}_{g,h}(m)) = 5k-\ell+4(g-1)(h-1)-8, \tag{12}$$

$$2\chi(M^{k,\ell}_{g,h}(m))-3\tau(M^{k,\ell}_{g,h}(m)) = 5\ell-k+88+4(g-1)(h-1). \tag{13}$$

Lemma 10. *Consider the connected sum (11) in the case where $g=h=3$, i.e., $M^{k,\ell}_{3,3}(m)$. Then the following inequality holds if both (6) and (7) are satisfied:*

$$2\chi(M^{k,\ell}_{3,3}(m))-3|\tau(M^{k,\ell}_{3,3}(m))| > \frac{1}{1296\pi^2}||M^{k,\ell}_{3,3}(m)|| \neq 0. \tag{14}$$

Similarly, the following holds if (8) is satisfied:

$$C(X_{k,\ell})+C(Y_m)+C(\Sigma_3\times\Sigma_3) < 24, \tag{15}$$

where $C(X) := 2\chi(X)+3\tau(X)$ for any closed 4-manifold X.

Proof. Notice that we have $C(X_{k,\ell}) = 5k-\ell$, $C(K3)=0$ and $C(\Sigma_3\times\Sigma_3) = 4\cdot 2\cdot 2 = 16$. Therefore, (15) is equivalent to $5k-\ell+16<24$. This is nothing but (8). On the other hand, the simplicial volume of any connected sum $M_1\#M_2$ satisfies $||M_1\#M_2|| = ||M_1||+||M_2||$. See [7, 13]. It is known that [7, 13] that any simply connected manifold has vanishing simplicial volume. In particular, we have $||Y_m||=0$. It is also [13] known that the simplicial volume vanishes for any closed manifold whose fundamental group is amenable. Since it is known that any abelian group is amenable, we have $||X_{k,\ell}|| = 0$ because the fundamental group of $X_{k,\ell}$ is $\mathbb{Z}$. Moreover, the following result is proved in [8]:

$$||\Sigma_h \times \Sigma_g|| = 24(g-1)(h-1).$$

Hence, we have $||M^{k,\ell}_{3,3}(m)|| = ||X_{k,\ell}|| + ||Y_m|| + ||\Sigma_3 \times \Sigma_3|| = 24 \cdot 2 \cdot 2$. In particular, $||M^{k,\ell}_{3,3}(m)|| \neq 0$. This implies

$$\frac{1}{1296\pi^2}||M^{k,\ell}_{3,3}(m)|| = \frac{24}{1296\pi^2}4 = \frac{12}{1296\pi^2}8.$$

By (12), we also have $2\chi(M^{k,\ell}_{3,3}(m))+3\tau(M^{k,\ell}_{3,3}(m)) = 5k-\ell+4\cdot2\cdot2-8 = 5k-\ell+8\cdot2-8$. Therefore,

$$2\chi(M^{k,\ell}_{3,3}(m)) + 3\tau(M^{k,\ell}_{3,3}(m)) > \frac{1}{1296\pi^2}||M^{k,\ell}_{3,3}(m)||$$

is equaivalent to

$$5k - \ell + 8\cdot 2 - 8 > \frac{12}{1296\pi^2}8,$$

namely,

$$5k - \ell + 8\Big(2 - \frac{12}{1296\pi^2}\Big) > 8.$$

Notice that this is the inequality (7). Similarly, by (13), we also have $2\chi(M^{k,\ell}_{3,3}(m)) - 3\tau(M^{k,\ell}_{3,3}(m)) = 5\ell - k + 88 + 4\cdot 2\cdot 2 = 5k - \ell + 88 + 8\cdot 2$. Hence,

$$2\chi(M^{k,\ell}_{3,3}(m)) - 3\tau(M^{k,\ell}_{3,3}(m)) > \frac{1}{1296\pi^2}||M^{k,\ell}_{3,3}(m)||$$

is equivalent to

$$5\ell - k + 88 + 8\cdot 2 > \frac{12}{1296\pi^2}8,$$

namely,

$$5\ell - k + 88 + 8\Big(2 - \frac{12}{1296\pi^2}\Big) > 0.$$

This is nothing but the inequality (6). Therefore, (14) holds if both (6) and (7) are satisfied. □

