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SPACELIKE FOLIATIONS BY $(n-1)$-UMBILICAL HYPERSURFACES IN SPACETIMES*

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Abstract. We consider the problem of whether a given spacetime admits a foliation by $(n-1)$-umbilical spacelike hypersurfaces. We introduce the notion of a timelike closed partially conformal vector field in a spacetime and show that the existence of a vector field of this kind guarantees in turn the existence of that foliation. We then construct explicit examples of families of $(n-1)$-umbilical spacelike hypersurfaces in the de Sitter space. Imposing the further condition of having constant r-th mean curvature, we give the complete description of any leaf of a foliation of the de Sitter space by these hypersurfaces. Finally, in a spacetime foliated by $(n-1)$-umbilical spacelike hypersurfaces we characterize the immersed spacelike hypersurfaces which are $(n-1)$-umbilical.

Key words. Closed partially conformal vector field, $(n-1)$-umbilical spacelike hypersurface.

AMS subject classifications. Primary 53C12, Secondary 53A10.

1. Introduction. The study of the geometric behavior of Lorentzian manifolds and their hypersurfaces has important consequences in many instances; in particular, for their use as spacetimes models in general relativity. Research in this area was marked in 1977 by the appearance of Goddard's paper [15], in which he posed his now well-known conjecture: The only complete constant mean curvature spacelike hypersurfaces in the de Sitter space $\mathbb{S}_1^{n+1}$ are the totally umbilical ones.

As it is now known, Goddard's conjecture was proved false in general. In fact, the first counterexample was given by Dajczer and Nomizu [12] in 1981, with an example of a non-totally umbilical complete constant mean curvature spacelike surface in the 3-dimensional de Sitter space $\mathbb{S}_1^3$ (see also other examples in [1]).

It is worth mentioning that in 1988 Montiel proved that Goddard's conjecture is true for the compact case; that is, he showed that the only compact constant mean curvature spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ are the totally umbilical ones and he described all of them (see [23]). Since then, interest has been growing in constructing examples of non-totally umbilical spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$, study initiated by Dajczer and Nomizu and followed by Akutagawa [1], Ramanathan [31] and Ki, Kim and Nakagawa [21].

Later on, in [24] and [26], Montiel studied and characterized a family of non-totally umbilical complete constant mean curvature spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$, the so called hyperbolic cylinders (see also [21]).

We note that all these non-totally umbilical examples are in fact $(n-1)$-umbilical, notion defined in [5] and [11] for the Riemannian case and directly extended in [9] to the Lorentzian case: given a Lorentzian manifold $\bar{M}_1^{n+1}$, a spacelike hypersurface $M^n \subset \bar{M}_1^{n+1}$ is $(n-1)$–umbilical if there exists a $(n-1)$-dimensional distribution D on $\bar{M}^{n+1}$ and a positive function μ such that

$$(1) \qquad D = \{\ X \in \mathfrak{X}(M) \mid \alpha(X,Y) = \mu\langle X,Y\rangle N,\ \text{for all } Y \in \mathfrak{X}(M)\ \},$$

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where N is a smooth unit vector field everywhere normal to M and α is the second fundamental form of M. From the definition it is easy to see that a hypersurface is $(n-1)$-umbilical if and only if has $(n-1)$ equal principal curvatures at each point.

We must mention that in the Riemannian setting there are many papers studying $(n-1)$-umbilical hypersurfaces; see for example [18], [20], [22], [29], [30] and more recently, [10].

On the other hand, in 1995 Alías, Romero and Sánchez formulated in [3] the following question: when is a complete spacelike hypersurface in a Generalized Robertson-Walker spacetime totally umbilical or a slice? In [3] and the subsequent paper [4] they answered the question under some hypotheses such as the manifold being Einstein or the timelike convergence condition (i.e., the Ricci curvature is non-negative on timelike directions). They used integral formulas obtained by derivating the natural timelike closed conformal vector field of the ambient. An account of their contribution can be found in [6].

Since then increased the interest for the Goddard-type question:

Under what conditions a spacelike hypersurface with constant mean curvature is totally umbilical?

Montiel answered this question imposing two conditions on the ambient spacetime (see [26]). The first one is that the spacetime must admit a *closed conformal* timelike vector field. This condition is associated to the fact of the spacetime being expressed, locally, as a warped product as well as to the existence of a foliation whose leaves are totally umbilical.

The second condition on the spacetime is the *null convergence condition* (that is, the Ricci curvature is non-negative on null directions). Under both conditions, he proved that constant mean curvature compact spacelike hypersurfaces in such a spacetime must be totally umbilical.

Following these research lines, it is natural to study $(n-1)$-umbilical hypersurfaces in spacetimes and try to relate them to constant mean curvature hypersurfaces or, even more generally, to constant r-th mean curvature spacelike hypersurfaces. Recall that the r-th mean curvature H_r of a n-dimensional hypersurface is given in terms of its principal curvatures as

$$\binom{n}{r} H_r = \sum_{1\leq i_1<i_2<\cdots<i_r\leq n} \kappa_{i_1}\kappa_{i_2}\cdots\kappa_{i_r}. \tag{2}$$

Of course, for this theory to be meaningful the ambient space must have plenty of $(n-1)$-umbilical hypersurfaces. As an important example, we will define rotation spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ (see Section 5) and we use them to prove the existence of a 1-parameter family of $(n-1)$-umbilical complete spacelike hypersurfaces with constant r-th mean curvature in $\mathbb{S}_1^{n+1}$ (see Section 6).

In this context, it is then natural to pose the following questions:

Under what conditions a spacetime possesses a foliation whose leaves are $(n-1)$*-umbilical?*

In a spacetime admitting such a foliation, under what conditions an immersed spacelike hypersurface is $(n-1)$*-umbilical?*

Here we answer both questions. After some preliminaries, in Section 3 we introduce the notion of a timelike closed partially conformal vector field in a spacetime $\bar{M}_1^{n+1}$ (see Definition 3.1): A timelike vector field $K \in \mathfrak{X}(\bar{M}_1^{n+1})$ is *closed* *partially conformal* if there is a unit vector field W orthogonal to K and functions $\phi, \psi : \bar{M}_1^{n+1} \to \mathbb{R}$ such that

$$\bar{\nabla}_X K = \phi X + (\psi - \phi)\langle X, W\rangle W$$

for every $X \in \mathfrak{X}(\bar{M})$. In particular, *closed conformal* vector fields satisfy $\phi = \psi$ everywhere.

REMARK 1.1. *We note that the transformations associated to the flow of a closed partially conformal vector field are in fact partially conformal transformations in the sense of Tanno (see [32] and [33]). Also, our notion is closely related with that of a biconformal vector field (see, for example [16, 17]). A vector field K defined on $\bar{M}$ is said to be biconformal if there are complementary and mutually orthogonal projections π_1, π_2 on $T\bar{M}$ and differentiable functions ϕ, ψ on $\bar{M}$ such that*

$$L_K \pi_1^* \langle\ ,\ \rangle = 2\phi \pi_1^* \langle\ ,\ \rangle, \quad L_K \pi_2^* \langle\ ,\ \rangle = 2\psi \pi_2^* \langle\ ,\ \rangle,$$

where L_K is the Lie derivative relative to K. In fact, if K is a closed partially conformal vector field, and π_1 (resp. π_2) is the projection onto the orthogonal complement to W (resp. onto the space spanned by W), it is straightforward to show that K is biconformal; for example,

$$\begin{aligned} L_K \pi_1^* \langle X, Y\rangle &= K\langle \pi_1 X, \pi_1 Y\rangle - \langle [K, \pi_1 X], \pi_1 Y\rangle - \langle [K, \pi_1 Y], \pi_1 X\rangle \\ &= \langle \bar{\nabla}_{\pi_1 X} K, \pi_1 Y\rangle + \langle \bar{\nabla}_{\pi_1 Y} K, \pi_1 X\rangle \\ &= 2\phi \langle \pi_1 X, \pi_1 Y\rangle = 2\phi \pi_1^* \langle X, Y\rangle. \end{aligned}$$

The presence of a timelike closed partially conformal vector field guarantees the existence of the foliation we are interested in. In fact, the answer to the first of our questions is given by the following result, proved by the authors in the previous work [9]:

(Theorem 3.2 in [9]) Let $\bar{M}_1^{n+1}$ be a Lorentzian manifold possessing a timelike closed partially conformal vector field K. Then the distribution $K^\perp$ defined by taking the orthogonal complement of K at every point is involutive and each connected leaf of the foliation determined by this distribution is a $(n-1)$-umbilical hypersurface having $(n-1)$ equal and constant principal curvatures.

