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FIRST ORDER DEFORMATIONS OF PAIRS OF A RATIONAL CURVE AND A HYPERSURFACE*

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Abstract. Let X_0 be a smooth hypersurface (not assumed generic) in projective space $\mathbf{P}^n, n \geq 3$ over the complex numbers, and C_0 a smooth rational curve on X_0. We are interested in the deformations of the pair C_0, X_0. In this paper, we prove that if the first order deformations of the pair exist along certain first order deformations of the hypersurface X_0, then the twisted normal bundle

$$N_{C_0/X_0}(1) = N_{C_0/X_0} \otimes \mathcal{O}_{\mathbf{P}^n}(1)|_{C_0}$$

is generated by global sections.

Key words. Rational curve, hypersurface, twisted normal bundle.

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1. Introduction. Let X_0 be a smooth hypersurface in $\mathbf{P}^n$ over $\mathbb{C}$, and C_0 be a smooth rational curve on X_0. The main focus of this paper is on how the existence of the first order deformation of the pair $C_0 \subset X_0$ affects the twisted normal bundle $N_{C_0/X_0}(1)$ of the rational curve. Previously, there are many works on pairs C_0, X_0 by Albano and Katz ([1]), Clemens ([3], [4]), Katz ([7]), Pacienza ([8]), Voisin ([9], [10]), etc.(there are many important papers missing from this list). In this paper, we add more results to this list. But one of main differences of this paper from others, which also turns out to be the main difficulty, is our weaker first order assumption (1.3) in theorem 1.2 below. Such an assumption is useful in understanding the deformations of the pair. Deformations of pairs in more general setting were formulated and studied by Clemens ([5], [6]). Our results are not consequences of this study. To state the theorem in a precise way, let's give a formal description of the assumption. Throughout the paper varieties are over complex numbers. Let $H^0(\mathcal{O}_{\mathbf{P}^n}(h))$ denote the vector space of homogeneous polynomials of degree $h = deg(X_0)$ in $n+1$ variables for $n \geq 3$. Let $f_0 \in H^0(\mathcal{O}_{\mathbf{P}^n}(h))$ such that

$$X_0 = div(f_0)$$

is a smooth hypersurface. Let

$$[f_0] \in \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$$

denote the corresponding point of f_0 in the projectivization. Let

$$c_0 : \mathbf{P}^1 \to X_0 \subset \mathbf{P}^n$$

be an embedding of $\mathbf{P}^1$, whose image is C_0. Let

$$\mathbb{H}^1(T_{X_0} \to N_{C_0/X_0})$$

be the hypercohomology of the complex, that is isomorphic to the tangent space of the deformation space of the pair $C_0 \subset X_0$. Let $H^1(T_{X_0})$ be cohomology group that

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is isomorphic to the tangent space of the deformation space of hypersurfaces at the point X_0. There is a known diagram ([1]):

$$\begin{array}{ccc} & & \mathbb{H}^1(T_{X_0} \to N_{C_0/X_0}) \\ & & \downarrow\phi \\ T_{[f_0]}\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))) & \stackrel{\psi}{\to} & H^1(T_{X_0}) \end{array} \tag{1.1}$$

where the map ψ is the differential at $[f_0]$ from $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))$ to the deformation space of complex structures of the differential manifold of X_0. This diagram gives a relation between the first order deformations of the pair and the first order deformation of hypersurfaces at the level of moduli spaces (i.e. factoring out isomorphism). There is another version of ϕ without factoring out isomorphism (see (2.3) section 2).

Next we define a specific family of hypersurfaces.

Definition 1.1. *Let $L_i \in H^0(\mathcal{O}_{\mathbf{P}^n}(1))$, $i = 0, \cdots, h = deg(X_0)$ be some non-zero sections (not assumed generic) that satisfy*

$$\{L_i = 0\} \cap \{L_j = 0\} \cap C_0 = \emptyset, i \neq j. \tag{1.2}$$

We consider the family of sections in $H^0(\mathcal{O}_{\mathbf{P}^n}(h))$,

$$F(a_1, \cdots, a_h, x) = f_0(x) + \sum_{i=0}^{h} a_i L_0(x) \cdots \hat{L}_i(x) \cdots L_h(x), \quad (omit\ L_i),$$

parametrized by the coefficients $a_i, i = 0, \cdots, h$. Since the degree h hypersurfaces

$$L_0(x) \cdots \hat{L}_i(x) \cdots L_h(x), i = 0, \cdots, h$$

are linearly independent over $\mathbb{C}$, $\mathbb{C}^{h+1} = \{(a_0, \cdots, a_h)\}$ is the parameter space of this family. We define A to be the open set of $\mathbb{C}^{h+1} = \{(a_0, \cdots, a_h)\}$ that parametrizes the smooth hypersurfaces

$$\{x \in \mathbf{P}^n : F(a_0, \cdots, a_h, x) = 0\}.$$

Theorem 1.2. *If in the diagram (1.1)*

$$\phi \quad is \quad onto \quad \psi(T_{[f_0]}A), \tag{1.3}$$

for a specific set of sections L_i above, i.e. C_0 deforms with the hypersurface X_0 in all directions of $T_{[f_0]}A$ to the first order, then the twisted normal bundle

$$N_{C_0/X_0}(1), \tag{1.4}$$

is generated by global sections. In particular, if X_0 is a generic hypersurface and contains a smooth rational curve C_0,

$$N_{C_0/X_0}(1)$$

is generated by global sections.

In applying the theorem, we should note assumption (1.3) only requires ONE specific family A. An immediate corollary is

COROLLARY 1.3. *Assume X_0 is a smooth quintic threefold in $\mathbf{P}^4$, and assumption (1.3) in theorem 1.2 holds for some A defined in definition 1.1. Let $d = deg(\mathcal{O}_{\mathbf{P}^n}(1)|_{C_0})$. Then the splitting of the normal bundle N_{C_0/X_0} must be*

$$N_{C_0/X_0} \simeq \mathcal{O}_{\mathbf{P}^1}(k) \oplus O_{\mathbf{P}^1}(-2-k) \tag{1.5}$$

such that

$$-1 \leq k \leq d-2. \tag{1.6}$$

In particular, (1.5) and (1.6) hold for a quintic 3-fold X_0 that is generic and contains a smooth rational curve C_0.