Theorem 6, Lemma 9 and Lemma 10 imply

PROPOSITION 11. *Let (a, b) be any pair of integers satisfying*

$$a + b \equiv 0 \pmod 8),\ b \leq -2,\ 0 \leq 2a + 3b < 8. \tag{16}$$

Let $k = (a + b)/2$ and $\ell = (a - b)/2$. Then there exits a BF-admissible, irreducible symplectic 4-manifold $X_{k,\ell}$ with fundamental group $\mathbb{Z}$ which is homeomorphic to

$k\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}\#(S^1\times S^3)$. And the connected sum $M^{k,\ell}_{3,3}(m) := X_{k,\ell}\#Y_m\#(\Sigma_3\times\Sigma_3)$ satisfies both (14) and (15) for each k,ℓ,m.

Proof. First of all, notice that

$$5k-\ell = 5\frac{(a+b)}{2} - \frac{(a-b)}{2} = 2a+3b,\ 5\ell-k = 5\frac{(a-b)}{2} - \frac{(a+b)}{2} = 2a-3b.$$

Therefore, the condition (5) is equivalent to

$$-7 \le 2a+3b < 8,\ 2a-3b \ge -103. \tag{17}$$

Then, Lemma 9 and Lemma 10 tell us that, if (17) holds, then, for any closed 4-manifold $X_{k,\ell}$ which is homeomorphic to $k\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}\#(S^1\times S^3)$, the connected sum $M^{k,\ell}_{3,3}(m)$ satisfies both (14) and (15).

On the other hand, Theorem 6 tells us that, except possibly for (a,b) equal to $(11,-3)$, $(13,-5)$, or $(15,-7)$, for any pair (a,b) of integers satisfying

$$2a+3b \ge 0,\ a+b \equiv 0 \pmod 8, b \le -2, \tag{18}$$

there exists a BF-admissible, irreducible symplectic 4-manifold $X_{k,\ell}$ which is homeomorphic to $k\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}\#(S^1\times S^3)$. Notice that, under $2a+3b\ge 0$ and $b\le -2$, $2a-3b \ge -103$ always holds because $2a-3b \ge 2a+3b \ge 0$. Therefore, both (17) and (18) hold if (16) is satisfied. Notice also that $2a+3b \ge 9$ holds for $(a,b) = (11,-3),(13,-5),(15,-7)$. The desired result now follows. □

We are now in a position to prove the Case 1 in Theorem A. First of all, for any integer $0\le n\le 7$, we set

$$a = 17+n,\ b = -9-n. \tag{19}$$

In particular, we have $a+b=8$ and $2a+3b = 7-n$. Hence we have $a+b\equiv 0 \pmod 8$, $0\le 2a+3b = 7-n < 8$ and $b\le -2$. Notice also that

$$k = \frac{(a+b)}{2} = 4,\ \ell = \frac{(a-b)}{2} = 13+n.$$

Then, Proposition 11 tells us that there exists a BF-admissible, irreducible symplectic 4-manifold $X_{4,13+n}$ which is homeomorphic to $4\mathbb{C}P^2\#(13+n)\overline{\mathbb{C}P^2}\#(S^1\times S^3)$ and $M^{4,13+n}_{3,3}(m) := X_{4,13+n}\#Y_m\#(\Sigma_3\times\Sigma_3)$ satisfies both (14) and (15). Notice that $M^{4,13+n}_{3,3}(m)$ satisfies the strict Gromov-Hitchin-Thorpe type inequality by (14). Moreover, Theorem 8 in the case where $j=3$ tells us that there is no non-singular solution to the normalized Ricci flow on $M^{4,13+n}_{3,3}$ for any initial metric under (15), here notice that $X_{4,13+n}$, Y_m and $(\Sigma_3\times\Sigma_3)$ are all BF-admissible.