In Section 4 we will present explicit examples of timelike closed partially conformal vector fields in Lorentzian space forms of non-negative curvature. In the particular case of the de Sitter space $\mathbb{S}_1^{n+1}$, we obtain in Example 4.2 a foliation of the open subset of $\mathbb{S}_1^{n+1}$

$$\Omega = \left\{ (x_1, \ldots, x_{n+2}) \in \mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid -x_1^2 + x_2^2 < 0 \right\},$$

by the hyperbolic cylinders given by

$$\left\{ (x_1, \ldots, x_{n+2}) \in \mathbb{S}_1^{n+1} \;\middle|\; -x_1^2 + x_2^2 = \frac{1}{c_1},\ x_3^2 + \cdots + x_{n+2}^2 = \frac{1}{c_2} \right\},$$

where $c_1 < 0$, $c_2 > 0$ and $(1/c_1) + (1/c_2) = 1$.

In the same Section 4 we note that Ω may be expressed as a doubly warped product of the form $-I \times_f \mathbb{R} \times_g \mathbb{S}^{n-1}$ for some smooth positive functions $f, g : I \to \mathbb{R}^+$.

It is worth emphasizing that the inner Schwarzschild spacetime also can be expressed as a doubly warped product (see Section 5.5 of [19] and also [14]).

The close relation between spacetimes possessing timelike closed partially conformal vector fields and doubly warped products will be given by Proposition 4.3:

Every Lorentzian doubly warped product of the form

$$\bar{M}_1^{n+1} = -I \times_f F^{n-1} \times_g J, \quad \text{where } I, J \subset \mathbb{R},\ f, g : I \to \mathbb{R}^+,\ \text{and } f \neq g,$$

admits a timelike closed partially conformal vector field.

In consequence, the inner Schwarzschild spacetime admits a natural closed partially conformal vector field.

We have also a partial converse to Proposition 4.3 (see Proposition 4.5):

Let $\bar{M}_1^{n+1}$ be a spacetime which admits a timelike closed partially conformal vector field K with an associated vector field W orthogonal to K. Suppose W is parallel. Then $\bar{M}_1^{n+1}$ can be expressed locally as a doubly warped product of the form $-I \times_f F \times J$.

In Section 5 we give explicit examples of $(n-1)$-umbilical hypersurfaces in $\mathbb{S}_1^{n+1}$ by defining the spacelike rotation hypersurfaces. In section 6 we impose on these submanifolds the further condition of having constant r-th mean curvature to build 1-parameter families of hypersurfaces which, as far as we know, were not yet considered in the literature (see Theorem 6.1):

For every real number H_r there are three 1-parameter families of spacelike rotation hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant r-th mean curvature H_r, corresponding to the kind of rotation (spherical, parabolic or hyperbolic) considered. The family of parabolic rotation hypersurfaces contains the totally umbilical hypersurfaces of $\mathbb{S}_1^{n+1}$. Moreover,

1. *The family of spherical rotation hypersurfaces contains two cylinders if $r = 1$ and $2\sqrt{n-1}/n \leq H_1 < 1$, while it contains one cylinder if $r = 1$ and $H_1 > 1$.*
2. *The family of spherical rotation hypersurfaces contains one cylinder if $2 \leq r \leq n$ and $0 < H_r < 1$.*
3. *The family of hyperbolic rotation hypersurfaces contains one cylinder if $1 \leq r \leq n$ and $H_r > 1$.*

In the above statement, we call a hypersurface a *cylinder* if it is $(n-1)$-umbilical with two distinct and constant principal curvatures.

We use the above result to characterize any non-umbilical leaf of a foliation associated to a timelike closed partially conformal vector field defined on $\mathbb{S}_1^{n+1}$ and having constant r-th mean curvature as a cylinder (see Corollary 6.2):

Let K be a timelike closed partially conformal vector field defined on an open subset of $\mathbb{S}_1^{n+1}$. If a connected leaf of $\mathcal{K}^\perp$ has constant mean curvature, then the leaf is totally umbilical or a cylinder.

Moreover, let ϕ, ψ be the functions associated with K via equation (4). Suppose that $\phi \neq 0$ and that a connected leaf of $\mathcal{K}^\perp$ has constant r-th mean curvature for some $r > 1$. Then the leaf is totally umbilical or a cylinder.

In Theorem 6.3 we refine Theorem 6.1 to prove the existence of a subfamily of complete rotational hypersurfaces as follows:

For each of the following values of r and H_r there is a 1-parameter family of complete spherical rotation spacelike hypersurfaces with constant r-th mean curvature H_r in $\mathbb{S}_1^{n+1}$:

1. $r = 1$ *and* $2\sqrt{n-1}/n < H_1 < 1$;
2. $r \geq 2$ *and* $0 < H_r < 1$.

Finally, in section 7 we answer our second question, namely, under what conditions a hypersurface is $(n-1)$-umbilical, establishing restrictions on the class of the ambient spacetimes foliated by $(n-1)$-umbilical hypersurfaces. Then we give necessary and sufficient conditions for a immersed spacelike hypersurface to be $(n-1)$-umbilical (see Theorem 7.1):

Let $\bar{M}_1^{n+1} = -I \times_f F \times_g J$ be a doubly warped product with F compact. Let Σ be an immersed spacelike hypersurface everywhere transverse to K. For a given $s \in J$, let $\bar{M}_s = -I \times_f F \times_g \{s\}$ and $\Sigma_s = \Sigma \cap \bar{M}_s$. Suppose either of the following conditions true:

1. $\bar{M}$ *satisfies the null convergence condition*
$$\mathrm{Ric}_F \geq \sup(ff'' - f'^2),$$
where Ric *is the Ricci curvature of F and in addition, assume that Σ_s is a compact $(n-1)$-dimensional manifold immersed with constant mean curvature in $\bar{M}_s$ for every $s \in J$;*
2. $\bar{M}$ *satisfies the strong null convergence condition*
$$K_F \geq \sup(ff'' - f'^2),$$
where K_F stands for the sectional curvature of F; also assume that for each $s \in J$, Σ_s is a compact $(n-1)$-dimensional manifold immersed with constant r-th mean curvature H_r in $\bar{M}_s$ for some $2 \leq r \leq n$ and that Σ_s is contained in a slab $(t_1, t_2) \times_f F \times_g \{s\}$ where f' does not vanish.

Then Σ is $(n-1)$-umbilical in $\bar{M}$ if and only if the angle between Σ and $\bar{M}_s$ is constant for every $s \in J$.

2. Preliminaries. Let n, ν be integers such that $n \geq 2$ and $0 \leq \nu \leq n$. We will denote by $\bar{M}_\nu^n$ a n-dimensional semi-Riemannian manifold, endowed with a metric tensor of index ν denoted by $\langle\, ,\, \rangle$.

For example, let $\mathbb{R}_\nu^n$ be the n-dimensional semi-Euclidean space of index ν, i.e., the n-dimensional vector space with metric tensor

$$\langle x, y\rangle = -\sum_{i=1}^{\nu} x_i y_i + \sum_{j=\nu+1}^{n} x_j y_j,$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

For $n \geq 2$ and $c > 0$ we define

$$\mathbb{S}_\nu^n(c) = \left\{ x \in \mathbb{R}_\nu^{n+1} \;\middle|\; \langle x, x\rangle = \frac{1}{c} \right\}.$$

It is well known that $\mathbb{S}_\nu^n(c)$ has constant (positive) curvature c. We call $\mathbb{R}_\nu^{n+1}$ the *ambient space* of $\mathbb{S}_\nu^n(c)$. If $c = 1$, we simply denote this space as $\mathbb{S}_\nu^{n+1}$.

Similarly, for $c < 0$ we define

$$\mathbb{H}^n_\nu(c) = \left\{ \; x \in \mathbb{R}^{n+1}_{\nu+1} \;\middle|\; \langle x, x\rangle = \frac{1}{c} \; \right\},$$

which is a space with constant (negative) curvature c. We call $\mathbb{R}^{n+1}_{\nu+1}$ the *ambient space* of $\mathbb{H}^n_\nu(c)$. Note the index $\nu + 1$ in this case. Again, if $c = -1$ we denote this space as $\mathbb{H}^{n+1}_\nu$.

