

## A NEW PROOF OF ALMOST ISOMETRY THEOREM IN ALEXANDROV GEOMETRY WITH CURVATURE BOUNDED BELOW\*

YUSHENG WANG<sup>†</sup>, XIAOLE SU<sup>‡</sup>, AND HONGWEI SUN<sup>§</sup>

**Abstract.** In this paper we give a new proof (along the line of the original proof) for the almost isometry theorem in Alexandrov geometry with curvature bounded below in [1]. The motivation of the new proof is that we find that Lemma 9.11 in [1] is incorrect (see Example 1.3 below), while this lemma is a crucial step in the original proof.

**Key words.** Alexandrov spaces, GH-approximation, almost isometry.

**AMS subject classifications.** 53-C20.

In the seminal paper [1] on Alexandrov geometry with curvature bounded below, the almost isometry theorem (see Theorem 0.1 below) asserts (roughly) that if the Gromov-Hausdorff distance between two compact  $n$ -dimensional Alexandrov spaces with the same low curvature bound is sufficiently small and if one of them contains only  $(n, \delta)$ -strained points, then there is an almost isometry  $\bar{h}$  between the two spaces. We think that, in the original proof of the almost isometry theorem, Lemma 9.11 in [1] (Lemma 1.2 below) is incorrect (see Example 1.3 below). Precisely, the condition (1.1) of Lemma 9.11 is inadequate for its conclusion. An important part of the original proof of the almost isometry theorem is to verify that  $\bar{h}$  satisfies (1.1). In our new proof we replace the condition (1.1) with (2.3), and show that the conclusion of Lemma 9.11 still holds (see Lemma 2.1 below). Moreover, additional work is required to verify that  $\bar{h}$  satisfies (2.3) (via locally constructing an almost isometry using Lemma 2.1, and proving that the local map is sufficiently close to  $\bar{h}$ ).

**0. Notations and main theorem.** We first collect some notations almost all of which come from [1].

- $|xy|$  always denotes the distance between two points  $x$  and  $y$  in a metric space.
- For any three points  $p, q, r$  in a length space, we associate a triangle  $\Delta\tilde{p}\tilde{q}\tilde{r}$  on the  $k$ -plane (2-dimensional complete and simply-connected Riemannian manifold of constant curvature  $k$ ) with  $|\tilde{p}\tilde{q}| = |pq|$ ,  $|\tilde{p}\tilde{r}| = |pr|$  and  $|\tilde{r}\tilde{q}| = |rq|$ . For  $k \leq 0$  and for  $k > 0$  with  $|pq| + |pr| + |qr| \leq 2\pi/\sqrt{k}$ , such a triangle always exists. We denote by  $\tilde{\angle}pqr$  the angle of the triangle  $\Delta\tilde{p}\tilde{q}\tilde{r}$  at vertex  $\tilde{q}$ .
- $M$  always denotes an Alexandrov space with curvature bounded below by  $k$ , which is a length space and in which there exists a neighborhood  $U_x$  of any  $x \in M$  such that for any four (distinct) points  $(a; b, c, d)$  in  $U_x$

$$\tilde{\angle}bac + \tilde{\angle}bad + \tilde{\angle}cad \leq 2\pi.$$

- A point  $p \in M$  is called an  $(n, \delta)$ -strained point if there are  $n$  pairs of points

---

\*Received February 5, 2012; accepted for publication July 26, 2012.

<sup>†</sup>Corresponding author. School of Mathematical Sciences (and Lab. Math. Com. Sys.), Beijing Normal University, Beijing, 100875 P.R.C. (wwyusheng@gmail.com). Supported by NSFC 11001015 and 11171025.

<sup>‡</sup>School of Mathematical Sciences (and Lab. Math. Com. Sys.), Beijing Normal University, Beijing, 100875 P.R.C. (suxiaole@bnu.edu.cn). Supported by NSFC 10801011 and 11171025.

<sup>§</sup>School of Mathematical Sciences, Capital Normal University, Beijing, 100037 P.R.C. (hwsun@bnu.edu.cn).

distinct from  $p$ ,  $(a_i, b_i)$ , such that for  $i \neq j$

$$\begin{aligned} \tilde{Z}a_i p b_i &> \pi - \delta, \quad \tilde{Z}a_i p a_j > \pi/2 - \delta, \\ \tilde{Z}a_i p b_j &> \pi/2 - \delta, \quad \tilde{Z}b_i p b_j > \pi/2 - \delta. \end{aligned}$$

$\{(a_i, b_i)\}_{i=1}^n$  is called an  $(n, \delta)$ -strainer at  $p$  (which is obviously a generalization of a coordinate frame). We say that the  $(n, \delta)$ -strainer  $\{(a_i, b_i)\}_{i=1}^n$  at  $p$  is  $R$ -long if  $|a_i p| > \frac{R}{\delta}$  and  $|b_i p| > \frac{R}{\delta}$  for all  $i$ . And we denote by  $M(n, \delta, R)$  the set of points with  $R$ -long  $(n, \delta)$ -strainer in  $M$ .

• An important fact is that if any neighborhood of a point  $p \in M$  contains an  $(n, \delta)$ -strained point ( $\delta$  is sufficiently small) but no  $(n + 1, \delta)$ -strained point, then any neighborhood of any other point in  $M$  has the same property (see §6 in [1]). And it follows that the dimension of such  $M$  is defined to be  $n$ .

• We always denote by  $\varkappa(\cdot)$  or  $\varkappa(\cdot, \cdot)$  (resp.  $C$ ) a positive function which is infinitesimal at zero (e.g.  $\varkappa(\delta, \delta_1) \rightarrow 0$  as  $\delta, \delta_1 \rightarrow 0$ ) (resp. a constant depending only on  $n$ ); however we do not distinguish any two  $\varkappa$ -functions with the same parameters (resp. any two such constants) when we use them.

• A map  $f$  between metric spaces  $(X, d_1)$  and  $(Y, d_2)$  is called a  $\text{GH}_\epsilon$ -approximation if  $B_\epsilon(f(X)) = Y$  and  $|d_2(f(x_1), f(x_2)) - d_1(x_1, x_2)| < \epsilon$  for any  $x_1, x_2 \in X$ .

•  $f : (X, d_1) \rightarrow (Y, d_2)$  is called a  $\varkappa(\delta)$ -almost distance preserving map if

$$\left| 1 - \frac{|f(x)f(y)|}{|xy|} \right| < \varkappa(\delta) \text{ for any } x \neq y \in X;$$

and if in addition  $f$  is a bijection,  $f$  is called a  $\varkappa(\delta)$ -almost isometry.

• We say that  $\bar{f} : (X, d_1) \rightarrow (Y, d_2)$  is  $\nu$ -close to  $f$  if  $|f(x)\bar{f}(x)| < \nu$  for any  $x \in X$ .

Now we recall the almost isometry theorem stated in [1].

**THEOREM 0.1** (THEOREM 9.8 IN [1]). *Let  $M_1$  and  $M_2$  be two compact  $n$ -dimensional Alexandrov spaces with the same low curvature bound, and let  $h : M_1 \rightarrow M_2$  be a  $\text{GH}_\nu$ -approximation. Then for sufficiently small  $\delta$  and  $\frac{\nu}{R\delta^3}$ , there exists a  $\varkappa(\delta, \frac{\nu}{R\delta^3})$ -almost distance preserving map  $\bar{h} : M_1(n, \delta, R) \rightarrow M_2$  which is  $C\nu$ -close to  $h$ .*

Note that  $\bar{h}$  is a  $\varkappa(\delta, \frac{\nu}{R\delta^3})$ -almost isometry between  $M_1(n, \delta, R)$  and  $\bar{h}(M_1(n, \delta, R))$ . And it is not difficult to see that Theorem 0.1 implies the following corollary.