Proof. It is well-known that vector bundles over $\mathbf{P}^1$ can be decomposed as a direct sum of line bundles. Thus N_{C_0/X_0} must be

$$N_{C_0/X_0} \simeq \mathcal{O}_{\mathbf{P}^1}(k) \oplus O_{\mathbf{P}^1}(-2-k) \tag{1.7}$$

where $k \geq -1$. Apply theorem 1.2 to obtain that

$$N_{C_0/X_0}(1) \simeq \mathcal{O}_{\mathbf{P}^1}(d+k) \oplus O_{\mathbf{P}^1}(d-2-k) \tag{1.8}$$

is generated by global sections. Thus

$$d-2-k \geq 0.$$

This is the inequality in the corollary. In particular if X_0 is a quintic 3-fold that is generic and contains a smooth rational curve C_0, assumption (1.3) follows from lemma 2.2 below. □

REMARK. Assumption (1.3) stresses the importance of the first order of deformations of the pair. Most of previous results, such as those listed above, have the different assumption: X_0 is generic, which is strictly stronger than assumption (1.3). This classical genericity assumption can be explained in the following. Let M_d be the parameter space of embeddings $\mathbf{P}^1 \to \mathbf{P}^n$, whose image has degree d. So M_d is an open set of

$$\mathbf{P}(\oplus_{n+1}\mathcal{O}_{\mathbf{P}^n}(d)).$$

The map c_0 represents a point in M_d which is still denoted by c_0. Let

$$\begin{aligned} &\Gamma \subset M_d \times \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))) \\ &\Gamma = \{([c],[f]) : c^*(f) = 0\}. \end{aligned} \tag{1.9}$$

Then the assumption: X_0 is generic is equivalent to the assumption that there is an irreducible component of Γ containing $(c_0, [f_0])$, which dominates

$$\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))).$$

Our assumption (1.3) in this setting is equivalent to say that the Zariski tangent space $T_{(c_0,[f_0])}\Gamma$ is projected onto $T_{[f_0]}A$. It is not difficult to see the former assumption implies the latter one. We'll prove this assertion in section 2.

Example 5.2 in section 5 indicates assumption (1.3) is strictly weaker than the assumption: X_0 is generic.

The rest of the paper is organized as follows. In section 2, we give another description of first order deformation condition (1.3), that shows that it is weaker than the assumption: X_0 is generic. In section 3, we study a family of smooth hypersurfaces. This is the main technique for the paper. In section 4, we show that the first order assumption (1.3) leads to the positivity of the twisted normal bundle $N_{C_0/X_0}(1)$. This proves theorem 1.2. In section 5, we apply the result of theorem 1.2 to recover a classical result by Clemens, and give examples concerning our weaker assumption (1.3).

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2. First order deformations of the pair. In this section, we give another description of assumption (1.3), which will be used throughout. This is the description of the same map ϕ without factoring out isomorphism.

Let

$$S \subset \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$$

be an irreducible subvariety that contains $[f_0]$ and is smooth at $[f_0]$. Let

$$\mathcal{X}_S \subset \mathbf{P}^n \times S, \tag{2.1}$$

$$\mathcal{X}_S = \{(x, [f]) : [f] \in S, f(x) = 0\}. \tag{2.2}$$

be the universal hypersurface for $S \subset \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$.

Let

$$\begin{aligned} \bar{c}_0 : &\mathbf{P}^1 \to C_0 \times \{[f_0]\} \subset \mathcal{X}_S \\ &t \to (c_0(t), [f_0]) \end{aligned}$$

be the embedding determined by above embedding c_0. The projection

$$P_S : \mathcal{X}_S \to S$$

has a differential map

$$T_{(q,[f_0])}\mathcal{X}_S \quad \to \quad T_{[f_0]}S, \quad q \in C_0$$

which can be extended to a bundle map

$$(P_S)_* : \bar{c}_0^*(T_{\mathcal{X}_S}) \quad \to \quad T_{[f_0]}S \otimes \mathcal{O}_{\mathbf{P}^1}.$$

At last we obtain a morphism on the vector spaces

$$P_S^s : H^0(\bar{c}_0^*(T_{\mathcal{X}_S})) \to T_{[f_0]}S, \tag{2.3}$$

where $T_{[f_0]}S \simeq H^0(T_{[f_0]}S \otimes \mathcal{O}_{\mathbf{P}^1})$ is the space of global sections of the trivial bundle whose each fibre is $T_{[f_0]}S$.

Lemma 2.1.

$$\psi(T_{[f_0]}S) \subset image(\phi)$$

if and only if P_S^s is surjective.

Proof. Recall that M_d is the parameter space of embeddings $\mathbf{P}^1 \to \mathbf{P}^n$, whose image has degree d. Let $\mathcal{X}_n$ be the universal hypersurface for $S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$ (defined in formula (2.2)). Recall

$$\begin{aligned} &\Gamma \subset M_d \times \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))) \\ &\Gamma = \{(c, [f]) : c^*(f) = 0\}. \end{aligned} \tag{2.4}$$

be the incidence scheme containing the point $(c_0, [f_0])$. Let $T_{(c_0,[f_0])}\Gamma$ be the Zariski tangent space of Γ. Let e be the evaluation map

$$\begin{aligned} e: \quad & \Gamma \times \mathbf{P}^1 \to \mathcal{X}_n \\ & (c, [f], t) \to (c(t), [f]). \end{aligned}$$

Its differential map induces a bundle map

$$e_*: \quad T_{(c_0,[f_0])}\Gamma \otimes \mathcal{O}_{\mathbf{P}^1} \to c_0^*(T_{\mathcal{X}_n}).$$