On the other hand, we have $C(X_{4,13+n})+C(Y_m)+C(\Sigma_3\times\Sigma_3) = 5k-\ell+0+16 = 2a+3b+16 = 23-n > 0$. Therefore, we obtain the following bound on Perelman's $\overline{\lambda}$ invariant by (3):

$$\bar{\lambda}(M^{4,13+n}_{3,3}(m)) \le -4\pi\sqrt{2(23-n)} < 0.$$

Finally, for each n, we shall show that the following sequence

$$\{M^{4,13+n}_{3,3}(m)\}_{m\in\mathbb{N}} \tag{20}$$

contains infinitely many diffeo types. First of all, notice that the connected sum $X_{4,13+n}$ has non-trivial stable cohomotopy Seiberg-Witten invariants by Theorem 3. In particular, $M^{3,12+n}_{3,3}(m)$ has monopole classes which are given by

$$\pm c_1(X_{4,13+n}) \pm c_1(Y_m) \pm c_1(\Sigma_3 \times \Sigma_3), \tag{21}$$

where $c_1(X)$ denotes the first Chern class of the canonical line bundle of a closed symplectic 4-manifold X and we have $c_1(Y_m) = 2m\mathfrak{f}$. Suppose now that the sequence (20) contains only finitely many diffeomorphism types. Namely, suppose that there exists a positve integer m_0 such that $M^{4,13+n}_{3,3}(m_0)$ is diffemorphic to $M^{4,13+n}_{3,3}(m)$ for any integer $m \geq m_0$. Then, by taking $m \to \infty$, we see that the set of monopole classes of the 4-manifold $M^{4,13+n}_{3,3}(m_0)$ is unbounded by (21). However, this is a contradiction because the set of monopole classes of any given smooth 4-manifold with $b^+ > 1$ must be finite. Therefore, the sequence (20) must contain infinitely many diffeomorphism types. For any m, since $M^{4,13+n}_{3,3}(m)$ is homeomorphic to $M(n) := 4\mathbb{C}P^2\#(13+n)\overline{\mathbb{C}P^2}\#(S^1 \times S^3)\#K3\#(\Sigma_3 \times \Sigma_3)$, we are able to conclude that $M(n)$ has ∞-property $\mathcal{R}$ as desired. Case 1 in Theorem A now follows.

3.2. Case 2. In this subsection, we shall prove Case 2 in Theorem A. The strategy of the proof in this case is similar to that of Case 1.

LEMMA 12. *For any pair (k, ℓ) of positive integers satisfying*

$$-11 \leq 5k - \ell < 4, \ 5\ell - k \geq -107, \tag{22}$$

the following conditions are satisfied simultaneously:

$$5\ell - k + 92 + 8\Big(2 - \frac{12}{1296\pi^2}\Big) > 0, \tag{23}$$

$$5k - \ell + 8\Big(2 - \frac{12}{1296\pi^2}\Big) > 4, \tag{24}$$

$$5k - \ell < 4. \tag{25}$$

Proof. One can check that

$$-92 - 8\Big(2 - \frac{12}{1296\pi^2}\Big) < -107.$$

Hence, if $5\ell - k \geq -107$ holds, (23) is also satisfied. Similarly, we have

$$4 - 8\Big(2 - \frac{12}{1296\pi^2}\Big) < -11.$$

Hence, (24) holds if $5k - \ell \geq -11$. This tells us that both (24) and (25) are satisfied under $-11 \leq 5k - \ell < 4$. □

Let $Z_{k,\ell}$ be any 4-manifold which is homeomorphic to $k\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}\#N_p$. Then, we have

$$2\chi(Z_{k,\ell}) + 3\tau(Z_{k,\ell}) = 5k - \ell + 4, \ 2\chi(Z_{k,\ell}) - 3\tau(Z_{k,\ell}) = 5\ell - k + 4. \tag{26}$$