**COROLLARY 0.2** ([1]). *In Theorem 0.1 if, in addition, each point of  $M_2$  is  $(n, \delta)$ -strained, then there exists a  $\varkappa(\delta, \nu)$ -almost isometry  $\bar{h} : M_1 \rightarrow M_2$  which is  $C\nu$ -close to  $h$ .*

Theorem 0.1 (and Corollary 0.2) is of importance in connection with the Gromov-Hausdorff convergence of a sequence of  $n$ -dimensional Alexandrov spaces with the same low curvature bound.

In this paper we give a proof of the following sharpened version of Theorem 0.1.

**THEOREM A.** *Let  $M_1$  and  $M_2$  be two compact  $n$ -dimensional Alexandrov spaces with the same low curvature bound, and let  $h : M_1 \rightarrow M_2$  be a  $\text{GH}_\nu$ -approximation.*

Then for sufficiently small  $\delta$  and  $\nu < \delta^2 R$ , there exists a  $\varkappa(\delta)$ -almost distance preserving map  $\bar{h} : M_1(n, \delta, R) \rightarrow M_2$  which is  $C\nu$ -close to  $h$ .

Note that in Theorem 0.1  $\nu \ll R\delta^3$ , while we improve that to  $\nu < R\delta^2$  in Theorem A. This is obtained by sharpening the corresponding estimates in [1].

The proof of Theorem A follows the line of the original proof of Theorem 0.1 in [1]. We first construct  $\bar{h}$  (see Section 3) which is the same as in the original proof, and then verify that  $\bar{h}$  is an almost distance preserving map (see Section 4). The main difference between our proof and the original proof is in the latter part.

REMARK 0.3. Yamaguchi has proved an almost Lipschitz submersion theorem ([2]), which also implies Corollary 0.2. In [2], the basic idea to construct the almost Lipschitz submersion is to embed an Alexandrov space  $M$  with curvature bounded below into  $L^2(M)$ . Compared with it, the basic idea to construct  $\bar{h}$  in Theorem 0.1 (and A) is to embed  $M_1(n, \delta, R)$  locally and almost isometrically into the  $n$ -dimensional Euclidean space (see Theorem 1.1 below).

**1. Center of mass and a lemma in [1].** In Theorem 0.1 (and A), the main tool in the construction of  $\bar{h}$  is *center of mass* ([1]). Recall that the center of mass of a point set  $Q = \{q_1, q_2, \dots, q_l\} \subset \mathbb{R}^n$  with weights  $W = (w_1, w_2, \dots, w_l)$  (where  $\sum_{j=1}^l w_j = 1$  and  $w_j \geq 0$ ) is defined to be

$$Q_W = \sum_{j=1}^l w_j q_j.$$

The construction of the center of mass for a point set in  $M$  is based on the following important result.

THEOREM 1.1 (Theorem 9.4 in [1]). *Let  $M$  be an  $n$ -dimensional Alexandrov space with curvature bounded below, and let  $\{(a_i, b_i)\}_{i=1}^n$  be an  $(n, \delta)$ -strainer at  $p \in M$ . Then there exist neighborhoods  $U$  and  $V$  of  $p$  and  $(|a_1 p|, |a_2 p|, \dots, |a_n p|) \in \mathbb{R}^n$  respectively such that*

$$f : U \rightarrow V \subset \mathbb{R}^n \text{ defined by } f(q) = (|a_1 q|, |a_2 q|, \dots, |a_n q|)$$

*is a  $\varkappa(\delta, \delta_1)$ -almost isometry, where  $\delta_1 = \max_{1 \leq i \leq n} \{|pa_i|^{-1}, |pb_i|^{-1}\} \cdot \text{diam } U$ .*

In fact, from the proof of Theorem 1.1 (p.35 and 19 in [1]) we can conclude the following refined version of Theorem 1.1.

THEOREM 1.1'. *Let  $M$  be an  $n$ -dimensional Alexandrov space with curvature bounded below, and let  $\{(a_i, b_i)\}_{i=1}^n$  be an  $(n, \delta)$ -strainer at  $p \in M$ . Then there exists a constant  $c$  such that if a neighborhood  $U$  of  $p$  satisfies  $\max_{1 \leq i \leq n} \{|pa_i|^{-1}, |pb_i|^{-1}\} \cdot \text{diam } U < c\delta$ , then  $f : M \rightarrow \mathbb{R}^n$  defined by  $f(q) = (|a_1 q|, |a_2 q|, \dots, |a_n q|)$  maps  $U$   $\varkappa(\delta)$ -isometrically to a domain in  $\mathbb{R}^n$ .*

In Theorem 1.1', the constant  $c$  depends only on the dimension  $n$ , the low bound  $k$  of the curvature, and the length  $\ell \triangleq \min_{1 \leq i \leq n} \{|pa_i|, |pb_i|\}$  if  $k < 0$  (the larger  $\ell$  is, the smaller  $c$  is).

In Theorem 1.1 (and 1.1'), if  $f(U)$  is convex in  $\mathbb{R}^n$ , then the center of mass of  $Q = \{q_1, q_2, \dots, q_l\} \subset U$  with weights  $W$  is defined to be ([1])

$$Q_W = f^{-1} \left( \sum_{j=1}^l w_j f(q_j) \right).$$

Obviously,  $Q_W$  depends on the choice of the  $(n, \delta)$ -strainer at  $p$ .

Now we formulate the lemma in [1] mentioned at the beginning of the paper.

LEMMA 1.2 (Lemma 9.11 in [1]). *Let  $p, U, \{(a_i, b_i)\}_{i=1}^n$  and  $f$  be the same as in Theorem 1.1, and let  $\{(s_i, t_i)\}_{i=1}^n$  be another  $(n, \delta)$ -strainer at  $p$  with*

$$\delta_1 = \max \left\{ \frac{\text{diam } U}{\min_i \{|pa_i|, |pb_i|\}}, \frac{\max_i \{|pa_i|, |pb_i|\}}{\min_i \{|ps_i|, |pt_i|\}} \right\}.$$

And let  $Q = \{q_1, \dots, q_l\}$  and  $R = \{r_1, \dots, r_l\}$  be two point sets in  $U$  with

$$\max_j |q_j r_j| < (1 + \delta) \min_j |q_j r_j| \text{ and } |\max_j \tilde{Z}_{s_i} q_j r_j - \min_j \tilde{Z}_{s_i} q_j r_j| < \delta \text{ for } i = 1, \dots, n.$$

Assume that  $f(U)$  is convex in  $\mathbb{R}^n$ . Then for any weights  $W^1$  and  $W^2$  satisfying

$$(1.1) \quad \|W^1 - W^2\| < \delta_1,$$

the centers of mass  $Q_{W^1}$  and  $R_{W^2}$  (with respect to the strainer  $\{(a_i, b_i)\}$ ) satisfy

$$\left| 1 - \frac{|q_j r_j|}{|Q_{W^1} R_{W^2}|} \right| < \varkappa(\delta, \delta_1)$$

and  $|\tilde{Z}_{s_i} q_j r_j - \tilde{Z}_{s_i} Q_{W^1} R_{W^2}| < \varkappa(\delta, \delta_1)$  for  $j = 1, \dots, l$  and  $i = 1, \dots, n$ .

The following counter example shows that this lemma is incorrect.

