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THE OVERCONVERGENT FROBENIUS*

ROBERT F. COLEMAN†

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We will improve some estimates of Dwork and Gouvêa concerning the the U-operators on overconvergent forms of integral weight. One consequence of our estimates that is not evident from earlier results is that the U-operator applied to an overconvergent form of integral weight bounded by one on a neighborhood of the ordinary locus is still bounded by one on a neighborhood of the ordinary locus.

Let K be a complete local field contained in $\mathbf{C}_p$ with ring of integers R_K. Fix N, $(N,p)=1$. For $r \in R_K$, $Z(N,r)$ will denote the affinoid subdomain of $X_1(N)$ defined over K where $|E_{p-1}| \geq |r|$ (so a neighborhood of the component of the ordinary locus containing the cusp ∞). Let ϕ: $Z(N,r) \to Z(N,r^p)$ be the canonical Frobenius, which is defined when $v(r) < 1/(p+1)$. Let $S(N,r) := S(R_K,N,r)$ denote the R_K-module of forms of weight 0 on $Z(N,r)$ of absolute value at most 1, $S(K,N,r) = S(R_K,N,r) \otimes K$, $Z(r) = Z(1,r)$ and $S(r) = S(1,r)$. For $\alpha \in R_K/pR_K$, we set $v(\alpha) = v(\tilde{\alpha})$ for any $\tilde{\alpha} \in R_K$ which reduces to α, if $\alpha \neq 0$ and $v(\alpha) = \infty$ otherwise.

PROPOSITION 1. *When $N = 1$, ϕ is defined on $Z(r)$, $v(r) < p/(p+1)$. Let $h(j)$ denote the Hasse invariant of any elliptic curve modulo p with j-invariant j mod p. Then*

(*i*) $$|\phi(j) - j^p| \leq |p/h(j)|$$

(*ii*) $$Tr_\phi(S(r)) \subseteq pr^{-(p+1)}S(r^p).$$

Proof. For a supersingular point e let $i_e = 3$ if $j(e) = 0$, $i_e = 2$ if $j(e) = 1728$ and $i_e = 1$ otherwise. Dwork asserts, at formula (7.8) of "p-adic Cycles," that

$$\phi(j) = j^p + pk(j) + \sum_e \sum_{n=1}^{\infty} \frac{A_{e,n}}{(j-\beta_e)^n}$$

where $k(j)$ is a polynomial in j of degree at most $p-1$ over $\mathbf{Z}_p$, e runs over the supersingular points modulo p, β_e is a point in the residue class above e defined over $\mathbf{Q}_p^{unr}$ such that $\beta_{\bar{a}} = a$ when $a = 0$ or 1728 and $A_{e,n} \in \mathbf{Q}_p^{unr}$ such that

$$v(A_{e,n}) \geq \frac{1}{p+1} + i_e n \left(\frac{p}{p+1}\right).$$

Now $v(j-\beta_e) = i_e v(h(j))$, if $e = \bar{j}$ is supersingular and $0 < v(h(j)) < 1$.

Thus

$$v\left(\frac{A_{e,n}}{(j-\beta_e)^n}\right) \geq 1 + (ni_e - 1)(\frac{p}{p+1} - v(h(j))) - v(h(j)),$$

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†Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA (coleman@math.berkeley.edu).

and (i) follows.

Part (ii) follows from part (i). Indeed, suppose $v(r) = H$ and $s = r^{i_e}$. For a supersingular point e, let $Y_e(s)(j) = (j - \beta_e)/s$ so that $Y_e(s)$ is a parameter on the annulus $A_e(r) = A[\beta_e, |s|] = \{x \in X(1) : |j(x) - \beta_e| = |s|\}$ around β_e. Then (i) implies ϕ induces a rigid analtytic morphism from $A_e(r)$ to $A_{e^p}(r^p)$ such that $|\phi^*(Y_{e^p}(s^p)) - Y_e(s)^p| \le |p/s^{p+1}|$. It follows that $A^0(A_e(r))$ is finite and flat of degree p over $A^0(A_{e^p}(s^p))$ and the corresponding trace map $Tr_e(s)$ is 0 modulo p/s^{p+1}. We know by [Ka] that $A(Z(r))$ is finite and flat over $A(Z(r^p))$ and there is a trace map $Tr(r)$. As the diagram

$$\begin{array}{ccc} A(Z(r)) & \xrightarrow{Tr(r)} & A(Z(r^p)) \\ \downarrow & & \downarrow \\ A(A_e(r^{i_e})) & \xrightarrow{Tr_e(r^{i_e})} & A(A_{e^p}(r^{pi_e})) \end{array}$$

commutes for all supersingular points e, it follows that if $f \in A^0(Z(r))$,

$$(Tr(r)f)|_{A_e(r^p)} \in (p/r^{p+1})A^0(A_{e^p}(r^p))$$

for all e and thus by the maximum principle $Tr(r)f \in (p/r^{p+1})A^0(Z(r^p))$.

Corollary 2. *$Tr_\phi(S(N,r)) \subset pr^{-(p+1)}S(N,r^p)$.*

Proof. This follows from part (i) of the proposition and the fact that maps of residue disks on $X_1(N)$ to residue disks on the $X(1)$ are finite, ramified at 0 or 1728 where they have ramification indices 3 or 2. (See de Shalit's [dS] for finer results.)