It further induces a homomorphism on the cohomology groups:

$$e^s: \quad T_{(c_0,[f_0])}\Gamma \to H^0(c_0^*(T_{\mathcal{X}_n})),$$

where $T_{(c_0,[f_0])}\Gamma = H^0(T_{(c_0,[f_0])}\Gamma \otimes \mathcal{O}_{\mathbf{P}^1})$. Also there is a surjective map η:

$$T_{(C_0,[f_0])}\Gamma \quad \twoheadrightarrow \quad \mathbb{H}^1(T_{X_0} \to N_{C_0/X_0}),$$

such that the following diagram commutes

$$\begin{array}{ccccc} T_{(C_0,[f_0])}\Gamma & = & T_{(C_0,[f_0])}\Gamma & \stackrel{\eta}{\twoheadrightarrow} & \mathbb{H}^1(T_{X_0} \to N_{C_0/X_0}) \\ \downarrow e^s & & \downarrow & & \downarrow \phi \\ H^0(\bar{c}_0^*(T_{\mathcal{X}_n})) & \stackrel{P_n^s}{\to} & T_{[f_0]}\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))) & \stackrel{\psi}{\to} & H^1(T_{X_0}), \end{array} \tag{2.5}$$

where P_n^s is the corresponding map in formula (2.3). Because

$$T_{c_0}M_d \to H^0(c_0^*(T_{\mathbf{P}^n}))$$

is surjective (it is an isomorphism), e^s has to be surjective. Then the lemma is true for $S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$. Now we consider the subvariety $S \subset \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$ in the lemma. If $\psi(T_{[f_0]}S) \subset image(\phi)$, for any $\alpha \in T_{[f_0]}S$, we apply the diagram to find a section $\sigma \in H^0(\bar{c}_0^*(T_{\mathcal{X}_n}))$ such that $P_n^s(\sigma) = \alpha$. Because $P_n^s(\sigma) = \alpha \in T_{[f_0]}S$, σ must be in the subspace $H^0(\bar{c}_0^*(T_{\mathcal{X}_S}))$ of $H^0(\bar{c}_0^*(T_{\mathcal{X}_n}))$. Thus P_S^s is surjective. Conversely we suppose P_S^s is surjective. For any $\alpha \in T_{[f_0]}S$, using the commutative diagram, we obtain

$$\psi(\alpha) \in \phi \circ \eta \circ (e^s)^{-1} \circ (P_S^s)^{-1}(\alpha).$$

We complete the proof. □

LEMMA 2.2. *If X_0 is generic and contains a smooth rational curve C_0, or equivalently there is an irreducible component Γ_0 of the incidence scheme*

$$\{(c, [f]) \in M_d \times \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))) : c^*(f) = 0\},$$

such that Γ_0 dominates $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$ and $(c_0, [f_0]) \in \Gamma_0$ is generic, then ϕ is surjective.

Proof. In this proof, we consider the entire space of hypersurfaces, i.e.

$$S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))).$$

As before $\mathcal{X}_n$ denotes the universal hypersurface corresponding to $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$.

Let c_0 be as above and

$$\bar{c}_0 : \mathbf{P}^1 \to X_0 \times \{[f_0]\} \subset \mathcal{X}_n$$

be the morphism that lifts the image C_0 to $\mathcal{X}_n$. The projection

$$P : \mathcal{X}_n \to S$$

induces a map on the sections of bundles over $\mathbf{P}^1$,

$$P^s : H^0(\bar{c}_0^*(T_{\mathcal{X}_n})) \to T_{[f_0]}S, \tag{2.6}$$

where $T_{[f_0]}S \simeq H^0(T_{[f_0]}S \otimes \mathcal{O}_{\mathbf{P}^1})$ is the space of global sections of the trivial bundle whose each fibre is $T_{[f_0]}S$. Observe the commutative diagram

$$\begin{array}{ccc} T_{(c_0,[f_0])}\Gamma & \stackrel{(e_\Gamma)_*}{\to} & H^0(\bar{c}_0^*(T_{\mathcal{X}_n})) \\ \downarrow{\scriptstyle(\pi_\Gamma)_*} & & \downarrow{\scriptstyle P_n^s} \\ T_{[f_0]}S & = & T_{[f_0]}S. \end{array}$$

(see (2.5) for P_n^s) where $(e_\Gamma)_*$ is induced from the differential of the evaluation e_Γ:

$$\begin{array}{ccc} e_\Gamma : \Gamma \times \mathbf{P}^1 & \to & \mathcal{X}_n \\ (c, [f], t) & \to & c(t) \times \{[f]\}. \end{array}$$

Since f_0 is generic and π_Γ is dominant (by the assumption of the lemma), then $(c_0, [f_0]) \in \Gamma$ is a generic point in Γ_0. Then the dominance of π_Γ implies the surjectivity of $(\pi_\Gamma)_*$. Thus P_n^s is surjective. By lemma 2.1, we proved lemma 2.2. □

3. Deformation of hypersurfaces. In this section we do not assume that there is a rational curve C_0 as in section 1. So we continue with the notations in the section 1, however we do not assume the condition (1.2) because there is no rational curve C_0 in this section. Instead we assume

$$div(L_i) \neq div(L_j), i \neq j.$$

Recall that

$$F(a_1, \cdots, a_h, x) = f_0(x) + \sum_{i=0}^{h} a_i L_0(x) \cdots \hat{L}_i(x) \cdots L_h(x), \quad (omit\ L_i)$$

is the universal polynomial. Thus

$$\{F = 0\} = \mathcal{X}_A \subset \mathbf{P}^n \times A.$$

is the universal hypersurface, which is smooth. Let $W \subset \mathbf{P}^n$ denote the complement of the proper subvariety

$$\cup_{h \geq j > i \geq 0}\{L_i = L_j = 0\}.$$

Let

$$\mathcal{X}_W = \mathcal{X}_A \cap (W \times A). \tag{3.1}$$

Let

$$u_i = L_0 \frac{\partial}{\partial a_0} - L_i \frac{\partial}{\partial a_i}, i = 1, \cdots, h \tag{3.2}$$

be sections of $\mathcal{O}_{\mathbf{P}^n}(1) \otimes T_A$. Since u_i annihilate F, they are tangent to $\mathcal{X}_W$. So let

$$G(1) \subset T_{\mathcal{X}_W}(1) \tag{3.3}$$

be the vector bundle of rank h over $\mathcal{X}_W$ that is generated by the sections u_i.

We then have

THEOREM 3.1.

$$\frac{T_{\mathcal{X}_W}(1)}{G(1)} \simeq T_{(W\times A)/A}(1)|_{\mathcal{X}_W}, \tag{3.4}$$

where $T_{(W\times A)/A}(1) = (T_W(1) \oplus \{0\})$ *is the twisted relative tangent bundle of the projection* $W \times A \to A$.