EXAMPLE 1.3. 1. In fact, if  $q_j = r_j$  for  $j = 1, \dots, l$  and  $W^1 \neq W^2$ , then  $Q_{W^1} \neq R_{W^2}$  and thus

$$\left| 1 - \frac{|q_j r_j|}{|Q_{W^1} R_{W^2}|} \right| = |1 - 0| > \varkappa(\delta, \delta_1).$$

2. If  $|q_j r_j| \ll \|W^1 - W^2\|$  for all  $j$ , then “ $\left| 1 - \frac{|q_j r_j|}{|Q_{W^1} R_{W^2}|} \right| < \varkappa(\delta, \delta_1)$ ” does not hold.

As indicated in the example, the condition (1.1) is inadequate for Lemma 1.2 to hold.

**2. A modified lemma.** In this section we will prove a modified version of Lemma 1.2.

LEMMA 2.1. *Let  $p, U, \{(a_i, b_i)\}_{i=1}^n$  and  $f$  be the same as in Theorem 1.1', and let  $\{(s_i, t_i)\}_{i=1}^n$  be another  $(n, \delta)$ -strainer at  $p$  with*

$$\frac{\max_i \{|pa_i|, |pb_i|\}}{\min_i \{|ps_i|, |pt_i|\}} < \varkappa(\delta).$$

And let  $Q = \{q_1, \dots, q_l\}$  and  $R = \{r_1, \dots, r_l\}$  be two point sets in  $U$ . Assume that  $f(U)$  is convex in  $\mathbb{R}^n$ , and assume that

$$(2.1) \quad \max_j |q_j r_j| < (1 + \varkappa(\delta)) \min_j |q_j r_j| \text{ and}$$

$$(2.2) \quad |\max_j \tilde{Z}_{s_i} q_j r_j - \min_j \tilde{Z}_{s_i} q_j r_j| < \varkappa(\delta) \text{ for } i = 1, \dots, n.$$

Then for any weights  $W^1$  and  $W^2$  satisfying

$$(2.3) \quad \|W^1 - W^2\| < \frac{\min_j \{|q_j r_j|\}}{\max_{j,j'} \{|r_j r_{j'}|\}} \varkappa(\delta),$$

the centers of mass  $Q_{W^1}$  and  $R_{W^2}$  (with respect to the strainer  $\{(a_i, b_i)\}$ ) satisfy

$$\left| 1 - \frac{|q_j r_j|}{|Q_{W^1} R_{W^2}|} \right| < \varkappa(\delta) \text{ and}$$

$$|\tilde{Z}_{s_i} q_j r_j - \tilde{Z}_{s_i} Q_{W^1} R_{W^2}| < \varkappa(\delta) \text{ for } j = 1, \dots, l \text{ and } i = 1, \dots, n.$$

REMARK 2.2. The main change from Lemma 1.2 to Lemma 2.1 is that the condition (1.1) turns into (2.3). Under the condition (2.3), the smaller  $|q_j r_j|$  is, the smaller  $\|W^1 - W^2\|$  is (in particular, if  $|q_j r_j| = 0$ , then  $W^1 = W^2$  and thus  $Q_{W^1} = R_{W^2}$ ). In the proof of Theorem A, the essential difficulty in verifying that  $\bar{h}$  is an almost distance preserving map is to solve the case that  $|q_j r_j|$  is sufficiently small (see the second paragraph and the proof of Claim 1 in Section 4).

REMARK 2.3. From the original proof of Lemma 1.2 in [1], we can conclude that the following two statements are equivalent:

- (1)  $|\tilde{Z}_{a_i} q_j r_j - \tilde{Z}_{a_i} q_{j'} r_{j'}| < \varkappa(\delta)$  for  $i = 1, \dots, n$ ;
- (2)  $|\tilde{Z}_{s_i} q_j r_j - \tilde{Z}_{s_i} q_{j'} r_{j'}| < \varkappa(\delta)$  for  $i = 1, \dots, n$ .

For the convenience of readers we give its proof in Appendix.

In the proof of Lemma 2.1, we will use Lemmas 2.4 and 2.6.

LEMMA 2.4. Let  $Q = \{q_1, \dots, q_l\}$  and  $R = \{r_1, \dots, r_l\}$  be two point sets in  $\mathbb{R}^n$ , and let  $W^i = (w_1^i, \dots, w_l^i)$  be two weights with  $i = 1, 2$ . Then

$$\overrightarrow{Q_{W^1} R_{W^2}} = \sum_{j=1}^l w_j^1 \overrightarrow{q_j r_j} + \sum_{j=1}^l (w_j^2 - w_j^1) \overrightarrow{r_{j_0} r_j}$$

for any  $j_0 \in \{1, 2, \dots, l\}$ .

*Proof.* Straightforward computation gives

$$\begin{aligned} \overrightarrow{Q_{W^1} R_{W^2}} &= \sum_{j=1}^l w_j^2 r_j - \sum_{j=1}^l w_j^1 q_j \\ &= \sum_{j=1}^l w_j^1 (r_j - q_j) + \sum_{j=1}^l (w_j^2 - w_j^1) r_j \\ &= \sum_{j=1}^l w_j^1 \overrightarrow{q_j r_j} + \sum_{j=1}^l (w_j^2 - w_j^1) r_j - \sum_{j=1}^l (w_j^2 - w_j^1) r_{j_0} \\ &= \sum_{j=1}^l w_j^1 \overrightarrow{q_j r_j} + \sum_{j=1}^l (w_j^2 - w_j^1) \overrightarrow{r_{j_0} r_j}. \quad \square \end{aligned}$$

In order to simplify further discussions, we introduce the following definition.

DEFINITION 2.5. Given point sets  $Q = \{q_1, q_2\}$  and  $R = \{r_1, r_2\}$  in  $\mathbb{R}^n$ , we say that  $\overrightarrow{q_1 r_1}$  is  $\varkappa(\delta)$ -almost parallel to  $\overrightarrow{q_2 r_2}$  if

$$\angle(\overrightarrow{q_1 r_1}, \overrightarrow{q_2 r_2}) < \varkappa(\delta);$$

and if, in addition,

$$\left| 1 - \frac{|q_1 r_1|}{|q_2 r_2|} \right| < \varkappa(\delta),$$

we say that  $\overrightarrow{q_1 r_1}$  is  $\varkappa(\delta)$ -almost equal to  $\overrightarrow{q_2 r_2}$ .

LEMMA 2.6. Let  $p, U, \{(a_i, b_i)\}_{i=1}^n$  and  $f$  be the same as in Lemma 2.1. Then for any points  $x_1, x_2, y_1, y_2 \in U$ , the following two statements are equivalent:

- (1)  $|\angle a_i x_1 y_1 - \angle a_i x_2 y_2| < \varkappa(\delta)$  for  $i = 1, 2, \dots, n$ ;
- (2)  $\overrightarrow{f(x_1)f(y_1)}$  is  $\varkappa(\delta)$ -almost parallel to  $\overrightarrow{f(x_2)f(y_2)}$ .

Lemma 2.6 can be obtained from the original proof of Theorem 0.1 in [1] (we will give its proof in Appendix).

*Proof of Lemma 2.1.* Note that inequalities (2.1) and (2.2) imply that  $\overrightarrow{f(q_j)f(r_j)}$  are  $\varkappa(\delta)$ -almost equal each other for  $j = 1, \dots, l$  (see Remark 2.3, Lemma 2.6 and Theorem 1.1). It then follows from Lemma 2.4 that  $\overrightarrow{f(Q_{W^1})f(R_{W^2})}$  is  $\varkappa(\delta)$ -almost equal to  $\overrightarrow{f(q_j)f(r_j)}$  for every  $j$  (note that  $f(Q_{W^1}) = \sum_{j=1}^l w_j^1 f(q_j)$  and  $f(R_{W^2}) = \sum_{j=1}^l w_j^2 f(r_j)$ , and  $\|W^1 - W^2\| \cdot \max_{j, j'} \{|r_j r_{j'}|\} < \varkappa(\delta) \min_j \{|q_j r_j|\}$ ). Hence, the conclusion of Lemma 2.1 follows from Lemma 2.6, Remark 2.3 and the fact that  $f$  is a  $\varkappa(\delta)$ -almost isometry.  $\square$

We end this section with a corollary of Remark 2.3 and Lemma 2.6.