To standardize our notation, we set $\mathbb{R}^{n+1}_\nu$ as the ambient space of $\mathbb{R}^n_\nu$, that is,

$$\mathbb{R}^n_\nu = \{ \; x \in \mathbb{R}^{n+1}_\nu \mid x_{n+1} = 0 \; \}.$$

If the index ν is equal to 1, we say that $\bar{M}^n_1$ is a *Lorentzian manifold* or a *spacetime*. We will denote by $\mathbb{Q}^{n+1}_1(c)$ the standard Lorentzian space forms; that is, the $(n+1)$-dimensional simply connected spacetimes of constant curvature c. Then for $c = 0$ we have the *Lorentz-Minkowski space* $\mathbb{R}^{n+1}_1$; for $c > 0$ the *de Sitter space* $\mathbb{S}^{n+1}_1(c)$, and for $c < 0$ the *anti-de Sitter space* $\mathbb{H}^{n+1}_1(c)$.

In this paper we will also use the notion and basic properties of doubly warped products, which we include here for completeness.

Definition 2.1. *Let (B, g_B), (F_1, g_1) and (F_2, g_2) be semi-Riemannian manifolds and $f_1, f_2 > 0$ differentiable functions on B. The* doubly warped product $B \times_{f_1} F_1 \times_{f_2} F_2$ *is the product manifold $B \times F_1 \times F_2$ endowed with the metric*

$$g = \pi^*(g_B) + (f_1 \circ \pi)^2 \sigma_1^*(g_1) + (f_2 \circ \pi)^2 \sigma_2^*(g_2), \tag{3}$$

where π, σ_1, σ_2 are the projections from $B \times F_1 \times F_2$ onto B, F_1 and F_2, respectively.

Remark 2.2. *In Sections 4 and 7 we will use a doubly warped product of the form $-I \times_f F \times_g J$, where $-I$ is an open interval with negative definite metric, F is a Riemannian manifold, J is an open interval with positive definite metric and f, g are smooth, positive functions defined on I.*

As in any product manifold (see [28], p. 24ff), we may lift to a given doubly warped product $B \times_{f_1} F_1 \times_{f_2} F_2$ functions, individual tangent vectors or even vector fields defined on any of the factors using the corresponding projection. We denote by $\mathfrak{L}(B)$ the space of lifts of vector fields in $\mathfrak{X}(B)$; $\mathfrak{L}(F_1)$ and $\mathfrak{L}(F_2)$ have analogous meanings. Unless otherwise stated, we will use the same notation for vector fields defined on any of the factors and their lifts.

It is easy to see that the Lie bracket of lifts from different factors is always zero; for example, if $X \in \mathfrak{L}(B)$ and $W \in \mathfrak{L}(F_2)$, then $[X, W] = 0$. (See [28], Corollary 44, p. 25.)

In the following proposition we include the formulae for covariant derivatives of vector fields in a doubly warped product (see also [34]). We denote by ∇^B, ∇^1, ∇^2 and $\bar\nabla$ the semi-Riemannian connection on B, F_1, F_2 and $B \times_{f_1} F_1 \times_{f_2} F_2$, respectively.

Proposition 2.3. *Given a doubly warped product $B \times_{f_1} F_1 \times_{f_2} F_2$ and vector fields $X, Y \in \mathfrak{L}(B)$, $V \in \mathfrak{L}(F_i)$, $W \in \mathfrak{L}(F_j)$, then*

1. $\bar\nabla_X Y = \nabla^B_X Y$.
2. $\bar\nabla_X W = \bar\nabla_W X = \dfrac{X(f_j)}{f_j} W$.

$$3.\ \bar{\nabla}_V W = \begin{cases} 0, & i \neq j, \\ \nabla^i_V W - \dfrac{g(V,W)}{f_i}\operatorname{grad}_B(f_i), & i = j. \end{cases}$$

The proof of Proposition 2.3 is completely analogous to that of Proposition 35 in ([28], p. 206) and will be omitted.

3. Timelike closed partially conformal vector fields. Here we define the class of vector fields we are interested in. It will be seen that the definition below is a natural extension of the concept of closed conformal vector field analyzed by Montiel in [25] and [26].

DEFINITION 3.1. *Let $\bar{M}^{n+1}_1$ be a spacetime with connection $\bar{\nabla}$ and K be a smooth timelike vector field (thus everywhere different from zero). We say that K is* closed partially conformal *in $\bar{M}^{n+1}_1$ if there is a unit vector field W orthogonal to K and functions $\phi, \psi : \bar{M}^{n+1}_1 \to \mathbb{R}$ such that*

$$\bar{\nabla}_X K = \phi X + (\psi - \phi)\langle X, W\rangle W \tag{4}$$

for every $X \in \mathfrak{X}(\bar{M})$.

In [9] and [10], the authors defined and analyzed the basic properties of closed partially conformal vector field both in the Riemannian and Lorentzian settings. In the following theorem we summarize these results and refer the reader to the cited papers.

THEOREM 3.2 (Theorem 3.2 in [9]). *Let $\bar{M}^{n+1}_1$ be a Lorentzian manifold possessing a timelike closed partially conformal vector field K. Then the distribution $K^{\perp}$ defined by taking the orthogonal complement of K at every point is involutive and each connected leaf of the foliation determined by this distribution is a $(n-1)$-umbilical hypersurface having $(n-1)$ equal and constant principal curvatures.*

For future reference, we give here the explicit expression for the principal curvatures of a leaf of the foliation determined by $K^{\perp}$. We denote $\widehat{K} = K/|K|$, where $|K| = \sqrt{-\langle K, K\rangle}$. It is easily proved that the principal curvatures of the leaf with respect to $\widehat{K}$ are given by

$$\kappa_i = -\frac{\phi}{|K|}, \quad i = 1, \ldots, n-1; \quad \text{and} \quad \kappa_n = -\frac{\psi}{|K|}. \tag{5}$$

The authors proved in [9] that ϕ and $|K|$ are constant along each connected leaf of the foliation determined by $K^{\perp}$, which implies in turn that the principal curvatures κ_i, $i = 1, \ldots, n-1$ are also constant along such a leaf. One may also ask whether the function ψ is constant along the leaf. We give an answer to this question in the following proposition.

PROPOSITION 3.3. *If either* div K*, the function ψ or the mean curvature H is constant along a leaf of the foliation determined by $K^{\perp}$, then the same happens with the other two quantities.*

Each of the above conditions imply that every r-th mean curvature H_r, $r > 1$, is constant along a leaf of the foliation determined by $K^{\perp}$. Conversely, if there exists $r > 1$ such that H_r is constant along a leaf of the foliation determined by $K^{\perp}$ and $\phi \neq 0$, then each of the above quantities is constant along that leaf.

Proof. Let $E_1, \ldots, E_{n-1}, E_n = W$ be an orthonormal frame on the leaf. Then from (4) we have that $\bar{\nabla}_{E_i} K = \phi E_i$ for $i = 1, \ldots, n-1$ and $\bar{\nabla}_{E_n} K = \psi E_n$. Let us calculate the divergence of K using this frame and the timelike unit vector field $\widehat{K}$:

$$\begin{aligned} \operatorname{div} K &= \sum_{i=1}^{n-1} \langle \bar{\nabla}_{E_i} K, E_i \rangle + \langle \bar{\nabla}_{E_n} K, E_n \rangle - \langle \bar{\nabla}_{\widehat{K}} K, \widehat{K} \rangle \\ &= \sum_{i=1}^{n-1} \phi \langle E_i, E_i \rangle + \psi \langle E_n, E_n \rangle - \phi \langle \widehat{K}, \widehat{K} \rangle = n\phi + \psi. \end{aligned}$$

Using (2) and (5), we obtain that the r-th mean curvature of the leaf is given by

$$\binom{n}{r} H_r = \binom{n-1}{r-1} \left(-\frac{\phi}{|K|} \right)^{r-1} \left(-\frac{\psi}{|K|} \right) + \binom{n-1}{r} \left(-\frac{\phi}{|K|} \right)^r,$$

so that

$$n|K|^r H_r = (-1)^r (r\phi^{r-1}\psi + (n-r)\phi^r).$$

As ϕ and $|K|$ are constant along the leaf, these formulas prove our claim. □

For further use, we give the following useful definition.

DEFINITION 3.4. *Let $\bar{M}_1^{n+1}$ be a Lorentzian manifold and M a hypersurface of $\bar{M}_1^{n+1}$. M is a* cylinder *if it is a $(n-1)$-umbilical hypersurface with two distinct and constant principal curvatures.*

4. Examples and properties of spacetimes with a timelike closed partially conformal vector field. The spacetimes we are interested in are those which admit (at least in an open subset) a timelike closed partially conformal vector field. We will give some examples for the Lorentzian space forms $\mathbb{Q}_1^{n+1}(c)$, $c \geq 0$.