REMARK. This theorem does not require additional assumptions. This is a fact about this special type of family of hypersurfaces.

Proof. Consider the exact sequence

$$0 \to \frac{T_{\mathcal{X}_W}(1)}{G(1)} \to \frac{T_{(W\times A)}(1)}{G(1)} \to \mathcal{D} \to 0. \tag{3.5}$$

of bundles over $\mathcal{X}_W$, where $\mathcal{D}$ is some quotient bundle over $\mathcal{X}_W$. It is easy to see that

$$c_1(\mathcal{D}) = c_1(\mathcal{O}_{\mathbf{P}^n}(h+1))|_{\mathcal{X}_W}. \tag{3.6}$$

Let s be a generic section of $\mathcal{O}_{\mathbf{P}^n}(1)$. Let σ be the reduction of $s\frac{\partial}{\partial a_0}$ in $\frac{T_{(W\times A)}(1)}{G(1)}$. Notice that the zero-locus of σ is given by

$$div(\sigma) = div(sL_1 \cdots L_h). \tag{3.7}$$

Since $sL_1 \cdots L_h \in H^0(\mathcal{O}_{\mathbf{P}^n}(h+1))$, σ splits the sequence (3.5). If $L_s \subset \frac{T_{(W\times A)}(1)}{G(1)}$ is the line bundle generated by σ,

$$L_s \oplus \frac{T_{\mathcal{X}_W}(1)}{G(1)} = \frac{T_{(W\times A)}(1)}{G(1)}, \tag{3.8}$$

as bundles over $\mathcal{X}_W$. Secondly, we have another exact sequence

$$0 \to T_{(W\times A)/A}(1) \to \frac{T_{(W\times A)}(1)}{G(1)} \to \mathcal{D}' \to 0. \tag{3.9}$$

of bundles over $\mathcal{X}_W$, where $\mathcal{D}'$ is some quotient bundle over $\mathcal{X}_W$. By direct computation (note $G(1)$ is a trivial bundle), we obtain:

$$c_1(\mathcal{D}') = c_1(T_{W\times A/W}(1)) = (h+1)(c_1(\mathcal{O}_{\mathbf{P}^n}(1)))|_{\mathcal{X}_W}.$$

As above, σ splits this sequence (3.9). Hence

$$L_s \oplus T_{(W\times A)/A}(1) = \frac{T_{(W\times A)}(1)}{G(1)}. \tag{3.10}$$

Comparing (3.8), (3.10), we obtain

$$\frac{T_{\mathcal{X}_W}(1)}{G(1)} \simeq T_{(W\times A)/A}(1), \tag{3.11}$$

over $\mathcal{X}_W$. □

4. Positivity of the twisted normal bundle. In this section we prove theorem 1.2. We continue with the notations in section 1. In particular C_0 is a smooth rational curve in X_0.

Proof of Theorem 1.2. Denote $C_0 \times \{[f_0]\}$ by $\bar{C}_0$. Because of our assumption (1.2), C_0 completely lies in W. Thus we have the exact sequence of bundles

$$0 \to \bar{c}_0^*(T_{\bar{C}_0}(1)) \to \bar{c}_0^*(\tfrac{T_{\mathcal{X}_A}(1)}{G(1)}) \to \bar{c}_0^*(\tfrac{T_{\mathcal{X}_A}(1))}{G(1)+T_{\bar{C}_0}(1)}) \to 0. \tag{4.1}$$

By theorem 3.1, we have two exact sequences,

$$\begin{array}{ccccccccc} 0 & \to & \bar{c}_0^*(T_{\bar{C}_0}(1)) & \to & \bar{c}_0^*(\frac{T_{\mathcal{X}_A}(1)}{G(1)}) & \to & \bar{c}_0^*(\frac{T_{\mathcal{X}_A}(1))}{G(1)+T_{\bar{C}_0}(1)}) & \to & 0 \\ & & \| & & \downarrow I & & & & \\ 0 & \to & \bar{c}_0^*(T_{\bar{C}_0}(1)) & \to & \bar{c}_0^*(T_{(\mathbf{P}^n\times A)/A}(1)) & \to & c_0^*(N_{C_0/\mathbf{P}^n}(1)) & \to & 0. \end{array} \tag{4.2}$$

where I is the isomorphism in theorem 3.1. Notice in theorem 3.1, the isomorphism I is restricted to the identity map on $\bar{c}_0^*(T_{\bar{C}_0}(1))$. Notice that the first half of the diagram (4.2),

$$\begin{array}{ccccc} 0 & \to & \bar{c}_0^*(T_{\bar{C}_0}(1)) & \to & \bar{c}_0^*(\frac{T_{\mathcal{X}_A}(1)}{G(1)}) \\ & & \| & & \downarrow I \\ 0 & \to & \bar{c}_0^*(T_{\bar{C}_0}(1)) & \to & \bar{c}_0^*(T_{(\mathbf{P}^n\times A)/A}(1)). \end{array} \tag{4.3}$$

is commutative. So we obtain

$$c_0^*(N_{C_0/\mathbf{P}^n}(1)) \simeq \bar{c}_0^*(\frac{T_{\mathcal{X}_A}(1))}{G(1)+T_{\bar{C}_0}(1)}). \tag{4.4}$$

This isomorphism gives us another exact sequence

$$0 \to \bar{c}_0^*(G(1)) \to \bar{c}_0^*(\tfrac{T_{\mathcal{X}_A}(1)}{T_{\bar{C}_0}(1)}) \to c_0^*(N_{C_0/\mathbf{P}^n}(1)) \to 0. \tag{4.5}$$

To see the positivity of bundles, we observe that since $H^1(c_0^*G(1)) = 0$ ($c_0^*G(1)$ is a trivial bundle over $\mathbf{P}^1$), sections of $c_0^*(N_{C_0/\mathbf{P}^n}(1))$ can be lifted to sections of

$$\bar{c}_0^*(\frac{T_{\mathcal{X}_A}(1)}{T_{\bar{C}_0}(1)}).$$

Then $\bar{c}_0^*(\frac{T_{\mathcal{X}_A}(1)}{T_{\bar{C}_0}(1)})$ must be generated by global sections because $c_0^*(N_{C_0/\mathbf{P}^n}(1))$ and $c_0^*G(1)$ are. The proof is now completed with the last observation that $\bar{c}_0^*(\frac{T_{\mathcal{X}_A}(1)}{T_{\bar{C}_0}(1)})$ is mapped onto $N_{C_0/X_0}(1)$. This is the only place where assumption (1.3) is used. The following is the argument for this surjection.