COROLLARY 2.7. Let  $p, U, \{(a_i, b_i)\}_{i=1}^n, f$  and  $\{(s_i, t_i)\}_{i=1}^n$  be the same as in Lemma 2.1. Let  $\{(a'_i, b'_i)\}_{i=1}^n$  be an  $(n, \delta)$ -strainer at another point  $p'$ , and let  $f'$  and  $U'$  be the map and the neighborhood of  $p'$  determined by Theorem 1.1' (with respect to  $\{(a'_i, b'_i)\}$ ). Moreover we assume that  $\{(s_i, t_i)\}_{i=1}^n$  is also an  $(n, \delta)$ -strainer at  $p'$  with

$$\frac{\max_i \{|p' a'_i|, |p' b'_i|\}}{\min_i \{|p' s_i|, |p' t_i|\}} < \varkappa(\delta).$$

Then for any points  $x_1, x_2, y_1, y_2 \in U_1 \cap U_2$ , the following two statements are equivalent:

- (1)  $\overrightarrow{f(x_1)f(y_1)}$  is  $\varkappa(\delta)$ -almost equal to  $\overrightarrow{f(x_2)f(y_2)}$ ;
- (2)  $\overrightarrow{f'(x_1)f'(y_1)}$  is  $\varkappa(\delta)$ -almost equal to  $\overrightarrow{f'(x_2)f'(y_2)}$ .

**3. Construction of  $\bar{h}$ .** In this section we construct the  $\bar{h}$  in Theorems 0.1 and A (ref. [1]).

Since the closure of  $M_1(n, \delta, R)$  is compact, we can select  $x_j \in M_1(n, \delta, R)$  with  $j = 1, \dots, N_1$  such that

$$(3.0) \quad \bigcup_{j=1}^{N_1} B_{x_j}(\delta R) \supset \bigcup_{j=1}^{N_1} B_{x_j}(\frac{1}{3}\delta R) \supset M_1(n, \delta, R).$$

Since  $x_j \in M_1(n, \delta, R)$ , there exists an  $R$ -long  $(n, \delta)$ -strainer  $\{(s_i^j, t_i^j)\}_{i=1}^n$  at  $x_j$  (with  $\min_i\{|x_j s_i^j|, |x_j t_i^j|\} > \frac{R}{\delta}$ ), and thus there exists ([1]) an  $(n, \delta)$ -strainer  $\{(a_i^j, b_i^j)\}_{i=1}^n$  at  $x_j$  with  $\min_i\{|x_j a_i^j|, |x_j b_i^j|\} = \min\{\frac{2R}{c}, 1\}$  such that

$$(3.1) \quad \frac{\max_i\{|x_j a_i^j|, |x_j b_i^j|\}}{\min_i\{|x_j s_i^j|, |x_j t_i^j|\}} < \frac{2\delta}{c},$$

where  $c$  is the constant in Theorem 1.1' (if the low bound of the curvature  $k < 0$ , then  $c = c(n, k, 1)$  (for  $c(n, k, 1)$  please refer to the comments after Theorem 1.1')). Let  $f_j : M_1 \rightarrow \mathbb{R}^n$  be the map defined by  $f_j(q) = (|qa_1^j|, \dots, |qa_n^j|)$ . According to Theorem 1.1',  $f_j|_{B_{x_j}(\delta R)}$  is a  $\varkappa(\delta)$ -almost isometry (note that  $\max_i\{|x_j a_i^j|^{-1}, |x_j b_i^j|^{-1}\} \cdot \text{diam}(B_{x_j}(\delta R)) < c\delta$ ). Then we can select  $U_j$  with

$$(3.2) \quad B_{x_j}(3\delta R/4) \subset U_j \subset B_{x_j}(\delta R)$$

such that  $f_j(U_j)$  is convex in  $\mathbb{R}^n$ . Obviously,

$$f_j|_{U_j} \text{ is a } \varkappa(\delta)\text{-almost isometry.}$$

On the other hand, since  $h$  is a  $\text{GH}_\nu$ -approximation with  $\nu < R\delta^2$ ,

$$(3.3) \quad \{(h(a_i^j), h(b_i^j))\}_{i=1}^n \text{ and } \{(h(s_i^j), h(t_i^j))\}_{i=1}^n \text{ are } (n, 2\delta)\text{-strainers at } h(x_j);$$

and according to Theorem 1.1', the map  $g_j$  defined by  $g_j(q) = (|qh(a_1^j)|, \dots, |qh(a_n^j)|)$  maps  $h(B_{x_j}(\delta R))$   $\varkappa(\delta)$ -almost isometrically to a domain in  $\mathbb{R}^n$ . Moreover, without loss of generality, we can assume that  $f_j(U_j) \subset g_j(h(B_{x_j}(\delta R)))$ . It is clear that

$$h_j = g_j^{-1} \circ f_j \text{ is a } \varkappa(\delta)\text{-almost isometry on each } U_j.$$

Note that for any  $x \in U_j$ ,

$$\begin{aligned} |h_j(x)h(x)| &= (1 + \varkappa(\delta))|g_j(h_j(x))g_j(h(x))| \\ &= (1 + \varkappa(\delta))|f_j(x)g_j(h(x))| \\ &= (1 + \varkappa(\delta))\sqrt{(|a_1^j x| - |h(a_1^j)h(x)|)^2 + \dots + (|a_n^j x| - |h(a_n^j)h(x)|)^2} \\ &< (1 + \varkappa(\delta))\sqrt{n\nu} \text{ (note that } h \text{ is a } \text{GH}_\nu\text{-approximation),} \end{aligned}$$

i.e. each  $h_j$  is  $C\nu$ -close to  $h$  on  $U_j$ .

We will use center of mass to glue all these local almost isometries  $h_j$  to a global one. We first define weight functions<sup>1</sup>  $\phi_j : M_1 \rightarrow \mathbb{R}$  by

$$\phi_j(x) = \begin{cases} 1 - \frac{2|x x_j|}{\delta R}, & x \in B_{x_j}(\delta R/2) \\ 0, & x \in M_1 \setminus B_{x_j}(\delta R/2) \end{cases}.$$

Then for an arbitrary point  $z \in M_1(n, \delta, R)$  we define a sequence  $\{z_j\}_{j=1}^{N_1} \subset M_2$  :

$$z_j = \begin{cases} g_j^{-1} \left( \frac{\sum_{j-1}(z)}{\sum_j(z)} g_j(z_{j-1}) + \frac{\phi_j(z)}{\sum_j(z)} g_j(h_j(z)) \right), & z \in U_j \\ z_{j-1}, & z \notin U_j \end{cases},$$

<sup>1</sup>The original definition in [1] is  $\phi_j(x) = (1 - 2|x x_j|/(\delta R))^N$  if  $x \in B_{x_j}(\delta R/2)$ , where  $N$  is the multiplicity of the cover  $\{B_{x_j}(\delta R)\}$ .