EXAMPLE 4.1. *If $c = 0$, i.e., if our manifold is the Lorentz-Minkowski space $\mathbb{R}_1^{n+1}$ with standard coordinates $x_1, \ldots, x_{n+1}$, we may work in the following open subset:*

$$\Omega = \{\ x \in \mathbb{R}_1^{n+1} \mid -x_1^2 + x_2^2 + \cdots + x_n^2 < 0\ \}$$

and define the following vector field

$$K = \sum_{i=1}^{n} x_i \partial_i, \quad \text{where} \quad \partial_i = \frac{\partial}{\partial x_i}.$$

It is easy to see that

$$\bar{\nabla}_{\partial_j} K = \partial_j, \text{ for } j = 1, \ldots, n \quad \text{and} \quad \bar{\nabla}_{\partial_{n+1}} K = 0,$$

which means that K is a closed partially conformal vector field. Note that K is timelike in Ω. The foliation of Ω associated to K is given by the 1-parameter family of spacelike cylinders *defined for $c < 0$ as*

$$\mathbb{H}^{n-1}(c) \times \mathbb{R} = \left\{ (x, y) \in \mathbb{R}_1^{n+1} \;\middle|\; -x_1^2 + x_2^2 + \cdots + x_n^2 = \frac{1}{c} \right\}$$

which are already present in the literature; see [21], for example.

EXAMPLE 4.2. *We now define a timelike closed partially conformal vector field in the open subset of* $\mathbb{S}_1^{n+1}$ *given by*

$$\Omega = \left\{ (x_1, \dots, x_{n+2}) \in \mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid -x_1^2 + x_2^2 < 0, x_1 > 0 \right\}.$$

The open set Ω *may be parametrized by*

$$P(t, u_1, \dots, u_{n-1}, s) = (\sinh t \cosh s, \sinh t \sinh s, \cosh t \cdot \Phi(u_1, \dots, u_{n-1})),$$

where $t > 0$, $s \in \mathbb{R}$ *and* Φ *is an orthogonal parametrization of the unit sphere* $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

We denote by $\bar{\nabla}$ *the connection on* $\mathbb{R}_1^{n+2}$ *and* ∇ *the induced connection on* $\mathbb{S}_1^{n+1}$. *Recall that the position vector* P *as given above is precisely the unit vector field everywhere normal to* $\mathbb{S}_1^{n+1}$ *and that the de Sitter space is totally umbilical in* $\mathbb{R}_1^{n+2}$*; in fact we have*

$$\bar{\nabla}_X Y = \nabla_X Y - \langle X, Y \rangle P$$

for every $X, Y \in \mathfrak{X}(\mathbb{S}_1^{n+1})$. *Using this expression we obtain*

$$\nabla_{\partial_t}\partial_t = 0, \quad \nabla_{\partial_i}\partial_t = (\tanh t)\partial_i \quad \text{and} \quad \nabla_{\partial_s}\partial_t = (\coth t)\partial_s,$$

where $\partial_i = \partial P/\partial u_i$, $i = 1, \dots, n-1$. *Now we define the (timelike) vector field* $K = (\cosh t)\partial_t$ *which satisfies*

$$\nabla_{\partial_t} K = (\sinh t)\partial_t; \quad \nabla_{\partial_i}\partial_t = (\sinh t)\partial_i \quad \text{and} \quad \nabla_{\partial_s} K = (\cosh t \coth t)\partial_s.$$

where again $i = 1, \dots, n-1$. *These expressions show that* K *is a timelike closed partially conformal vector field defined in* Ω. *The foliation associated to* K *is that of the* hyperbolic cylinders $\mathbb{H}^1(c_1) \times \mathbb{S}^{n-1}(c_2) \subset \mathbb{S}_1^{n+1}$ *given by*

$$\left\{ (x_1, \dots, x_{n+2}) \in \mathbb{S}_1^{n+1} \;\middle|\; -x_1^2 + x_2^2 = \frac{1}{c_1},\ x_3^2 + \dots + x_{n+2}^2 = \frac{1}{c_2} \right\},$$

where $c_1 < 0$, $c_2 > 0$ *and* $(1/c_1) + (1/c_2) = 1$.

We remark that a similar approach may be followed to define a timelike closed partially conformal vector field in an open subset of the anti-de Sitter space, which will be included elsewhere.

In each case we can express the domain of K as a doubly warped product (see Definition 2.1). Explicitly, for the Example 4.1 in $\mathbb{R}_1^{n+1}$, the domain Ω may be parametrized as

$$(t \cdot \Phi(u_1, \dots, u_{n-1}), s), \quad t > 0,$$

where Φ is an orthogonal parametrization of the unit hyperbolic space $\mathbb{H}^{n-1}$. Using this parametrization and the notation $d\sigma^2$ for the metric on $\mathbb{H}^{n-1}$, we express the metric on Ω as

$$-dt^2 + t^2 d\sigma^2 + ds^2,$$

which shows that Ω is the doubly warped product

$$-I \times_f \mathbb{H}^{n-1} \times_g \mathbb{R}, \quad I = (0, \infty),$$

with $f, g : I \to \mathbb{R}^+$ given by $f(t) = t$ and $g(t) \equiv 1$.

On the other hand, for the Example 4.2 in $\mathbb{S}_1^{n+1}$ we use the parametrization there given for the domain Ω to write the metric as

$$-dt^2 + (\cosh^2 t)\, d\sigma^2 + (\sinh^2 t)\, ds^2,$$

where $d\sigma^2$ is the standard metric on the unit sphere $\mathbb{S}^{n-1}$. In this case, Ω is the doubly warped product

$$-I \times_f \mathbb{S}^{n-1} \times_g \mathbb{R},$$

where $I = (0, \infty)$, $f(t) = \cosh t$ and $g(t) = \sinh t$.

These examples suggest a strong relation between the existence of a timelike closed partially conformal vector field on a given spacetime and a certain doubly warped product structure on the spacetime. In fact, we have the following result.

PROPOSITION 4.3. *Every Lorentzian doubly warped product of the form*

$$\bar{M}_1^{n+1} = -I \times_f F^{n-1} \times_g J, \quad \text{where } I, J \subset \mathbb{R},\ f, g : I \to \mathbb{R}^+,\ \text{and } f \neq g,$$

admits a timelike closed partially conformal vector field.

Proof. Take t, s standard coordinates on I, J respectively, and $u_1, \ldots, u_{n-1}$ coordinates on F, with corresponding vector fields $\partial_t, \partial_s, \partial_1 \ldots, \partial_{n-1}$. Using the formulae of Proposition 2.3 we have that

$$\bar{\nabla}_{\partial_t}\partial_t = 0, \quad \bar{\nabla}_{\partial_i}\partial_t = \frac{f'}{f}\partial_i \quad \text{and} \quad \bar{\nabla}_{\partial_s}\partial_t = \frac{g'}{g}\partial_s,$$

where $i = 1, \ldots, n-1$. We define $K = f\,\partial_t$ and obtain that

$$\bar{\nabla}_{\partial_t} K = f'\partial_t, \quad \bar{\nabla}_{\partial_i} K = f'\partial_i \quad \text{and} \quad \bar{\nabla}_{\partial_s} K = f\frac{g'}{g}\partial_s,$$

where again $i = 1, \ldots, n-1$. Since $f \neq g$, we have that K is a timelike closed partially conformal vector field. □

REMARK 4.4. *The unit vector field W associated to $K = f\,\partial t$ is given by*

$$W = \frac{1}{g}\,\partial_s.$$

We may use again Proposition 2.3 to conclude easily that it satisfies

$$\bar{\nabla}_{\partial_t} W = \frac{g'}{g} W \quad \text{and} \quad \bar{\nabla}_V W = 0 \tag{6}$$

for $\partial_t \in \mathfrak{L}(I)$ and each $V \in \mathfrak{L}(F)$. Also, if we fix $s \in J$ and denote $\bar{M}_s = -I \times_f F \times \{s\}$, we observe that $\bar{M}_s$ is totally geodesic in $\bar{M}$. In fact, let α_s denote the second fundamental form of $\bar{M}_s$ in $\bar{M}$. We use the fact that W is everywhere normal to $\bar{M}_s$ to have for every $X, Y \in \mathfrak{X}(\bar{M}_s)$

$$\langle \alpha_s(X, Y), W\rangle = \langle \bar{\nabla}_X Y, W\rangle = -\langle Y, \bar{\nabla}_X W\rangle.$$

Now write X as $a\partial_t + V$, where a is a smooth function and $V \in \mathfrak{L}(F)$. By equation (6),

$$\langle \alpha_s(X,Y), W\rangle = -\frac{ag'}{g}\langle Y, W\rangle = 0,$$

which implies our assertion that $\bar{M}_s$ is totally geodesic in $\bar{M}$.