Because ϕ is onto $\psi(A)$, P_A^s is onto $T_{f_0}A$. This gives us a natural bundle decomposition of $\bar{c}_0^*(T_{\mathcal{X}_A}(1))$ in the following way: Note $\bar{c}_0^*(T_{\mathcal{X}_A}(1))$ has sections $(P_A^s)^{-1}(\frac{\partial}{\partial a_j}), j = 0, \cdots, h$. Let $\sigma_i \in (P_A^s)^{-1}(\frac{\partial}{\partial a_j})$, $i = 0, \cdots, h$ be a vector in each inverse $(P_A^s)^{-1}(\frac{\partial}{\partial a_j})$. Then all sections $\{\sigma_j\}_j$ generate a trivial subbundle $\mathcal{E}$

$$\mathcal{E} = \oplus_{h+1}\mathcal{O}_{\mathbf{P}^1}.$$

This subbundle gives a decomposition

$$\bar{c}_0^*(T_{\mathcal{X}_A}) \simeq \mathcal{E} \oplus \bar{c}_0^*(T_{\mathcal{X}_{A/A}}), \tag{4.6}$$

Tensoring it with $\bar{c}_0^*(\mathcal{O}_{\mathbf{P}^n}(1))$, we obtain

$$\bar{c}_0^*(T_{\mathcal{X}_A}(1)) \simeq \mathcal{E}(1) \oplus \bar{c}_0^*(T_{\mathcal{X}_{A/A}}(1)). \tag{4.7}$$

Notice

$$\frac{\bar{c}_0^*(T_{\mathcal{X}_A}(1))}{\mathcal{E}(1) \oplus \bar{c}_0^*(T_{\bar{C}_0}(1))} \simeq c_0^*(N_{C_0/X_0}(1)).$$

Thus there is an exact sequence

$$\bar{c}_0^*\left(\frac{T_{\mathcal{X}_A}(1)}{T_{\bar{C}_0}(1)}\right) \quad \rightarrow \quad c_0^*(N_{C_0/X_0}(1)) \quad \rightarrow \quad 0. \tag{4.8}$$

Because $\bar{c}_0^*(\frac{T_{\mathcal{X}_A}(1)}{T_{\bar{C}_0}(1)})$ is generated by global sections, then so is $c_0^*(N_{C_0/X_0}(1))$. This completes the proof. □

5. Application.

COROLLARY 5.1. *Assume X_0 is a smooth hypersurface in $\mathbf{P}^n$ of degree h. Also assume assumption (1.3) in theorem 1.2 holds.*

Then

$$h \leq 2n - 2.$$

Proof. By the isomorphism in theorem 3.1, we have the exact sequence

$$0 \quad \rightarrow \quad \bar{c}_0^*(G(1)) \quad \rightarrow \quad \bar{c}_0^*(T_{\mathcal{X}_A}(1)) \quad \rightarrow \quad c_0^*(T_{\mathbf{P}^n}(1)) \quad \rightarrow \quad 0. \tag{5.1}$$

As before

$$H^1(\bar{c}_0^*G(1)) = 0. \tag{5.2}$$

Hence

$$0 \quad \rightarrow \quad H^0(\bar{c}_0^*G(1)) \quad \rightarrow \quad H^0(\bar{c}_0^*(T_{\mathcal{X}_A}(1))) \quad \rightarrow \quad H^0(c_0^*(T_{\mathbf{P}^n}(1))) \quad \rightarrow \quad 0.$$

Therefore

$$h^0(\bar{c}_0^*(T_{\mathcal{X}_A}(1))) = h^0(c_0^*(T_{\mathbf{P}^n}(1))) + h. \tag{5.3}$$

To calculate $H^0(c_0^*(T_{\mathbf{P}^n}(1)))$, we consider the twisted Euler sequence

$$(5.4) \qquad 0 \to c_0^*(\mathcal{O}_{\mathbf{P}^n}(1)) \to c_0^*(\oplus_{n+1}\mathcal{O}_{\mathbf{P}^n}(2)) \to c_0^*(T_{\mathbf{P}^n}(1)) \to 0.$$

Because $H^1(c_0^*(\mathcal{O}_{\mathbf{P}^n}(1))) = H^0(\mathcal{O}_{\mathbf{P}^1}(-d-2)) = 0$, we obtain

$$0 \to H^0(c_0^*(\mathcal{O}_{\mathbf{P}^n}(1))) \to H^0(c_0^*(\oplus_{n+1}\mathcal{O}_{\mathbf{P}^n}(2))) \to H^0(c_0^*(T_{\mathbf{P}^n}(1))) \to 0.$$

Hence we find

$$\begin{aligned} h^0(c_0^*(T_{\mathbf{P}^n}(1))) &= h^0(c_0^*(\oplus_{n+1}\mathcal{O}_{\mathbf{P}^n}(2))) - h^0(c_0^*(\mathcal{O}_{\mathbf{P}^n}(1))) \\ &= (2d+1)(n+1) - (d+1). \end{aligned} \tag{5.5}$$

To calculate $H^0(\bar{c}_0^*(T_{\mathcal{X}_A}(1)))$, we note that ϕ is onto $\psi(T_{[f_0]}A)$. Then we obtain the decomposition (4.6):

$$\bar{c}_0^*(T_{\mathcal{X}_A}(1)) = \mathcal{E}(1) \oplus c_0^*(T_{X_0}(1)). \tag{5.6}$$

where

$$\mathcal{E}(1) \simeq \oplus_{h+1} c_0^*(\mathcal{O}_{\mathbf{P}^n}(1)) \simeq \oplus_{h+1}\mathcal{O}_{\mathbf{P}^1}(d).$$

Hence

$$h^0(\bar{c}_0^*(T_{\mathcal{X}_A}(1))) = (h+1)(d+1) + h^0(c_0^*(T_{X_0}(1))). \tag{5.7}$$