where  $z_0 = h(z)$ ,  $\Sigma_0(z) = 0$  and  $\Sigma_j(z) = \sum_{l=1}^j \phi_l(z)$  for  $j \geq 1$ . A basic fact is that

$$(3.4) \quad \Sigma_{N_1}(z) > \frac{1}{3} \quad (\text{see (3.0)}).$$

Now we define the desired map  $\bar{h} : M_1(n, \delta, R) \rightarrow M_2$  in Theorem A by

$$\bar{h}(z) = z_{N_1} \text{ for any } z \in M_1(n, \delta, R).$$

We first show that  $\bar{h}$  is  $C\nu$ -close to  $h$ . Since each  $B_{x_j}(\delta R)$  is  $\varkappa(\delta)$ -almost isometric to a domain in  $\mathbb{R}^n$ , without loss of generality, we can assume that the multiplicity of the cover  $\{B_{x_j}(\delta R)\}$  is bounded by a number  $N$  depending only on the dimension  $n$  (ref. [1]). Since each  $h_j$  is  $C\nu$ -close to  $h$  and  $z$  belongs to at most  $N$  pieces of  $B_{x_j}(\delta R)$ , it is easy to see that

$$(3.5) \quad |h_j(z)h_{j'}(z)| < C\nu \text{ and } |z_j h_{j'}(z)| < C\nu,$$

and thus

$$\bar{h} \text{ is } C\nu\text{-close to } h.$$

In the next section we will verify that  $\bar{h}$  is a  $\varkappa(\delta)$ -almost distance preserving map.

**4. Verifying that  $\bar{h}$  is an almost distance preserving map.** In this section, we will show that  $\bar{h}$  (constructed in Section 3) is a  $\varkappa(\delta)$ -almost distance preserving map, i.e. for any  $y, z \in M_1(n, \delta, R)$ ,

$$(4.1) \quad \left| 1 - \frac{|\bar{h}(y)\bar{h}(z)|}{|yz|} \right| < \varkappa(\delta) \text{ or } \left| |\bar{h}(y)\bar{h}(z)| - |yz| \right| < \varkappa(\delta)|yz|,$$

and thus the proof of Theorem A is completed.

We first observe that we only need to consider the case  $|yz| < R\delta^{3/2}$ . In fact, if  $|yz| \geq R\delta^{3/2}$ , then  $||\bar{h}(y)\bar{h}(z)| - |yz|| < C\nu < CR\delta^2 < |yz|\varkappa(\delta)$  (i.e. (4.1) holds) because  $\bar{h}$  is  $C\nu$ -close to  $h$  and  $h$  is a  $\text{GH}_\nu$ -approximation.

Without loss of generality, we assume that  $\phi_j(y) + \phi_j(z) \neq 0$  for  $1 \leq j \leq N_2$ , and  $\phi_j(y) + \phi_j(z) = 0$  for  $N_2 < j \leq N_1$ . Note that if  $\phi_j(y) \neq 0$  (i.e.,  $y \in B_{x_j}(\delta R/2)$ ), then  $z \in B_{x_j}(\delta 2R/3)$  because  $|yz| < R\delta^{3/2}$  ( $\delta$  is sufficiently small). Hence

$$y, z \in B_{x_j}(2\delta R/3) \text{ for } j = 1, \dots, N_2 \text{ and } y, z \notin B_{x_j}(\delta R/2) \text{ for } j > N_2,$$

which implies that  $N_2 \leq N$  (the multiplicity of the cover  $\{B_{x_j}(\delta R)\}$ ). Since  $B_{x_j}(2\delta R/3) \subset B_{x_j}(3\delta R/4) \subset U_j$  (see (3.2)), we have that

$$(4.2) \quad \bar{h}(y) = y_{N_2} \text{ and } \bar{h}(z) = z_{N_2};$$

moreover, since each  $h_j$  is a  $\varkappa(\delta)$ -almost isometry and  $|h_j(y)h_{j'}(y)| < C\nu < CR\delta^2$  (see (3.5)), we have

$$\{h_1(y), h_1(z), \dots, h_{N_2}(y), h_{N_2}(z)\} \subset h_j(U_j) \text{ for any } j = 1, \dots, N_2.$$

Then we can define two sequences  $\{\bar{y}_j\}_{j=1}^{N_2}$  and  $\{\bar{z}_j\}_{j=1}^{N_2}$  in  $M_2$  (which are not introduced in [1]):

$$\bar{y}_j = g_j^{-1} \left( \sum_{l=1}^j \frac{\phi_l(y)}{\Sigma_j(y)} g_j(h_l(y)) \right) \text{ and } \bar{z}_j = g_j^{-1} \left( \sum_{l=1}^j \frac{\phi_l(z)}{\Sigma_j(z)} g_j(h_l(z)) \right).$$

Note that  $\bar{y}_j$  (resp.  $\bar{z}_j$ ) is the center of mass of  $\{h_l(y)\}_{l=1}^j$  (resp.  $\{h_l(z)\}_{l=1}^j$ ) with weights  $W_y^j = \left(\frac{\phi_1(y)}{\Sigma_j(y)}, \dots, \frac{\phi_j(y)}{\Sigma_j(y)}\right)$  (resp.  $W_z^j = \left(\frac{\phi_1(z)}{\Sigma_j(z)}, \dots, \frac{\phi_j(z)}{\Sigma_j(z)}\right)$ ) with respect to the  $(n, 2\delta)$ -strainer  $\{(h(a_i^j), h(b_i^j))\}_{i=1}^n$  at  $h(x_j)$ . And note that

$$(4.3) \quad \bar{y}_j = y_j \text{ and } \bar{z}_j = z_j \text{ for } j = 1, 2.$$

Now we give two claims.

CLAIM 1<sup>2</sup>.

$$|\bar{y}_{N_2} \bar{z}_{N_2} - |yz|| < \varkappa(\delta) |yz|.$$

CLAIM 2.

$$|y_{N_2} z_{N_2} - \bar{y}_{N_2} \bar{z}_{N_2}| < \varkappa(\delta) |yz|.$$

It is clear that Claims 1 and 2 (together with (4.2)) imply (4.1). Hence, we only need to verify Claims 1 and 2.

*The proof of Claim 1.* Since  $h$  is a  $\text{GH}_\nu$ -approximation with  $\nu < R\delta^2$ , we can obtain that

$$(4.4) \quad \begin{aligned} & \{h(s_i^j), h(t_i^j)\}_{i=1}^n \text{ is an } R\text{-long } (n, 2\delta)\text{-strainer at } h(x_{j'}) \\ & \text{with } \frac{\max_i\{|h(x_{j'})h(a_i^{j'})|, |h(x_{j'})h(b_i^{j'})|\}}{\min_i\{|h(x_{j'})h(s_i^j)|, |h(x_{j'})h(t_i^j)|\}} < \varkappa(\delta) \text{ (see (3.1))} \end{aligned}$$

for  $j, j' = 1, \dots, N_2$  (note that  $|x_j x_{j'}| < 2R\delta$  because  $U_j \cap U_{j'} \neq \emptyset$ ). Then according to Lemma 2.1, Claim 1 follows from the following three properties.

(i) Since each  $h_j$  is a  $\varkappa(\delta)$ -almost isometry, we have

$$(4.5) \quad \max_j\{|h_j(y)h_j(z)|\} < (1 + \varkappa(\delta)) \min_j\{|h_j(y)h_j(z)|\}.$$

(ii) For any fixed  $j$  (especially for  $j = N_2$ ),

$$(4.6) \quad \left| \max_l \tilde{Z}h(s_i^j)h_l(y)h_l(z) - \min_l \tilde{Z}h(s_i^j)h_l(y)h_l(z) \right| < \varkappa(\delta) \text{ for } i = 1, \dots, n.$$

We will give its proof in Appendix (which is drawn from the original proof of Theorem 0.1 ([1])).