The following proposition is a partial converse of the above facts.

PROPOSITION 4.5. *Let $\bar{M}_1^{n+1}$ be a spacetime which admits a timelike closed partially conformal vector field K with an associated vector field W orthogonal to K. Suppose W is parallel. Then $\bar{M}_1^{n+1}$ can be expressed locally as a doubly warped product of the form $-I \times_f F \times J$.*

Proof. Let us prove first that the distribution $W^\perp$ everywhere orthogonal to W is involutive. If $X, Y \in W^\perp$, then

$$\begin{aligned}\langle [X,Y], W\rangle &= \langle \bar{\nabla}_X Y - \bar{\nabla}_Y X, W\rangle \\ &= X\langle Y, W\rangle - \langle Y, \bar{\nabla}_X W\rangle - (Y\langle X, W\rangle - \langle X, \bar{\nabla}_Y W\rangle) = 0.\end{aligned}$$

Now note that K is a timelike closed conformal field when restricted to each integral manifold of $W^\perp$, so by Proposition 2 in [26] this integral manifold may be expressed as a warped product $-I \times_f F$. As W is parallel, it is also Killing, and its flow is by isometries. Following this flow we may write locally $\bar{M}_1^{n+1}$ as $-I \times_f F \times J$. □

5. Rotation hypersurfaces. The examples in the previous section were quite particular. As we shall see, the hypersurfaces involved are nothing but rotation hypersurfaces. A general definition of this kind of hypersurfaces was given in the Riemannian case by do Carmo and Dajczer in [13], which was extended later to some spacetimes (see for example [27]). For the sake of completeness, we give the definition of these rotation hypersufaces in $\mathbb{Q}_1^{n+1}(c)$.

Recall from the introduction that each $\mathbb{Q}_1^{n+1}(c)$ has an ambient space of the form $\mathbb{R}_\nu^{n+2}$, $\nu = 1, 2$. We say that an *orthogonal transformation* of $\mathbb{R}_\nu^{n+2}$ is a linear map that preserves the metric. By restriction, these orthogonal transformations induce all isometries of $\mathbb{Q}_1^{n+1}(c)$.

Let P^k be a k-dimensional vector subspace of $\mathbb{R}_\nu^{n+2}$. $O(P^k)$ will denote the set of orthogonal transformations of $\mathbb{R}_\nu^{n+2}$ with positive determinant that leave P^k pointwise fixed.

Choose P^2 and $P^2 \subset P^3$, and let C be a regular, spacelike curve in $\mathbb{Q}_1^{n+1}(c) \cap (P^3 - P^2)$, parametrized by arc length. The orbit of C under $O(P^2)$ is called the *rotation spacelike hypersurface M in $\mathbb{Q}_1^{n+1}(c)$ generated by C*. M is *spherical* (*hyperbolic, parabolic*, resp.) whenever the restriction of the metric to P^2 is Lorentzian (Riemannian, degenerate, resp.).

We are interested in rotation hypersurfaces in $\mathbb{Q}_1^{n+1}(c)$ with H_r constant. Spherical rotation hypersurfaces with H_1 and H_2 constant in the de Sitter space $\mathbb{S}_1^{n+1}$ were described in previous works (see [7] and [8]). Thus, we will describe here in detail the hyperbolic rotation hypersurfaces in $\mathbb{S}_1^{n+1}$ and will make some comments about the spherical and parabolic cases.

Let $\{e_1, e_2, \ldots, e_{n+2}\}$ be the canonical orthonormal basis of $\mathbb{R}_1^{n+2}$, so that

$$\langle e_1, e_1\rangle = -1 \quad \text{and} \quad \langle e_i, e_i\rangle = 1 \text{ for } i > 1.$$

Also, let $P^2 = \mathrm{span}(e_{n+1}, e_{n+2})$ and $P^3 = \mathrm{span}(e_1, e_{n+1}, e_{n+2})$. The profile curve generating the rotation hypersurface is given by

$$(y_1(s), 0, \ldots, 0, y_{n+1}(s), y_{n+2}(s)),$$

where

$$-y_1^2 + y_{n+1}^2 + y_{n+2}^2 = 1 \qquad \text{and} \qquad -\dot{y}_1^2 + \dot{y}_{n+1}^2 + \dot{y}_{n+2}^2 = 1.$$

Here the dots denote derivative with respect to s.

Following [13], we may describe $O(P^2)$ as follows. The matrix of an element of $O(P^2)$ with respect to the canonical basis has the form

$$\begin{pmatrix} B & & & & \\ & A_1 & & & \\ & & \ddots & & \\ & & & A_{(n-2)/2} & \\ & & & & I \end{pmatrix}$$

for n even, or the form

$$\begin{pmatrix} B & & & & & \\ & A_1 & & & & \\ & & \ddots & & & \\ & & & A_{(n-3)/2} & & \\ & & & & 1 & \\ & & & & & I \end{pmatrix}$$

for n odd. Here I denotes the identity matrix 2×2, the matrix B is given by

$$B = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}.$$

and, for each i, the matrix A_i is given by

$$A_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}.$$

Now take $\Phi(t_1, \ldots, t_{n-1}) = (\varphi_1, \ldots, \varphi_n)$ as an orthogonal parametrization of the unit hyperbolic space $\mathbb{H}^{n-1} \subset \mathbb{R}_1^n$, so that

$$-\varphi_1^2 + \varphi_2^2 + \cdots + \varphi_n^2 = -1.$$

Thus,

$$f(t_1, \ldots, t_{n-1}, s) = (y_1(s)\Phi(t_1, \ldots, t_{n-1}), y_{n+1}(s), y_{n+2}(s)) \tag{7}$$

is the desired parametrization of the spacelike hyperbolic rotation hypersurface generated by the curve $(y_1(s), 0, \ldots, 0, y_{n+1}(s), y_{n+2}(s))$.

We will calculate the principal curvatures of the rotation hypersurfaces parametrized by (7). Differentiating this equation, we have

$$\frac{\partial f}{\partial s} = (\dot{y}_1\Phi, \dot{y}_{n+1}, \dot{y}_{n+2}) \quad \text{and} \quad \frac{\partial f}{\partial t_j} = \left(y_1\frac{\partial \Phi}{\partial t_j}, 0, 0\right),$$

so that

$$\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s} \right\rangle = 1, \; \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t_j} \right\rangle = 0 \quad \text{and} \quad \left\langle \frac{\partial f}{\partial t_i}, \frac{\partial f}{\partial t_j} \right\rangle = y_1^2 \left\langle \frac{\partial \Phi}{\partial t_i}, \frac{\partial \Phi}{\partial t_j} \right\rangle.$$

Differentiating again, we obtain

$$\begin{aligned} \frac{\partial^2 f}{\partial s^2} &= (\ddot{y}_1 \Phi, \ddot{y}_{n+1}, \ddot{y}_{n+2}), \\ \frac{\partial^2 f}{\partial s \partial t_j} &= \left(\dot{y}_1 \frac{\partial \Phi}{\partial t_j}, 0, 0 \right), \\ \frac{\partial^2 f}{\partial t_i \partial t_j} &= \left(y_1 \frac{\partial^2 \Phi}{\partial t_i \partial t_j}, 0, 0 \right). \end{aligned}$$

We choose the unit normal vector N as

$$(-(\dot{y}_{n+1} y_{n+2} - y_{n+1} \dot{y}_{n+2})\Phi, (y_1 \dot{y}_{n+2} - \dot{y}_1 y_{n+2}), (\dot{y}_1 y_{n+1} - y_1 \dot{y}_{n+2})).$$

We proceed to calculate the principal curvatures in terms of the coordinates of the generating curve. The coordinate curves are lines of curvature and the principal curvatures along the t_i–curves are

$$\kappa_i = \frac{\langle N, \partial^2 f / \partial t_i^2 \rangle}{\langle \partial f / \partial t_i, \partial f / \partial t_i \rangle} = -\frac{\dot{y}_{n+1} y_{n+2} - y_{n+1} \dot{y}_{n+2}}{y_1}. \tag{8}$$

We use $-y_1^2 + y_{n+1}^2 + y_{n+2}^2 = 1$ to write these principal curvatures in terms of y_3 alone. Let

$$y_{n+1} = \sqrt{1 + y_1^2} \cos\theta \quad \text{and} \quad y_{n+2} = \sqrt{1 + y_1^2} \sin\theta, \tag{9}$$

for an unknown function θ, which may be obtained deriving the above expressions and using $-\dot{y}_1^2 + \dot{y}_{n+1}^2 + \dot{y}_{n+2}^2 = 1$. We have

$$\dot{\theta}^2 = \frac{y_1^2 + \dot{y}_1^2 + 1}{y_1^2 + 1}.$$

We differentiate y_{n+1} and y_{n+2} in (9), use the above expression for $\dot{\theta}$ and (8) to express the principal curvatures κ_i of M in terms of y_1 as

$$\kappa_i = \frac{\sqrt{y_1^2 + \dot{y}_1^2 + 1}}{y_1}.$$

The expression for κ_n is obtained in a similar but longer way, differentiating $\dot{y}_{n+1}$ and $\dot{y}_{n+2}$, using (8), (9) and the expression for $\dot{\theta}$. In the following proposition we summarize this analysis, including the spherical and parabolic cases which may be treated similarly (see for example [27] for the case $n = 2$).