Combining formulas (5.3), (5.5) and (5.7), we obtain that

$$h^0(c_0^*(T_{X_0}(1))) + (h-2n)d - (n-1) = 0. \tag{5.8}$$

Since

$$\begin{aligned} h^0(c_0^*(T_{X_0}(1))) &= h^0(N_{C_0/X_0}(1)) + h^0\big(T_{\mathbf{P}^1} \otimes c_0^*(\mathcal{O}_{\mathbf{P}^n}(1))\big) \\ &= h^0(N_{C_0/X_0}(1)) + d + 3, \end{aligned} \tag{5.9}$$

formula (5.8) becomes

$$(h-2n+1)d + h^0(N_{C_0/X_0}(1)) - (n-4) = 0. \tag{5.10}$$

To show $h \le 2n-2$, it suffices to prove that

$$h^0(N_{C_0/X_0}(1)) - (n-4) > 0.$$

Applying theorem 1.2, we obtain that

$$h^0(N_{C_0/X_0}(1)) \geq Rank(N_{C_0/X_0}(1)) = n-2.$$

This completes the proof. □

Remark. There are two previously well-known results that are related to corollary 5.1:

(1). H. Clemens proved a theorem in [3], that implies if X_0 is a generic hypersurface containing an immersed rational curve C_0, then

$$deg(X_0) \leq 2n-2.$$

The result later was improved by C. Voisin [9], [10]:

(2) If X_0 is a generic hypersurface containing C_0 which is any rational curve, then

$$deg(X_0) \leq 2n-3, \quad for\ n \geq 4$$

and the equality holds for a line in a generic hypersurface of degree $2n-3$.

Both authors in their papers addressed more general situations.

Our theorem 1.2 is valid for immersed rational curves on hypersurfaces. Then using lemma 2.2 and corollary 5.1, we recover H. Clemens' result mentioned in (1). So corollary 5.1 implies Clemens' result (1), but they are not equivalent. Even though Clemens' bound in corollary 5.1 is worse than Voisin's, it is still sharp under our weaker assumption. Please see the following example 5.2 for this.

Example 5.2 is in the case where assumption (1.3) holds for a specific family A, but it does not hold for all such families A. Example 5.3 is in the case where assumption (1.3) does not hold for any family A[1].

Example 5.2. This example constructs X_0, C_0, A of theorem 1.2 satisfying assumption (1.3), and they further satisfy

(1) $deg(X_0) = 2n-2$ (This is the minimum degree of X_0 by corollary 5.1).

(2) C_0 does not deform to all hypersurfaces to the first order.

(assertion (2) will be proved elsewhere).

Let $x_0, \cdots, x_n$ be homogeneous coordinates for $\mathbf{P}^n$. The construction is based on the break-down of $\mathbf{P}^n$ to smaller subspaces. Let

$$\mathbf{P}^1_{sub} \subset \mathbf{P}^2_{sub} \subset \mathbf{P}^n$$

be subspaces defined as follows:

$$\mathbf{P}^2_{sub} = \{x_0 = \cdots = x_{n-3} = 0\}, \mathbf{P}^1_{sub} = \mathbf{P}^2_{sub} \cap \mathbf{P}^{n-1}_{sub}, \tag{5.11}$$

where

$$\mathbf{P}^{n-1}_{sub} = \{x_n = 0\}. \tag{5.12}$$

In the following, we always regard homogeneous polynomials on subspaces

$$\mathbf{P}^i_{sub}, i = 1, 2, n-1$$

as polynomials of the same degree on $\mathbf{P}^n$, i.e. we use the natural inclusion

$$H^0(\mathcal{O}_{\mathbf{P}^i_{sub}}(r)) \subset H^0(\mathcal{O}_{\mathbf{P}^n}(r)), for\ i = 1, 2, n-1.$$

Next we construct three varieties: X_0, C_0, A in theorem 1.2.

(1) X_0:

Let $g_0, \cdots, g_{n-3}$ be generic sections in $H^0(\mathcal{O}_{\mathbf{P}^{n-1}_{sub}}(2n-3))$.

Let

$$q_1 = \sum_{i=0}^{2} x^2_{n-i} \in H^0(\mathcal{O}_{\mathbf{P}^2_{sub}}(2)). \tag{5.13}$$

[1]It turns out that these two examples cover all situations regarding assumption (1.3), but excluding the assumption: X_0 is generic.

Let $q_2 \in H^0(\mathcal{O}_{\mathbf{P}^n}(2n-4))$ be a section

$$q_2 = x_n^{2(n-2)} + p, \tag{5.14}$$

where $p \in H^0(\mathcal{O}_{\mathbf{P}^{n-1}_{sub}}(2n-4))$ is generic.

Now we construct the smooth hypersurface $X_0 = div(f_0)$ of degree $2n-2$: Let

$$f_0 = q_1 q_2 + \sum_{k=0}^{n-3} x_k g_k. \tag{5.15}$$

Such $X_0 = div(f_0)$ is smooth (see Appendix).

(2) C_0:
Let the rational curve C_0 be

$$C_0 = \{x_0 = \cdots = x_{n-3} = q_1 = 0\}. \tag{5.16}$$

Since C_0 is a smooth plane conic in X_0, it is a smooth rational curve. Let

$$c_0 : \mathbf{P}^1 \to C_0 \subset X_0$$

be an isomorphism.

(3) A:
Let $L_0, \cdots, L_h$ be generic $h+1$ sections in the pencil

$$span(x_{n-2}, x_{n-1}) = H^0(\mathcal{O}_{\mathbf{P}^1_{sub}}(1)) \subset H^0(\mathcal{O}_{\mathbf{P}^n}(1)),$$

where $h = 2n-2$. Then the condition (1.2) is satisfied, i.e.,

$$\{L_i = 0\} \cap \{L_j = 0\} \cap C_0 = \emptyset, \quad i \neq j.$$

Let $A \subset H^0(\mathcal{O}_{\mathbf{P}^n}(h))$ be constructed with $L_j, j = 0, \cdots, h$ as in definition 1.1.