(iii)

$$(4.7) \quad \left\| W_y^{N_2} - W_z^{N_2} \right\| < \frac{\min_j\{|h_j(y)h_j(z)|\}}{\max_{j,j'}\{|h_j(z)h_{j'}(z)|\}} \varkappa(\delta).$$

In order to prove inequality (4.7), we first give an estimate

$$(4.8) \quad \left| \frac{\phi_l(y)}{\Sigma_j(y)} - \frac{\phi_l(z)}{\Sigma_j(z)} \right| \leq \frac{C|yz|}{\delta R \Sigma_j(y)} \text{ for } 1 \leq l \leq j \leq N_2.$$

<sup>2</sup>The present proof is mainly inspired by this observation.

In fact, we have  $|\phi_l(y) - \phi_l(z)| = 2 \frac{||zx_l| - |yx_l||}{\delta R} \leq \frac{2|yz|}{\delta R}$  for any  $1 \leq l \leq N_2$ , and thus

$$\begin{aligned} \left| \frac{\phi_l(y)}{\Sigma_j(y)} - \frac{\phi_l(z)}{\Sigma_j(z)} \right| &= \frac{1}{\Sigma_j(y)} \left| \phi_l(y) - \frac{\phi_l(z)\Sigma_j(y)}{\Sigma_j(z)} \right| \\ &= \frac{1}{\Sigma_j(y)} \left| \phi_l(y) - \phi_l(z) - \phi_l(z) \frac{\Sigma_j(y) - \Sigma_j(z)}{\Sigma_j(z)} \right| \\ &\leq \frac{1}{\Sigma_j(y)} \max_l \{|\phi_l(y) - \phi_l(z)|\} \cdot N_2 \\ &\leq \frac{C|yz|}{\delta R \Sigma_j(y)}. \end{aligned}$$

It is clear that inequality (4.7) follows from (4.8) because  $\Sigma_{N_2}(y) > \frac{1}{3}$  (see (3.4)) and  $|h_j(z)h_{j'}(z)| < C\nu < CR\delta^2$  (see (3.5)).

The proof of Claim 2. Put  $\vec{\alpha}_j = \overrightarrow{g_j(y_j)g_j(z_j)} - \overrightarrow{g_j(\tilde{y}_j)g_j(\tilde{z}_j)}$  for  $j = 1, \dots, N_2$ . Since each  $g_j$  is a  $\varkappa(\delta)$ -almost isometry, Claim 2 is equivalent to

$$(4.9) \quad |\vec{\alpha}_{N_2}| < \varkappa(\delta)|yz|.$$

SUBCLAIM.

$$(4.10) \quad |\vec{\alpha}_j| \leq \frac{C|yz|\nu}{\delta R \Sigma_j(y)} + \frac{\Sigma_{j-1}(y)}{\Sigma_j(y)}(1 + \varkappa(\delta))|\vec{\alpha}_{j-1}| + \varkappa(\delta)|yz| \text{ for } j = 2, \dots, N_2.$$

It follows from the subclaim that

$$\begin{aligned} |\vec{\alpha}_{N_2}| &\leq \frac{C|yz|\nu}{\delta R \Sigma_{N_2}(y)} + \varkappa(\delta)|yz| + \frac{\Sigma_{N_2-1}(y)}{\Sigma_{N_2}(y)}(1 + \varkappa(\delta))|\vec{\alpha}_{N_2-1}| \\ &\leq \frac{C|yz|\nu}{\delta R \Sigma_{N_2}(y)} + \varkappa(\delta)|yz| + \frac{\Sigma_{N_2-2}(y)}{\Sigma_{N_2}(y)}(1 + \varkappa(\delta))|\vec{\alpha}_{N_2-2}| \\ &\leq \dots \\ &\leq \frac{C|yz|\nu}{\delta R \Sigma_{N_2}(y)} + \varkappa(\delta)|yz| + \frac{\Sigma_2(y)}{\Sigma_{N_2}(y)}(1 + \varkappa(\delta))|\vec{\alpha}_2| \\ &< \varkappa(\delta)|yz| \text{ (note that } \Sigma_{N_2}(y) > \frac{1}{3}, \nu < R\delta^2 \text{ and } |\vec{\alpha}_2| = 0 \text{ (see (4.3)))}. \end{aligned}$$

That is (4.9) holds, so we only need to verify the subclaim.

To simplify notations in the following computations, let us denote  $g_j(x)$  by  $\tilde{x}$  for any  $x \in U_j$ .

Recall that

$$\tilde{y}_j = \frac{\Sigma_{j-1}(y)}{\Sigma_j(y)}\tilde{y}_{j-1} + \frac{\phi_j(y)}{\Sigma_j(y)}\widetilde{h_j(y)} \text{ and } \tilde{\tilde{y}}_j = \sum_{l=1}^j \frac{\phi_l(y)}{\Sigma_j(y)}\widetilde{h_l(y)}$$

( $\tilde{z}_j$  and  $\tilde{\tilde{z}}_j$  have the same forms respectively). Through straightforward computation, one can get

$$\vec{\alpha}_j = \frac{\Sigma_{j-1}(y)}{\Sigma_j(y)} \left( \overrightarrow{\tilde{y}_{j-1}\tilde{z}_{j-1}} - \sum_{l=1}^{j-1} \frac{\phi_l(y)}{\Sigma_{j-1}(y)} \overrightarrow{\widetilde{h_l(y)}\widetilde{h_l(z)}} \right) + \sum_{l=1}^{j-1} \left( \frac{\phi_l(z)}{\Sigma_j(z)} - \frac{\phi_l(y)}{\Sigma_j(y)} \right) \overrightarrow{\widetilde{h_l(z)}\tilde{z}_{j-1}}.$$

Put  $\vec{\beta} = \overrightarrow{\tilde{y}_{j-1}\tilde{z}_{j-1}} - \sum_{l=1}^{j-1} \frac{\phi_l(y)}{\Sigma_{j-1}(y)} \overrightarrow{h_l(y)h_l(z)}$  and  $\vec{\gamma} = \sum_{l=1}^{j-1} \left( \frac{\phi_l(z)}{\Sigma_j(z)} - \frac{\phi_l(y)}{\Sigma_j(y)} \right) \overrightarrow{h_l(z)\tilde{z}_{j-1}}$ , and thus

$$(4.11) \quad \vec{\alpha}_j = \frac{\Sigma_{j-1}(y)}{\Sigma_j(y)} \vec{\beta} + \vec{\gamma}.$$

It follows from inequalities (4.8) and (3.5) that

$$(4.12) \quad |\vec{\gamma}| \leq \sum_{l=1}^{j-1} \frac{C|yz|}{\delta R \Sigma_j(y)} \cdot C\nu \leq \frac{C|yz|\nu}{\delta R \Sigma_j(y)}.$$

In order to estimate  $|\vec{\beta}|$ , we introduce two points  $\tilde{z}'_{j-1}$  and  $z'_{j-1}$  such that

$$\tilde{z}'_{j-1} = g_{j-1}^{-1} \left( \sum_{l=1}^{j-1} \frac{\phi_l(y)}{\Sigma_{j-1}(y)} g_{j-1}(h_l(z)) \right)$$

and

$$(4.13) \quad \overrightarrow{g_{j-1}(y_{j-1})g_{j-1}(z'_{j-1})} = \overrightarrow{g_{j-1}(\tilde{y}_{j-1})g_{j-1}(\tilde{z}_{j-1})}.$$