THEOREM 5.1. *The principal curvatures of a rotation hypersurface in $\mathbb{S}_1^{n+1}$ are given by*

$$\kappa_i = \frac{\sqrt{y^2 + \dot{y}^2 - \delta}}{y} \quad \text{and} \quad \kappa_n = \frac{\ddot{y} + y}{\sqrt{y^2 + \dot{y}^2 - \delta}}, \tag{10}$$

where $i = 1, \ldots, n-1$, $y^2 \geq \delta$ and $\delta = -1, -1, 0$ depending on whether the rotation hypersurface is spherical, hyperbolic or parabolic, respectively.

Using (2) and (10) we obtain the following formula for the r-th mean curvature H_r:

$$\binom{n}{r} H_r = \binom{n-1}{r} \left(\frac{\sqrt{y^2 + \dot{y}^2 - \delta}}{y} \right)^r + \\ + \binom{n-1}{r-1} \left(\frac{\sqrt{y^2 + \dot{y}^2 - \delta}}{y} \right)^{r-1} \frac{\ddot{y} + y}{\sqrt{y^2 + \dot{y}^2 - \delta}},$$

which simplifies to

$$nH_r y^r = (n-r)\left(y^2 + \dot{y}^2 - \delta\right)^{r/2} + r\left(y^2 + \dot{y}^2 - \delta\right)^{(r-2)/2} (\ddot{y} + y)y. \tag{11}$$

We will study this equation in detail in Section 6. To close this section, we give a useful characterization of rotation hypersurfaces. The proof of this Theorem is the same as in [13] and we shall omit it.

PROPOSITION 5.2. *Let M^n, $n \geq 3$, be an arbitrary connected, spacelike hypersurface in $\mathbb{Q}_1^{n+1}(c)$. Assume that the principal curvatures $\kappa_1, \ldots, \kappa_n$ satisfy $\kappa_1 = \cdots = \kappa_{n-1} = -\lambda \neq 0$ and $\kappa_n = -\mu = -\mu(\lambda)$, where $\lambda \neq \mu$. Then M^n is contained in a rotation hypersurface.*

6. Rotation hypersurfaces with H_r constant in $\mathbb{S}_1^{n+1}$. Let us suppose that H_r is constant in Equation (11). Then a first integral is given by

$$G_r(y, \dot{y}) = y^{n-r}\left(\left(y^2 + \dot{y}^2 - \delta\right)^{r/2} - H_r y^r\right). \tag{12}$$

The analysis of the level curves of G_r will give us a full classification of the spacelike rotation hypersurfaces with H_r constant in $\mathbb{S}_1^{n+1}$. In particular, we will prove the existence of 1-parameter families of such hypersurfaces, which will include a cylinder (Definition 3.4) depending on the value of H_r. For $r = 1$, we re-obtain the well-known constant mean curvature cylinders already cited in this paper.

First we study very briefly the parabolic case $\delta = 0$ in (12), namely,

$$G_r(y, \dot{y}) = y^{n-r}\left(\left(y^2 + \dot{y}^2\right)^{r/2} - H_r y^r\right).$$

The critical points of G_r have special relevance. Solving the equations $\partial G_r/\partial y = 0$ and $\partial G_r/\partial \dot{y} = 0$ we obtain easily that

- G_r has no critical points other than $(0, 0)$ for $H_r \neq 1$, and
- Every point of the form $(y, 0)$ with $y \geq 0$ is a critical point of G_r for $H_r = 1$.

To each critical point $(y, 0)$ with $y > 0$ we associate a parabolic rotation hypersurface. By (10), the principal curvatures of such a hypersurface are all equal, so we obtain a totally umbilical hypersurface, which of course is also $(n-1)$-umbilical.

From now on we suppose that $\delta = \pm 1$. Additionally, we will analyze later the case $H_r = 0$, so that we will suppose $H_r \neq 0$ and separate our analysis in the following cases according to the value of r: $r = 1$ (the constant mean curvature case), $r = 2$, $2 < r < n$ odd, $2 < r < n$ even, and $r = n$.

1. Case of $r = 1$. Equation (12) takes the form

$$G_1(y, \dot{y}) = y^{n-1}\left(\sqrt{y^2 + \dot{y}^2 - \delta} - Hy\right), \quad H = H_1.$$

We will describe the level curves of G_1. As in the analysis of the parabolic rotation case, the critical points of G_1 have particular importance. Solving $\partial G_1/\partial y = 0$ and $\partial G_1/\partial \dot{y} = 0$ we obtain a critical point of the form $(y, 0)$, where

$$y^2 - nHy\sqrt{y^2 - \delta} + (n-1)(y^2 - \delta) = 0.$$

To solve it, we divide by $(y^2 - \delta)$ to get

$$\frac{y}{y^2 - \delta} - nH\frac{y}{\sqrt{y^2 - \delta}} + (n-1) = 0,$$

which we may write as

$$x^2 - nHx + (n-1) = 0, \quad x = \frac{y}{\sqrt{y^2 - \delta}},$$

with solutions

$$x_{\pm} = \frac{nH \pm \sqrt{n^2H^2 - 4(n-1)}}{2}. \tag{13}$$

From this expression we have the restriction $n^2H^2 - 4(n-1) \geq 0$, or $|H| \geq 2\sqrt{n-1}/n$. We may choose the vector normal to the hypersurface such that the mean curvature is positive, hence $H \geq 2\sqrt{n-1}/n$. In this case,

$$x_+ = \frac{nH + \sqrt{n^2H^2 - 4(n-1)}}{2} \geq \frac{nH}{2} \geq \sqrt{n-1} > 1,$$

and then there exists $y_+ > 1$ such that

$$x_+ = \frac{y_+}{\sqrt{y_+^2 - 1}} > 1. \tag{14}$$

In short, if $H \geq 2\sqrt{n-1}/n$ and $\delta = 1$, the function G_1 has one critical point of the form $(y_+, 0)$.
It is also easy to see that $2\sqrt{n-1}/n \leq H < 1$ implies $x_- > 1$, so we may write it analogously to (14) for some $y_- > 1$, obtaining another critical point for G_1 in the case $\delta = 1$.
On the other hand, if $H = 1$, then $x_- = 1$, which can not be expressed as $y/\sqrt{y^2 - \delta}$ for $\delta = \pm 1$.
Lastly, if $H > 1$ we have that $x_- < 1$. In this case, we use $\delta = -1$ and write

$$x_- = \frac{y_-}{\sqrt{y_-^2 + 1}} \quad \text{for some } y_- > 0. \tag{15}$$

2. Case of $r = 2$. The first integral equation (12) reads

$$G_2(y, \dot{y}) = y^{n-2}\left(y^2 + \dot{y}^2 - \delta - H_2y^2\right)$$

The critical points of the form $(y,0)$ are given by

$$y^{n-3}\big(n(1-H_2)y^2-(n-2)\delta\big)=0.$$

If $y\neq 0$, we obtain a critical point satisfying

$$y_+^2=\frac{(n-2)\delta}{n(1-H_2)}. \tag{16}$$

In consequence, the function G_2 has one critical point $(y_+,0)$ with $y_+>0$ either if $\delta=1$ and $H_2<1$, or if $\delta=-1$ and $H_2>1$. In the case $H_2=1$ we do not obtain critical points of this form.