For fixed generic q_2, q_3, g_k, corresponding maps ψ and ϕ in theorem 1.2 satisfy

$$\psi(A) \subset image(\phi). \tag{5.17}$$

See Appendix for the proof of it. The example shows our bound $2n-2$ is the sharp bound under assumption (1.3). Together with Voisin's result (2), this indicates assumption (1.3) is strictly weaker than the classical assumption: X_0 is generic.

Example 5.3. Let's consider the lines in the Fermat quintic threefold. Let $x_0, \cdots, x_4$ be the homogeneous coordinates for $\mathbf{P}^4$. We consider f_0 to be the Fermat quintic

$$x_0^5 + x_1^5 + \cdots + x_4^5.$$

Let C_0 be the line in $\mathbf{P}^4$ connecting two points

$$[1, -1, 0, 0, 0], [0, 0, a_1, a_2, a_3]$$

with $a_1^5 + a_2^5 + a_3^5 = 0$. Then $C_0 \subset X_0 = div(f_0)$. We can find

$$N_{C_0} X_0 \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(3).$$

Corollary 1.3 says that if assumption (1.3) holds, then

$$N_{C_0} X_0 \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1).$$

(because $d = 1$). This contradiction says (C_0, f_0) does not deform to all hypersurfaces in A to the first order, i.e. assumption (1.3) does not hold. This result is stronger than Albano and Katz's result, Prop. 2.1, in [1], which says (C_0, f_0) does not globally deform to all hypersurfaces.

Appendix. (1) X_0 is smooth. To see that, we specialize it at

$$g_k = x_k^{2n-3}, g_0 = q_3 + x_0^{2n-3}, q_2 = x_n^{2n-4}$$

where

$$q_3 \in H^0(\mathcal{O}_{\mathbf{P}^1}(2n-3))$$

is generic. Then if $z \in X_0$ is a singular point, it must satisfy

$$\begin{cases} x_1 = \cdots = x_{n-3} = 0, & (\frac{\partial f_0}{\partial x_k}|_z = 0, k = 1, \cdots, n-3)) \\ \frac{\partial (q_1 q_2 + x_0(x_0^{2n-3} + q_3))}{\partial \alpha}|_z = 0, & (\alpha \in T_q\mathbb{C}^4,\ \mathbb{C}^4 = (x_0, x_n, x_{n-1}, x_{n-2})) \\ (q_1 q_2 + x_0(x_0^{2n-3} + q_3))|_z = 0, & (f_0(z) = 0). \end{cases}$$

It is easy to see such point z must be a zero of

$$q_1 = x_n = x_0 = q_3 = 0, \quad and\ x_0 = \cdots = x_{n-3} = 0$$

which does not exist for generic q_3. Hence X_0 is smooth in this case. Now moving q_2, g_k to generic sections, we obtain that if all

$$q_2, g_k, k = 0, \cdots, n-3$$

are generic, X_0 is a smooth hypersurface.

(2) Proof of (5.17). The general idea follows from the direct calculation. We compute $image(\phi)$ to be the set

$$(\sum_{i=0}^{2n-2} z_i x_{n-1}^i x_{n-2}^{2n-2-i})|_{C_0}, for\ all\ complex\ numbers\ z_i$$

and $image(\nu^s)$ is the set

$$(\sum_{i=0}^{n} a_i(x_0, \cdots, x_n) \frac{\partial f_0}{\partial x_i})|_{C_0}, for\ all\ linear\ forms\ a_i(x_0, \cdots, x_n).$$

Assumption (1.3) says the former set is contained in the latter. We would like to show the former set of polynomials in x_{n-1}, x_{n-2} (not evaluated at C_0) is already contained in the latter set of polynomials (before evaluated at C_0). The following is the detailed computation.

First we'll build a diagram (5.21) below by the following maps: Let ν^s:

$$\begin{array}{ccc} H^0(c_0^*(T_{\mathbf{P}^n})) & \stackrel{\nu^s}{\to} & H^0(\mathcal{O}_{\mathbf{P}^1}(2h)) \\ [\alpha_0, \cdots, \alpha_n] & \to & \sum_{i=0}^n \alpha_i c_0^*(\frac{\partial f_0}{\partial x_i}). \end{array} \tag{5.18}$$

where $\alpha_i \in H^0(\mathcal{O}_{\mathbf{P}^1}(2))$ and $[\alpha_0, \cdots, \alpha_n]$ represents a vector in

$$H^0(c_0^*(T_{\mathbf{P}^n})) \simeq M_2.$$

Let μ^s:

$$\begin{array}{ccc} T_{[f_0]}A & \stackrel{\mu^s}{\to} & H^0(\mathcal{O}_{\mathbf{P}^1}(2h)) \\ \frac{\partial}{\partial a} & \to & -c_0^*(\frac{\partial F}{\partial a}), \end{array} \tag{5.19}$$

where F is the universal polynomial of the family of hypersurfaces determined by A (defined at the beginning of section 3). Let P_1^s be the obvious projection map

$$H^0(\bar{c}_0^*(T_{\mathcal{X}_A})) \stackrel{P_1^s}{\to} H^0(c_0^*(T_{\mathbf{P}^n})). \tag{5.20}$$

Then all these maps fit into the commutative diagram

$$\begin{array}{ccc} H^0(\bar{c}_0^*(T_{\mathcal{X}_A})) & \stackrel{P_A^s}{\to} & T_{[f_0]}A \\ \downarrow P_1^s & & \downarrow \mu^s \\ H^0(c_0^*(T_{\mathbf{P}^n})) & \stackrel{\nu^s}{\to} & H^0(\mathcal{O}_{\mathbf{P}^1}(2h)). \end{array} \tag{5.21}$$

By theorem 2.1, we need to show the surjectivity of P_A^s. This commutative diagram (5.21) shows that it suffices to show

$$image(\mu^s) \subset image(\nu^s). \tag{5.22}$$

Note

$$A = H^0(\mathcal{O}_{\mathbf{P}^1_{sub}}(h)) \subset H^0(\mathcal{O}_{\mathbf{P}^n}(h)),$$

where the inclusion is obtained by the natural extension of the sections of $\mathcal{O}_{\mathbf{P}^1_{sub}}(h)$ to $\mathcal{O}_{\mathbf{P}^n}(h)$ (as indicated before). By the definition of μ^s,

$$\mu^s(T_{[f_0]}A) = c_0^*(H^0(\mathcal{O}_{\mathbf{P}^1_{sub}}(h))). \tag{5.23}$$