Now we put

$$\begin{aligned} \vec{\beta}^1 &= \overrightarrow{\tilde{y}_{j-1}\tilde{z}_{j-1}} - \overrightarrow{\tilde{y}_{j-1}\tilde{z}'_{j-1}}, \\ \vec{\beta}^2 &= \overrightarrow{\tilde{y}_{j-1}\tilde{z}'_{j-1}} - \overrightarrow{\tilde{y}_{j-1}\tilde{z}_{j-1}}, \\ \vec{\beta}^3 &= \overrightarrow{\tilde{y}_{j-1}\tilde{z}_{j-1}} - \overrightarrow{\tilde{y}_{j-1}\tilde{z}'_{j-1}}, \\ \vec{\beta}^4 &= \overrightarrow{\tilde{y}_{j-1}\tilde{z}'_{j-1}} - \sum_{l=1}^{j-1} \frac{\phi_l(y)}{\Sigma_{j-1}(y)} \overrightarrow{h_l(y)h_l(z)}. \end{aligned}$$

Obviously,  $\vec{\beta} = \vec{\beta}^1 + \vec{\beta}^2 + \vec{\beta}^3 + \vec{\beta}^4$ . In the following, we will analyze  $|\vec{\beta}^i|$  one by one. Firstly,

$$\begin{aligned} |\vec{\beta}^1| &= \left| \overrightarrow{\tilde{z}'_{j-1}\tilde{z}_{j-1}} \right| = (1 + \varkappa(\delta)) |z'_{j-1}z_{j-1}| \\ &= (1 + \varkappa(\delta)) \left| \overrightarrow{g_{j-1}(z'_{j-1})g_{j-1}(z_{j-1})} \right| \\ &= (1 + \varkappa(\delta)) \left| \overrightarrow{g_{j-1}(y_{j-1})g_{j-1}(z_{j-1})} - \overrightarrow{g_{j-1}(y_{j-1})g_{j-1}(z'_{j-1})} \right| \\ (\text{by (4.13)}) &= (1 + \varkappa(\delta)) \left| \overrightarrow{g_{j-1}(y_{j-1})g_{j-1}(z_{j-1})} - \overrightarrow{g_{j-1}(\tilde{y}_{j-1})g_{j-1}(\tilde{z}_{j-1})} \right| \\ &= (1 + \varkappa(\delta)) |\vec{\alpha}_{j-1}|. \end{aligned}$$

Secondly,

$$\begin{aligned}
 |\vec{\beta}^3| &= \left| \overrightarrow{\widetilde{z}'_{j-1}\widetilde{z}_{j-1}} \right| = (1 + \varkappa(\delta))|\overrightarrow{z'_{j-1}z_{j-1}}| \\
 &= (1 + \varkappa(\delta))\overrightarrow{|g_{j-1}(z'_{j-1})g_{j-1}(z_{j-1})|} \\
 &= (1 + \varkappa(\delta)) \left| \sum_{l=1}^{j-1} \left( \frac{\phi_l(y)}{\Sigma_{j-1}(y)} - \frac{\phi_l(z)}{\Sigma_{j-1}(z)} \right) g_{j-1}(h_l(z)) \right| \\
 &= (1 + \varkappa(\delta)) \left| \sum_{l=1}^{j-1} \left( \frac{\phi_l(y)}{\Sigma_{j-1}(y)} - \frac{\phi_l(z)}{\Sigma_{j-1}(z)} \right) \overrightarrow{|g_{j-1}(h_l(z))g_{j-1}(h_l(z))|} \right| \\
 &\leq \frac{C|yz|\nu}{\delta R \Sigma_{j-1}(y)} \quad (\text{similar to getting (4.12)}).
 \end{aligned}$$

Thirdly, we estimate  $|\vec{\beta}^4|$ . Note that (4.5) and (4.6) (together with Remark 2.3 and Lemma 2.6) imply that for any  $1 \leq l, l_1, l_2 \leq N_2$

$$(4.14) \quad \overrightarrow{|g_l(h_{l_1}(y))g_l(h_{l_1}(z))|} \text{ is } \varkappa(\delta)\text{-almost equal to } \overrightarrow{|g_l(h_{l_2}(y))g_l(h_{l_2}(z))|},$$

and thus

$$\overrightarrow{|g_{j-1}(\overline{y}_{j-1})g_{j-1}(\overline{z}'_{j-1})|} \text{ is } \varkappa(\delta)\text{-almost equal to } \overrightarrow{|g_{j-1}(h_l(y))g_{j-1}(h_l(z))|}.$$

Then due to (4.4), we can apply Corollary 2.7 to get that

$$(4.15) \quad \overrightarrow{|\widetilde{y}_{j-1}\widetilde{z}'_{j-1}|} \text{ is } \varkappa(\delta)\text{-almost equal to } \overrightarrow{|h_l(y)h_l(z)|}.$$

On the other hand, by (4.14)

$$\sum_{l=1}^{j-1} \frac{\phi_l(y)}{\Sigma_{j-1}(y)} \overrightarrow{|h_l(y)h_l(z)|} \text{ is } \varkappa(\delta)\text{-almost equal to } \overrightarrow{|h_l(y)h_l(z)|}.$$

It therefore follows that

$$|\vec{\beta}^4| < \varkappa(\delta)|\overrightarrow{|h_l(y)h_l(z)|}| = \varkappa(\delta)|yz|.$$

Finally, we estimate  $|\vec{\beta}^2|$ . Note that it follows from (4.15) that  $|\overrightarrow{|\widetilde{y}_{j-1}\widetilde{z}'_{j-1}|}| < \varkappa(\delta)|yz|$ , and thus

$$|\overrightarrow{|\widetilde{y}_{j-1}\widetilde{z}'_{j-1}|}| \leq |\vec{\beta}^3| + |\overrightarrow{|\widetilde{y}_{j-1}\widetilde{z}'_{j-1}|}| < \frac{C|yz|\nu}{\delta R \Sigma_{j-1}(y)} + \varkappa(\delta)|yz|.$$

On the other hand, according to Corollary 2.7, it follows from (4.13) that

$$\overrightarrow{|\widetilde{y}_{j-1}\widetilde{z}'_{j-1}|} \text{ is } \varkappa(\delta)\text{-almost equal to } \overrightarrow{|\widetilde{y}_{j-1}\widetilde{z}_{j-1}|}.$$

Therefore, we have

$$|\vec{\beta}^2| \leq \varkappa(\delta)|\overrightarrow{|\widetilde{y}_{j-1}\widetilde{z}'_{j-1}|}| \leq \varkappa(\delta) \left( \frac{C|yz|\nu}{\delta R \Sigma_{j-1}(y)} + \varkappa(\delta)|yz| \right).$$

Now we conclude that

$$|\vec{\beta}| \leq |\vec{\beta}^1| + |\vec{\beta}^2| + |\vec{\beta}^3| + |\vec{\beta}^4| < (1 + \varkappa(\delta))|\vec{\alpha}_{j-1}| + \frac{C|yz|\nu}{\delta R \Sigma_{j-1}(y)} + \varkappa(\delta)|yz|.$$

Since  $\vec{\alpha}_j = \frac{\Sigma_{j-1}(y)}{\Sigma_j(y)}\vec{\beta} + \vec{\gamma}$  (see (4.11)), it is clear that the estimates of  $|\vec{\beta}|$  and  $|\vec{\gamma}|$  (see (4.12)) imply (4.10) (i.e. the Subclaim holds, so the whole proof is finished).  $\square$

**5. Appendix.** In Appendix, we give the proofs of Remark 2.3, Lemma 2.6 and (4.6). In the proof of Remark 2.3, we will use a result contained in Lemma 5.6 in [1].