3. Case of $2<r<n$ and r odd. We may suppose that $H_r>0$. Here the critical points of G_r of the form $(y,0)$ must satisfy

$$(n-r)\left(y^2-\delta\right)^{r/2}+ry^2\left(y^2-\delta\right)^{(r-2)/2}-nH_ry^r=0.$$

We divide by $\left(y^2-\delta\right)^{r/2}$ to obtain

$$h(x)=(n-r)+rx^2-nH_rx^r=0,\quad \text{where } x=\frac{y}{\sqrt{y^2-\delta}}>1. \tag{17}$$

An elementary analysis of this function shows that h is increasing in $[0,a]$ and decreasing in $[a,\infty)$, where

$$a^{r-2}=\frac{2}{nH_r}.$$

Since $h(0)=(n-r)>0$ and $h(x)\to-\infty$ when $x\to\infty$, h has a unique positive real root x_+. If $0<H_r<1$, $h(1)>0$ and x_+ must lie to the right of 1. This fact implies the existence of $y_+>1$ such that

$$x_+=\frac{y_+}{\sqrt{y_+^2-1}}.$$

Similarly, if $H_r>1$, $h(1)<0$, which means that x_+ lies to the left of 1 and we write instead

$$x_+=\frac{y_+}{\sqrt{y_+^2+1}}.$$

Once again, if $H_r=1$, $x_+=1$, which can not be written as $y/\sqrt{y^2-\delta}$ for $\delta=\pm1$.

4. Case of $2<r<n$ and r even. We obtain again the function $h(x)$ in (17). The difference is that we must consider separately the case $H_r<0$. In this case h increases to $+\infty$ when $x\to\infty$ and we do not get any critical points of G_r. The case $H_r>0$ is entirely analogous to that of the previous case.
5. Case of $r=n$. Equation (12) reads

$$G_n(y,\dot y)=(y^2+\dot y^2-\delta)^{n/2}-H_ny^n.$$

The critical points of the form $(y,0)$, $y>0$, must satisfy

$$(y^2-\delta)^{(n-2)/2}y-H_ny^{n-1}=0,$$

which does not have real roots if $H_n < 0$. On the other hand, if $H_n > 0$,

$$y^2 = \frac{\delta}{1 - H_n^{2/(n-2)}}.$$

If $\delta = 1$ ($\delta = -1$) we must impose the additional condition $H_n < 1$ ($H_n > 1$) to obtain critical points of G_n, similarly to the previous cases.

To complete the above analysis we must see what happens when $H_r = 0$. Equation (17) reads

$$h(x) = (n - r) + rx^2,$$

which never vanishes. This fact means in turn that G_r does not have critical points for $H_r = 0$.

Tracing our way back, each critical point $(y, 0)$ of G_r, $y > 0$, gives rise to a rotation hypersurface with *constant* principal curvatures given by equation (10), i.e., by

$$\kappa_i = \frac{\sqrt{y^2 - \delta}}{y} \quad \text{and} \quad \kappa_n = \frac{y}{\sqrt{y^2 - \delta}}. \tag{18}$$

If $\delta = \pm 1$ each critical point of G_r gives rise to a $(n-1)$-umbilical hypersurface with two distinct and constant principal curvatures; that is, a cylinder in the sense of our Definition 3.4. In general, we obtain a 1-parameter family of rotation hypersurfaces varying the values of the first integral G_r appearing in (12).

We summarize this relationship between the level curves of G_r and rotation hypersurfaces in the following result.

THEOREM 6.1. *For every real number H_r there are three 1-parameter families of spacelike rotation hypersurfaces $\mathbb{S}_1^{n+1}$ with constant r-th mean curvature H_r, corresponding to the kind of rotation (spherical, parabolic or hyperbolic) considered. The family of parabolic rotation hypersurfaces contains the totally umbilical hypersurfaces of $\mathbb{S}_1^{n+1}$. Moreover,*

1. *For $r = 1$, the family of spherical rotation hypersurfaces contains two cylinders if $2\sqrt{n-1}/n < H_1 < 1$, while it contains one cylinder if $H = 2\sqrt{n-1}/n$.*
2. *For $2 \leq r \leq n$, the family of spherical rotation hypersurfaces contains one cylinder if $0 < H_r < 1$.*
3. *For $1 \leq r \leq n$, the family of hyperbolic rotation hypersurfaces contains one cylinder if $H_r > 1$.*

We digress for a moment, taking advantage of the previous discussion to characterize any non-umbilical leaf of a foliation associated to a timelike closed partially conformal vector field defined on $\mathbb{S}_1^{n+1}$ and having constant r-th mean curvature as a cylinder.

COROLLARY 6.2. *Let K be a timelike closed partially conformal vector field defined on an open subset of $\mathbb{S}_1^{n+1}$. If a connected leaf of $\mathcal{K}^\perp$ has constant mean curvature, then the leaf is totally umbilical or a cylinder.*

Moreover, let ϕ, ψ be the functions associated with K via equation (4). Suppose that $\phi \neq 0$ and that a connected leaf of $\mathcal{K}^\perp$ has constant r-th mean curvature for some $r > 1$. Then the leaf is totally umbilical or a cylinder.

Proof. Suppose that a connected leaf of $\mathcal{K}^{\perp}$ has constant mean curvature. If ϕ, ψ are the functions associated with K, we know that ϕ and $|K|$ are constant along this leaf. Moreover, since the mean curvature of the leaf is constant, Proposition 3.3 implies that ψ is also constant along the leaf. If $\phi = \psi$, from equation (5) we have that all principal curvatures are equal and the leaf is totally umbilical.

On the other hand, if $\phi \neq \psi$, then ψ may be considered as a (constant) function of ϕ. Proposition 5.2 implies that the leaf is a rotation hypersurface. From our previous discussion, the only rotation hypersurfaces with constant principal curvatures are the cylinders, so the claim follows.

The argument is completely analogous for the case of constant r-th mean curvature. □

To close this section we will refine Theorem 6.1 by proving that some families of spherical rotation hypersurfaces contain a subfamily of *complete* hypersurfaces.

Theorem 6.3. *For each of the following values of r and H_r there is a 1-parameter family of complete spherical rotation spacelike hypersurfaces with constant r-th mean curvature H_r in $\mathbb{S}_1^{n+1}$:*

1. *$r = 1$ and $2\sqrt{n-1}/n < H < 1$;*
2. *$r \geq 2$ and $0 < H_r < 1$.*

Proof. Suppose $r = 1$ and $2\sqrt{n-1}/n < H < 1$. As proved before, we obtain two critical points $(y_\pm, 0)$ of G_1 given by equations (14) and (15), whereas $x_\pm$ satisfies (13). We will prove that $G_1(y, \dot{y})$ attains a minimum at $(y_+, 0)$. By an easy calculation we have

$$\frac{\partial^2 G_1}{\partial \dot{y}^2}(y_+, 0) > 0 \quad \text{and} \quad \frac{\partial^2 G_1}{\partial y \partial \dot{y}}(y_+, 0) = 0,$$

so that it remains to analyze the function

$$g_1(y) = G_1(y, 0) = y^{n-1}\left(\sqrt{y^2 - 1} - Hy\right).$$

We will prove that the coordinates $y_\pm$ belong to the closed interval $[1, y_0]$, where y_0 is the positive root of g_1 satisfying

$$y_0^2 - 1 = H^2 y_0^2.$$

Since $1 < y_- < y_+$, we just have to prove that $y_+ < y_0$. Recalling that $2\sqrt{n-1}/n < H < 1$, we have

$$2\sqrt{n-1} < nH < nH + \sqrt{n^2H^2 - 4(n-1)}$$

and then

$$\sqrt{n-1} < x_+ = \frac{y_+}{\sqrt{y_+^2 - 1}}.$$

On the other hand,

$$\frac{y_0}{\sqrt{y_0^2 - 1}} = \frac{1}{H} < \frac{n}{2\sqrt{n-1}},$$

but

$$\frac{n}{2\sqrt{n-1}} \leq \sqrt{n-1} \quad \text{for} \quad n \geq 2;$$

then

$$\frac{y_0}{\sqrt{y_0^2-1}} < \frac{y_+}{\sqrt{y_+^2-1}},$$

which implies $y_+ < y_0$, as desired.

We obtain easily the following properties of g_1:

$$g_1(1) < 0, \quad g_1(y_0) = 0, \quad \lim_{y \to 1^+} g_1'(y) = +\infty \quad \text{and} \quad g_1'(y_0) > 0.$$

From these properties and the fact that both critical points $y_\pm$ belong to $[1, y_0]$ it is easy to conclude that g_1 attains a local minimum at y_+.

Hence, G_1 attains a minimum at $(y_+, 0)$, which implies that all level curves of G_1 near to this critical point are closed. Figure 1 shows some level curves for a typical function G_1.