Consider the following connected sum

$$L^{k,\ell}_{g,h}(m) := Z_{k,\ell}\#Y_m\#(\Sigma_h \times \Sigma_g), \tag{27}$$

where Y_m is the homotopy $K3$ surface used in Section 3.1. Then we also have

$$2\chi(L^{k,\ell}_{g,h}(m)) + 3\tau(L^{k,\ell}_{g,h}(m)) = 5k - \ell + 4(g-1)(h-1) - 4, \tag{28}$$

$$2\chi(L^{k,\ell}_{g,h}(m)) - 3\tau(L^{k,\ell}_{g,h}(m)) = 5\ell - k + 4(g-1)(h-1) + 92. \tag{29}$$

LEMMA 13. *Consider the connected sum (27) in the case wher $g = h = 3$, i.e., $L^{k,\ell}_{3,3}(m)$. Then the following inequality holds if both (23) and (24) are satisfied:*

$$2\chi(L^{k,\ell}_{3,3}(m)) - 3|\tau(L^{k,\ell}_{3,3}(m))| > \frac{1}{1296\pi^2}||L^{k,\ell}_{3,3}(m)|| \neq 0. \tag{30}$$

Similarly, the following holds if (25) is satisfied:

$$C(Z_{k,\ell}) + C(Y_m) + C(\Sigma_3 \times \Sigma_3) < 24. \tag{31}$$

Proof. We have $C(Z_{k,\ell}) = 5k - \ell + 4$, $C(Y_m) = 0$ and $C(\Sigma_3 \times \Sigma_3) = 4 \cdot 2 \cdot 2 = 16$. Therefore, (31) is equivalent to $5k - \ell + 4 + 16 < 24$. This is (25). On the other hand, as the proof of Lemma 10, we have $||L^{k,\ell}_{3,3}(m)|| = ||Z_{k,\ell}|| + ||Y_m|| + ||\Sigma_3 \times \Sigma_3|| = 24 \cdot 2 \cdot 2$, where notice that $||Z_{k,\ell}|| = 0$ holds because the fundamental group of $Z_{k,\ell}$ is $\mathbb{Z}_p$ and hence this is amenable. From the above, we get

$$\frac{1}{1296\pi^2}||L^{k,\ell}_{3,3}(m)|| = \frac{24}{1296\pi^2}4 = \frac{12}{1296\pi^2}8.$$

In particular, $||L^{k,\ell}_{3,3}(m)|| \neq 0$. By (28), we also have $2\chi(L^{k,\ell}_{g,h}(m)) + 3\tau(L^{k,\ell}_{g,h}(m)) = 5k - \ell + 4 \cdot 2 \cdot 2 - 4 = 5k - \ell + 8 \cdot 2 - 4$. Therefore,

$$2\chi(L^{k,\ell}_{3,3}(m)) + 3\tau(L^{k,\ell}_{3,3}(m)) > \frac{1}{1296\pi^2}||L^{k,\ell}_{3,3}(m)||$$

is equivalent to

$$5k - \ell + 8 \cdot 2 - 4 > \frac{12}{1296\pi^2}8,$$

namely,

$$5k - \ell + 8\Big(2 - \frac{12}{1296\pi^2}\Big) > 4.$$

This is the inequality (24). Similarly, by (29), we also have $2\chi(L^{k,\ell}_{3,3}(m)) - 3\tau(L^{k,\ell}_{3,3}(m)) = 5\ell - k + 4 \cdot 2 \cdot 2 + 92 = 5\ell - k + 92 + 8 \cdot 2$. Hence,

$$2\chi(L^{k,\ell}_{3,3}(m)) - 3\tau(L^{k,\ell}_{3,3}(m)) > \frac{1}{1296\pi^2}||L^{k,\ell}_{3,3}(m)||$$

is equivalent to

$$5\ell - k + 92 + 8 \cdot 2 > \frac{12}{1296\pi^2}8,$$

namely,

$$5\ell - k + 92 + 8\Big(2 - \frac{12}{1296\pi^2}\Big) > 0.$$

This is nothing but (23). Therefore, (30) holds if both (23) and (24) are satisfied. □