Next we would like to see that the image of ν^s contains all polynomials in

$$c_0^*(H^0(\mathcal{O}_{\mathbf{P}^1_{sub}}(h))).$$

To see this, we'll build another diagram (5.27) below in the following way. First observe the natural inclusion:

$$H^0(\mathcal{O}_{\mathbf{P}^1_{sub}}(1)) \subset H^0(\mathcal{O}_{\mathbf{P}^n}(1)).$$

Then notice

$$H^0(\mathcal{O}_{\mathbf{P}^1_{sub}}(1))\frac{\partial}{\partial x_j}, j = 0, \cdots, n-1$$

are 2 dimensional sub-spaces of global sections of the bundle $T_{\mathbf{P}^n}$. Let

$$\mathcal{H} = \oplus_{j=0}^{n-1} H^0(\mathcal{O}_{\mathbf{P}^1_{sub}}(1))\frac{\partial}{\partial x_j} \subset H^0(T_{\mathbf{P}^n}). \tag{5.24}$$

Then $dim(\mathcal{H}) = 2n$.

There is a homomorphism

$$\begin{array}{ccc} \mathcal{H} & \stackrel{\xi}{\to} & H^0(\mathcal{O}_{\mathbf{P}^n}(h)) \\ \sum_{j=0}^{n-1} l_j(x_{n-1}, x_{n-2})\frac{\partial}{\partial x_j} & \to & \sum_{j=0}^{n-1} l_j(x_{n-1}, x_{n-2})\frac{\partial f_0}{\partial x_j}. \end{array} \tag{5.25}$$

Let Pr be the morphism of linear spaces

$$\oplus_{r=0}^{\infty} H^0(\mathcal{O}_{\mathbf{P}^n}(r)) \to \oplus_{r=0}^{\infty} H^0(\mathcal{O}_{\mathbf{P}^1_{sub}}(r)). \tag{5.26}$$

(Pr will not be the natural pullback map). To describe Pr, it suffices to define it on a basis. Let $(x_n)^m G$ be the monomials in $x_0, \cdots, x_n$ that form a basis for $H^0(\mathcal{O}_{\mathbf{P}^n}(r))$ under the $x_0, \cdots, x_n$ coordinates, where $G \in H^0(\mathcal{O}_{\mathbf{P}^{n-1}_{sub}}(r-m))$. Define

$$Pr((x_n)^m G) = 0, \quad if\ m\ is\ odd,$$

$$Pr((x_n)^m G) = \Big(-(x_{n-1}^2 + x_{n-2}^2)\Big)^{m/2} \tilde{G}, \quad if\ m\ is\ even,$$

where $\tilde{G} = G|_{x_0=\cdots=x_{n-3}=0}$. These maps form a diagram

$$\begin{array}{ccccc} \mathcal{H} & \stackrel{\xi}{\to} & H^0(\mathcal{O}_{\mathbf{P}^n}(h)) & = & H^0(\mathcal{O}_{\mathbf{P}^n}(h)) \\ \downarrow c_0^* & & & & \downarrow Pr \\ H^0(c_0^*(T_{\mathbf{P}^n})) & \stackrel{\nu^s}{\to} & H^0(\mathcal{O}_{\mathbf{P}^1}(2h)) & \stackrel{c_0^*}{\leftarrow} & H^0(\mathcal{O}_{\mathbf{P}^1_{sub}}(h)). \end{array} \tag{5.27}$$

that is commutative because of the choices of q_2 and g_k(No monomials of x_n-factor in g_k and only a monomial of x_n^{even}-factor in q_2).[2]

Our observation is that $Pr \circ \xi$ is surjective, because the image of the composition map $Pr \circ \xi$ is just

$$\sum_{i=1}^{2} l_{n-i}(x_{n-1}, x_{n-2}) Pr(\frac{\partial q_1}{\partial x_{n-i}} q_2) + \sum_{k=0}^{n-3} l_k(x_{n-1}, x_{n-2}) Pr(g_k),$$

(a polynomial in x_{n-1}, x_{n-2} only), where

$$\sum_{j=0}^{n-1} l_i \frac{\partial}{\partial x_j} \in \mathcal{H}.$$

[2]Notice in the diagram (5.27), the vertical c_0^* is different from the horizontal c_0^* because they are the pull-backs of different vector bundles over $\mathbf{P}^n$. Sorry for the abusing of notations.

Because all q_2, g_k are generic,

$$Pr(q_2), Pr(g_k)$$

are generic sections of $\mathcal{O}_{\mathbf{P}^1_{sub}}(2n-4)$ and $\mathcal{O}_{\mathbf{P}^1_{sub}}(2n-3)$ respectively.

Hence for the fixed q_2, g_k, the equation

$$\sum_{i=1}^{2} l_{n-i}(x_{n-1}, x_{n-2}) Pr(\frac{\partial q_1}{\partial x_{n-i}} q_2) + \sum_{k=0}^{n-3} l_k(x_{n-1}, x_{n-2}) Pr(g_k) = 0,$$

implies

$$\sum_{i=1}^{2} l_{n-i}(x_{n-1}, x_{n-2}) \frac{\partial q_1}{\partial x_{n-i}} = 0$$

and $l_k = 0, k = 0, \cdots, n-3$. Thus the kernel of $Pr \circ \xi$ is

$$\{(l_0, \cdots, l_{n-1}) : \sum_{i=1}^{2} l_{n-i}(x_{n-1}, x_{n-2}) \frac{\partial q_1}{\partial x_{n-i}} = 0, l_0 = \cdots = l_{n-3} = 0\}$$

which clearly has dimension 1. Thus $dim(image(Pr \circ \xi)) = 2n-1$, which is the dimension of $H^0(\mathcal{O}_{\mathbf{P}^1}(h))$. Hence $Pr \circ \xi$ is surjective (onto $H^0(\mathcal{O}_{\mathbf{P}^1_{sub}}(h))$). Thus

$$image(\nu^s) \supset image(\nu^s \circ c_0^*) = c_0^*(H^0(\mathcal{O}_{\mathbf{P}^1_{sub}}(h))).$$

By formula (5.23), we proved (5.17)

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