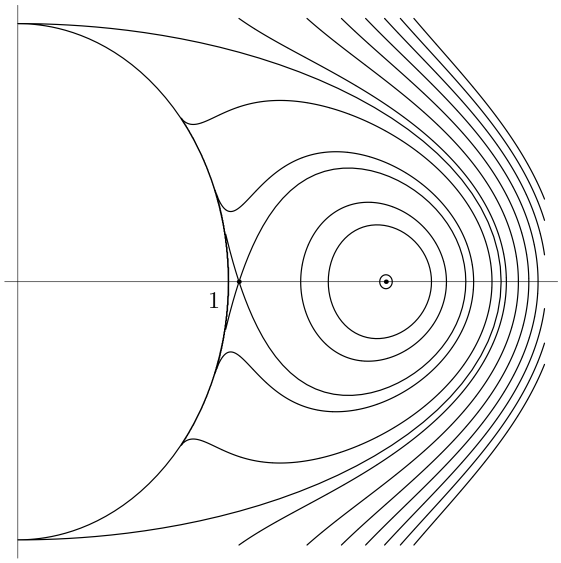

FIG. 1. *Several level curves for a typical function G_1. Note the two critical points $(y_\pm, 0)$ on the horizontal axis; one is a saddle point, while G_1 attains a minimum at the other.*

Now, each closed level curve is associated to a solution of the original equation (11) defined for all values of the parameter. In turn, this solution gives rise to a complete rotation hypersurface.

For the case $r = 2$ we analyze the function

$$g_2(y) = G_2(y, 0) = y^{n-2}((1 - H_2)y^2 - 1).$$

which has a critical point y_+ given in (16). Let $y_0 > 0$ be the point such that $g_2(y_0) = 0$, i.e., $(1 - H_2)y_0^2 = 1$. Then it is easy to show that $y_+ < y_0$. Also, from the facts that $g_2(0) = g_2(y_0) = 0$ and $g_2'(y_0) > 0$ we have that g_2 attains a minimum at y_+. The rest of the argument is similar to that of the case $r = 1$.

The analysis of the case $2 < r \leq n$ follows the same line of argumentation as the previous cases and will be omitted. ◻

7. Conditions for a hypersurface to be $(n-1)$-umbilical. Let $\bar{M}_1^{n+1}$ be a spacetime given as the doubly warped product $-I \times_f F^{n-1} \times_g J$ with $f \neq g$. By Proposition 4.3, $K = f\partial_t$ is a timelike closed partially conformal timelike vector field on $\bar{M}$ with associated vector field $W = \partial_s/g$.

THEOREM 7.1. *Let $\bar{M}_1^{n+1} = -I \times_f F \times_g J$ be a doubly warped product with F compact. Let Σ be a immersed spacelike hypersurface everywhere transverse to K. For a given $s \in J$, let $\bar{M}_s = -I \times_f F \times_g \{s\}$ and $\Sigma_s = \Sigma \cap \bar{M}_s$. Suppose either of the following conditions true:*

1. *$\bar{M}$ satisfies the null convergence condition*
$$\mathrm{Ric}_F \geq \sup(ff'' - f'^2),$$
where Ric *is the Ricci curvature of F and in addition, assume that Σ_s is a compact $(n-1)$-dimensional manifold immersed with constant mean curvature in $\bar{M}_s$ for every $s \in J$;*
2. *$\bar{M}$ satisfies the strong null convergence condition*
$$K_F \geq \sup(ff'' - f'^2),$$
where K_F stands for the sectional curvature of F; also assume that for each $s \in J$, Σ_s is a compact $(n-1)$-dimensional manifold immersed with constant r-th mean curvature H_r in $\bar{M}_s$ for some $2 \leq r \leq n$ and that Σ_s is contained in a slab $(t_1, t_2) \times_f F \times_g \{s\}$ where f' does not vanish.

Then Σ is $(n-1)$-umbilical in $\bar{M}$ if and only if the angle between Σ and $\bar{M}_s$ is constant for every $s \in J$.

Proof. Since K is timelike closed partially conformal, we have in particular that $\bar{\nabla}_X K = \phi X$ for every vector field tangent to $\bar{M}_s$. Denoting by ∇ the induced connection on $\bar{M}_s$, we have also $\nabla_X K = \phi X$, which means that K is a closed conformal timelike vector field when restricted to each $\bar{M}_s$.

Suppose the hypotheses of item 1 hold and fix $s \in J$. Then $\bar{M}_s$ is a spatially closed generalized Robertson-Walker spacetime satisfying the null convergence condition and Σ_s is a compact $(n-1)$-dimensional manifold immersed with constant mean curvature in $\bar{M}_s$. By Theorem 6 in [26], either Σ_s is a slice of the foliation of $\bar{M}_s$ determined by K or $\bar{M}_s$ is isometric to the de Sitter space $\mathbb{S}_1^n$ in a neighborhood of Σ_s and Σ_s is a round umbilical hypersphere. Each of these two alternatives implies that Σ_s is totally umbilical in $\bar{M}_s$.

Now suppose the hypotheses of item 2 hold and fix $s \in J$. Then $\bar{M}_s$ is a spatially closed generalized Robertson-Walker spacetime satisfying the strong null covergence condition and Σ_s is a compact $(n-1)$-dimensional manifold immersed with constant r-th mean curvature H_r in $\bar{M}_s$. Moreover, by the third hypothesis in item 2, Σ_s is contained in a slab where f' does not vanish. Then by Theorem 9.2 in [2], Σ_s is totally umbilical in $\bar{M}_s$.

Hence, under the hypoyheses of either item, we obtain that Σ_s is totally umbilical in $\bar{M}_s$; that is, if $Z \in \mathfrak{X}(\bar{M}_s)$ is a unit vector field everywhere normal to Σ_s, there exists a function λ such that

$$\nabla_X Z = \lambda X, \quad \text{for each } X \in \mathfrak{X}(\Sigma_s). \tag{19}$$

Also, by Remark 4.4, $\bar{M}_s$ is totally geodesic in $\bar{M}$ and in consequence equation (19) remains true if we substitute ∇ by the connection $\bar{\nabla}$ in $\bar{M}$.

Note that at each point of Σ_s we have that the 2-dimensional space generated by Z and W is orthogonal to the tangent space of Σ_s which has dimension $(n-1)$. Hence. if N is a unit timelike vector field everywhere normal to Σ, there is a hyperbolic angle θ such that

$$N = (\cosh\theta)Z + (\sinh\theta)W,$$

Let α denote the second fundamental form of Σ in $\bar{M}$. Take $X \in \mathfrak{X}(\Sigma_s)$, hence orthogonal to both K and W, and $Y \in \mathfrak{X}(\Sigma)$. Then

$$\begin{aligned}\langle \alpha(X,Y), N\rangle &= \langle \bar{\nabla}_X Y, N\rangle = -\langle Y, \bar{\nabla}_X N\rangle \\ &= -\langle Y, \bar{\nabla}_X((\cosh\theta)Z + (\sinh\theta)W)\rangle \\ &= -\langle Y, X(\cosh\theta)Z + (\cosh\theta)\bar{\nabla}_X Z + X(\sinh\theta)W\rangle;\end{aligned}$$

here $\bar{\nabla}_X W = 0$ by Proposition 2.3. By (19), we may write

$$\langle \alpha(X,Y), N\rangle = -\lambda\cosh\theta\langle X,Y\rangle - X(\theta)\langle Y, N^{\perp}\rangle, \tag{20}$$

where

$$N^{\perp} = (\sinh\theta)Z + (\cosh\theta)W \in \mathfrak{X}(\Sigma).$$

Suppose that θ is constant along Σ_s, i.e., $X(\theta) = 0$ for each $X \in \mathfrak{X}(\Sigma_s)$. By (20) this fact implies that

$$\alpha(X,Y) = -\lambda\cosh\theta\langle X,Y\rangle N, \tag{21}$$

for each $X \in \mathfrak{X}(\Sigma_s)$ and every $Y \in \mathfrak{X}(\Sigma)$. Since (21) holds for each Σ_s, we have that Σ is $(n-1)$-umbilical in $\bar{M}$.

Conversely, if Σ is $(n-1)$-umbilical in $\bar{M}$, there exists a function μ such that (20) can be written as

$$-\mu\langle X,Y\rangle = -\lambda\cosh\theta\langle X,Y\rangle - X(\theta)\langle Y, N^{\perp}\rangle,$$

for any $X \in \mathfrak{X}(\Sigma_s)$ and $Y \in \mathfrak{X}(\Sigma)$. Taking $Y = N^{\perp}$, we obtain $X(\theta) = 0$ and θ is constant along Σ_s, as desired. □

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