By Theorem 6, Lemma 12 and Lemma 13, we obtain

Proposition 14. *Let (a, b) be any pair of integers satisfying*

(32) $$a + b \equiv 0 \pmod 8,\ b \le -2,\ 0 \le 2a + 3b < 8.$$

And let

$$k = \frac{a+b}{2} - 1,\ \ell = \frac{a-b}{2} - 1.$$

Then there exits a BF-admissible, irreducible symplectic 4-manifold $Z_{k,\ell}$ with fundamental group $\mathbb{Z}_p$ which is homeomorphic to $k\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}\#N_p$, and the connected sum $L^{k,\ell}_{3,3}(m) := Z_{k,\ell}\#Y_m\#(\Sigma_3 \times \Sigma_3)$ satisfies (30) and (31) for each k, ℓ, m.

Proof. First of all, notice that

$$5k - \ell = 5\Big(\frac{(a+b)}{2} - 1\Big) - \Big(\frac{(a-b)}{2} - 1\Big) = 2a + 3b - 4,$$

$$5\ell - k = 5(\frac{(a-b)}{2} - 1) - (\frac{(a+b)}{2} - 1) = 2a - 3b - 4.$$

Therefore, the condition (22) is equivalent to

(33) $$-7 \le 2a + 3b < 8,\ 2a - 3b \ge -103.$$

By Lemma 12 and Lemma 13, if (33) holds, for any closed 4-manifold $Z_{k,\ell}$ which is homeomorphic to $k\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}\#N_p$, the connected sum $L^{k,\ell}_{3,3}(m)$ satisfies (30) and (31).

Moreover, Theorem 6 tells us that, except possibly for (a, b) equal to $(11, -3)$, $(13, -5)$, or $(15, -7)$, for any pair (a, b) of integers satisfying

(34) $$2a + 3b \ge 0,\ a + b \equiv 0 \pmod 8), b \le -2,$$

there exists a BF-admissible, irreducible symplectic 4-manifold $Z_{k,\ell}$ which is homeomorphic to $k\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}\#N_p$. Notice that $2a - 3b \ge -103$ always holds under $2a + 3b \ge 0$ and $b \le -2$. Therefore, both (33) and (34) hold if (32) is satisfied. The desired result now follows, where notice that $2a + 3b \ge 9$ holds for $(a, b) = (11, -3), (13, -5), (15, -7)$. □

We prove the Case 2 of Theorem A as follows: For any integer $0 \le n \le 7$, let $a = 17 + n$ and $b = -9 - n$. In particular, we have $a + b = 8 \equiv 0 \pmod 8$, $0 \le 2a + 3b = 7 - n < 8$ and $b \le -2$. We also have

$$k = \frac{(a+b)}{2} - 1 = 3,\ \ell = \frac{(a-b)}{2} - 1 = 12 + n.$$

Then, Proposition 14 tells us that there exists a BF-admissible, irreducible symplectic 4-manifold $Z_{3,12+n}$ which is homeomorphic to $3\mathbb{C}P^2\#(12+n)\overline{\mathbb{C}P^2}\#N_p$, and $L^{3,12+n}_{3,3}(m) := Z_{3,12+n}\#Y(m)\#(\Sigma_3\times\Sigma_3)$ satisfies (30) and (31). The connected sum $L^{3,12+n}_{3,3}(m)$ satisfies the strict Gromov-Hitchin-Thorpe type inequality by (14). And Theorem 8 implies that we have no non-singular solution to the normalized Ricci flow on $L^{3,12+n}_{3,3}(m)$ for any initial metric under (31). On the other hand, we have $C(Z_{3,12+n})+C(Y_m)+C(\Sigma_3\times\Sigma_3) = 7-n+16 = 23-n > 0$. Therefore, by (3), we obtain

$$\bar{\lambda}(L^{3,12+n}_{3,3}(m)) \leq -4\pi\sqrt{2(23-n)} < 0.$$

Finally, as the proof of Case 1 above, for each n, we are able to show that $\{L^{3,12+n}_{3,3}(m)\}_{m\in\mathbb{N}}$ contains infinitely many diffeo types by taking $m\to\infty$. For any m, notice that $L^{3,12+n}_{3,3}(m)$ is homeomorphic to $L(n) := 3\mathbb{C}P^2\#(12+n)\overline{\mathbb{C}P^2}\#N_p\#K3\#(\Sigma_3\times\Sigma_3)$. Therefore, we are able to conclude that $L(n)$ has ∞-property $\mathcal{R}$ as desired.