LEMMA 5.1. *Let  $p, q, r, s \in M$ . For sufficiently small  $\delta$ , if  $|qs| < \delta \cdot \min\{|pq|, |rq|\}$  and  $\tilde{\angle}pqr > \pi - \delta$ , then<sup>3</sup>  $|\tilde{\angle}pqs - \angle pqs| < \varkappa(\delta)$  and  $|\tilde{\angle}rqs - \angle rqs| < \varkappa(\delta)$ .*

*Proof of Remark 2.3.* According to Lemma 5.1, the conclusion in Remark 2.3 is equivalent to

$$(5.1) \quad |\angle a_i q_j r_j - \angle a_i q_{j'} r_{j'}| < \varkappa(\delta) \iff |\angle s_i q_j r_j - \angle s_i q_{j'} r_{j'}| < \varkappa(\delta) \text{ for } i = 1, \dots, n.$$

Using the law of cosine, it is not difficult to conclude that

$$|\tilde{\angle}uq_j v - \tilde{\angle}uq_{j'} v| < \varkappa(\delta) \text{ for } u \in \{s_i, t_i\}_{i=1}^n \text{ and } v \in \{a_i, b_i\}_{i=1}^n.$$

By Lemma 5.1 again,

$$(5.2) \quad |\angle uq_j v - \angle uq_{j'} v| < \varkappa(\delta).$$

Now we consider the direction space  $\Sigma_{q_j}$  at  $q_j$ . In the situation here,  $\Sigma_{q_j}$  is  $\varkappa(\delta)$ -almost isometric to a unit sphere  $\mathbb{S}^{n-1}$  (see Theorem 9.5 in [1]). Denote by  $\bar{a}_i \in \Sigma_{q_j}$  (resp.  $\bar{s}_i$  and  $\bar{r}_j$ ) the directions of geodesics  $q_j a_i$  (resp.  $q_j s_i$  and  $q_j r_j$ ) for  $i = 1, \dots, n$ . Note that  $|\bar{a}_i \bar{a}_{i'}| = \frac{\pi}{2} \pm \varkappa(\delta)$  and  $|\bar{s}_i \bar{s}_{i'}| = \frac{\pi}{2} \pm \varkappa(\delta)$  for  $i \neq i'$  (i.e.  $\{\bar{a}_i\}_{i=1}^n$  and  $\{\bar{s}_i\}_{i=1}^n$  roughly correspond to two orthogonal  $n$ -frames of  $\mathbb{S}^{n-1}$ ). Similarly, we also consider  $\Sigma_{q_{j'}}$ . Then it is not difficult to see that inequality (5.2) implies (5.1).  $\square$

*Proof of Lemma 2.6.* We first note that

$$(5.3) \quad \begin{aligned} & |\tilde{\angle}a_i x_1 y_1 - \tilde{\angle}a_i x_2 y_2| < \varkappa(\delta) \\ \iff & |\cos \tilde{\angle}a_i x_1 y_1 - \cos \tilde{\angle}a_i x_2 y_2| < \varkappa(\delta) \\ \iff & \left| \frac{|a_i x_1|^2 + |x_1 y_1|^2 - |a_i y_1|^2}{2|a_i x_1| \cdot |x_1 y_1|} - \frac{|a_i x_2|^2 + |x_2 y_2|^2 - |a_i y_2|^2}{2|a_i x_2| \cdot |x_2 y_2|} \right| < \varkappa(\delta) \\ \iff & \left| \frac{|a_i x_1| - |a_i y_1|}{|x_1 y_1|} - \frac{|a_i x_2| - |a_i y_2|}{|x_2 y_2|} \right| < \varkappa(\delta) \\ \iff & \left| \frac{|a_i x_1| - |a_i y_1|}{|f(x_1)f(y_1)|} - \frac{|a_i x_2| - |a_i y_2|}{|f(x_2)f(y_2)|} \right| < \varkappa(\delta) \text{ (} f \text{ is a } \varkappa(\delta)\text{-almost isometry),} \end{aligned}$$

where the second “ $\iff$ ” is due to that the neighborhood  $U$  in the lemma is sufficiently small. Recall that  $f(x) = (|a_1 x|, |a_2 x|, \dots, |a_n x|)$ . Hence,  $|\tilde{\angle}a_i x_1 y_1 - \tilde{\angle}a_i x_2 y_2| < \varkappa(\delta)$  for  $i = 1, 2, \dots, n \iff \angle(\overrightarrow{f(x_1)f(y_1)}, \overrightarrow{f(x_2)f(y_2)}) < \varkappa(\delta)$ .  $\square$

---

<sup>3</sup> $\angle pqs$  is the angle between geodesics  $qp$  and  $qs$  at  $q$ , which is well defined by  $\lim_{x, y \rightarrow q} \tilde{\angle}xqy$  with  $x \in qp$  and  $y \in qs$ .

*Proof of (4.6).* For any  $1 \leq l_1, l_2 \leq N_2$  and  $u \in \{s_i^j, t_i^j\}_{i=1}^n$ , we notice that

$$\begin{aligned}
 & \text{the inequality (4.6)} \\
 & \iff |\tilde{Z}h(u)h_{l_1}(y)h_{l_1}(z) - \tilde{Z}h(u)h_{l_2}(y)h_{l_2}(z)| < \varkappa(\delta) \\
 & \text{(obviously)} \iff |\tilde{Z}h(u)h_l(y)h_l(z) - \tilde{Z}uyz| < \varkappa(\delta) \text{ for } l = 1, \dots, N_2 \\
 & \text{(by Lemma 5.1)} \iff |\angle h(u)h_l(y)h_l(z) - \angle uyz| < \varkappa(\delta) \\
 (5.4) \quad & (?) \iff |\angle h(a_i^l)h_l(y)h_l(z) - \angle a_i^l yz| < \varkappa(\delta) \\
 & \text{(by Lemma 5.1)} \iff |\tilde{Z}h(a_i^l)h_l(y)h_l(z) - \tilde{Z}a_i^l yz| < \varkappa(\delta) \\
 & \text{(similar to (5.3))} \iff \left| \frac{|h(a_i^l)h_l(y)| - |h(a_i^l)h_l(z)|}{|h_l(y)h_l(z)|} - \frac{|a_i^l y| - |a_i^l z|}{|yz|} \right| < \varkappa(\delta),
 \end{aligned}$$

where the last inequality holds because  $|h(a_i^l)h_l(y)| = |a_i^l y|$ ,  $|h(a_i^l)h_l(z)| = |a_i^l z|$  (recall that  $h_l = g_l^{-1} \circ f_l$ ) and  $h_l$  is a  $\varkappa(\delta)$ -almost isometry.

Now we only need to verify the third ‘ $\iff$ ’ in (5.4). Similar to getting inequality (5.2), we can obtain that for any  $v \in \{a_i^l, b_i^l\}_{i=1}^n$

$$|\angle h(u)h_l(y)h(v) - \angle u y v| < \varkappa(\delta).$$

Then we can use the same argument as in the end of the proof of Remark 2.3 to conclude that the third ‘ $\iff$ ’ in (5.4) holds (taking into account that both  $\Sigma_y$  and  $\Sigma_{h_l(y)}$  are  $\varkappa(\delta)$ -almost isometric to  $S^{n-1}$ ).  $\square$

**Acknowledgement.** The authors are grateful to Professor Yuri Burago for his numerous constructive comments/suggestions on this paper.

REFERENCES

[1] YU. BURAGO, M. GROMOV, AND G. PEREL'MAN, *A. D. Alexandrov spaces with curvature bounded below*, Russian Math. Surveys, 47:2 (1992), pp. 1–58.  
 [2] T. YAMAGUCHI, *A convergence theorem in the geometry of Alexandrov spaces*, 1996, preprint.