3.3. Case 3. Finally, we shall prove Case 3 of Theorem A. Let $P_{k,\ell}$ be any 4-manifold which is homeomorphic to $k\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}$. Then, we have

$$2\chi(P_{k,\ell})+3\tau(P_{k,\ell}) = 5k-\ell+4,\ 2\chi(P_{k,\ell})-3\tau(P_{k,\ell}) = 5\ell-k+4. \tag{35}$$

Notice that we have $2\chi(P_{k,\ell})+3\tau(P_{k,\ell}) = 2\chi(Z_{k,\ell})+3\tau(Z_{k,\ell})$ and $2\chi(P_{k,\ell})-3\tau(P_{k,\ell}) = 2\chi(Z_{k,\ell})-3\tau(Z_{k,\ell})$ by (26) and (35). Consider the following connects sum

$$G^{k,\ell}_{g,h}(m) := P_{k,\ell}\#Y_m\#(\Sigma_h\times\Sigma_g),$$

where Y_m is again the homotopy $K3$ surface used as before. Then, by using Corollary 5 instead of Theorem 6 and using the same argument with that of Proposition 14, we are able to obtain

PROPOSITION 15. *Let (a,b) be any pair of integers satisfying*

$$a+b\equiv 0 \pmod 8),\ b\leq -2,\ 0\leq 2a+3b<8. \tag{36}$$

Let

$$k=\frac{a+b}{2}-1,\ \ell=\frac{a-b}{2}-1.$$

Then, there exits a BF-admissible, irreducible symplectic 4-manifold $P_{k,\ell}$ which is homeomorphic to $k\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}$ and for each k,ℓ,m, $G^{k,\ell}_{3,3}(m) := P_{k,\ell}\#Y_m\#(\Sigma_3\times\Sigma_3)$ satisfies

$$2\chi(G^{k,\ell}_{3,3}(m))-3|\tau(G^{k,\ell}_{3,3}(m))| > \frac{1}{1296\pi^2}||G^{k,\ell}_{3,3}(m)|| \neq 0, \tag{37}$$

$$C(P_{k,\ell})+C(Y_m)+C(\Sigma_3\times\Sigma_3) < 24. \tag{38}$$

Case 3 in Theorem A follows easily from Proposition 15. As before, for any integer $0\leq n\leq 7$, let $a=17+n$ and $b=-9-n$. Of course, we have $a+b\equiv 0 \pmod 8)$, $0\leq 2a+3b = 7-n<8$ and $b<-2$, $k=3$ and $\ell=12+n$. Then, Proposition 15 implies that there exists a BF-admissible, irreducible symplectic 4-manifold $P_{3,12+n}$ which is homeomorphic to $3\mathbb{C}P^2\#(12+n)\overline{\mathbb{C}P^2}$, and

$$G_{3,3}^{3,12+n}(m) := P_{3,12+n}\#Y(m)\#(\Sigma_3\times\Sigma_3)$$

satisfies (37) and (38). Hence, $G_{3,3}^{3,12+n}(m)$ satisfies the strict Gromov-Hitchin-Thorpe type inequality, and there is no non-singular solution to the normalized Ricci flow on $G_{3,3}^{3,12+n}(m)$ for any initial metric by Theorem 8. We are also able to show the following by (3):

$$\bar{\lambda}(G_{3,3}^{3,12+n}(m)) \le -4\pi\sqrt{2(23-n)} < 0.$$

Finally, by considering the sequence $\{G_{3,3}^{3,12+n}(m)\}_{m\in\mathbb{N}}$, we conclude that $3\mathbb{C}P^2\#(12+n)\overline{\mathbb{C}P^2}\#K3\#(\Sigma_3\times\Sigma_3)$ has ∞-property $\mathcal{R}$ for each n.

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