We can generalize the above to arbitrary weight and level. Let $S(N,k,r)$ the R_K-module of weight k modular forms bounded by one on $Z(N,r)$.

Lemma 3. *Suppose $k \ge 0$, $N > 4$, $(N,p) = 1$ and $v(r) < 1/p$. Then,*

$$U(S(N,k,r)) \subset \frac{1}{r^{p+1}} S(N,k,r^p).$$

Proof.

$$\begin{array}{ccc} X_1(N,r) & \xrightarrow{\phi} & X_1(N,r^p) \\ \downarrow & & \downarrow \\ X(1,r) & \xrightarrow{\phi} & X(1,r^p) \end{array}$$

is Cartesian. (Suppose $\alpha\colon \mu_N \to \phi(E)$ is an embedding. Let $\alpha'\colon \mu_N \to E$ be defined by $\alpha'(\zeta) = \check{\pi}(\alpha(\zeta)^{1/p})$ $A_1(N,r) = A(1,r) \otimes_\phi A(N,r^p)$

Suppose $N > 4$. We now follow §2 of [CO]. Let $f\colon E_1(N) \to X_1(N)$ be the universal elliptic curve and $E(N,r) = E_1(N)_{Z(N,r)}$. Then, since $v(r) < 1/(p+1)$, we have a commutative diagram

$$\begin{array}{ccc} E(N,r) & \xrightarrow{\Phi} & E(N,r^p) \\ \downarrow & & \downarrow \\ Z(N,r) & \xrightarrow{\phi} & Z(N,r^p) \end{array}$$

where ϕ is the Tate-Deligne map and if $f'\colon E'(r^p) \to Z(N,r)$ is the pullback of $E(N,r^p)$ to $Z(N,r)$, there is an isogeny over $Z(N,r)$, π, from $E(N,r)$ to $E'(r^p)$

(moding out by the canonical subgroup) such that Φ is the composition of π and the natural map from $E'(r^p)$ to $E(N, r^p)$. Now using the fact that

$$f'_*\Omega^1_{E'(r^p)/Z(N,r)} = \phi^* f_*\Omega^1_{E(N,r^p)/Z(N,r^p)}$$

and that $U\colon S(K, N, k, r^p) \to S(K, N, k, r^p)$ can be described as $\frac{1}{p}V \circ Res^{r^p}_r$ where V is the composition

$$\begin{array}{c} \omega^{\otimes k}(Z(N,r)) \stackrel{(\check{\pi}^*)^{\otimes k}}{\longrightarrow} A(Z(N,r)) \otimes_{A(Z(N,r^p))} \omega^{\otimes k}(Z(N,r^p)) \\ \downarrow Tr_\phi \otimes 1 \\ A(Z(N,r^p)) \otimes_{A(Z(N,r^p))} \omega^{\otimes k}(Z(N,r^p)) = \omega^{\otimes k}(Z(N,r^p)) \end{array}$$

and where $\omega = f_*\Omega^1_{E_1^{sm}(N)/X_1(N)}(\log f^{-1}C)$ (see [CO]§2). This can be checked easily on q-expansions. The lemma now follows from the previous corollary.

Using this we can conclude: Suppose now $N > 4$, $k \geq 0$ and if $k = 1$, $N \leq 11$.

Theorem 1. *If p is at least 5 and $0 < v(r) < 1/(p+1)$, the submodule.*

$$S(N, k, r^p) + r^{-2}S(N, k, r^p) \cap r^{2p-4}S(N, k, r)$$

of $S(N, k, r^p) \otimes K$ is stable under U.

Proof. Set $T(r) = S(N, k, r^p) + r^{-2}S(N, k, r^p) \cap r^{2p-4}S(N, k, r)$. First,

$$U(r^{2(p-2)}S(N, k, r)) \subset r^{p-5}S(N, k, r^p),$$

by the previous corollary. So since $p \geq 5$, we only have to show $U(S(N, k, r^p))$ is contained in $T(r)$. Suppose $f \in S(N, k, r^p)$. Then we may write

$$\begin{aligned} f &= \sum_{a=0}^{\infty} \frac{r^{pa}b_a}{E^a_{p-1}} \\ &= b_0 + \frac{r^p b_1}{E_{p-1}} + r^{2(p-1)}s \end{aligned}$$

where $b_a \in B(N, k, a) \subset \omega^{\otimes k+a(p-1)}(X(N))$ ((notation as in §2.6 of [Ka]) so has weight $k + a(p-1)$) and $s \in S(N, k, r)$. Then

$$U(f) \equiv U(\frac{r^p b_1}{E_{p-1}}) \bmod S(N, k, r^p),$$

using the previous corollary, because $2(p-1) \geq p+1$. And since $rb_1/E_{p-1} \in S(N, k, r)$

$$U(\frac{rb_1}{E_{p-1}}) = \frac{1}{r^{p+1}} \sum_{a=0}^{\infty} \frac{r^{pa}b'_a}{E^a_{p-1}}$$

where $b'_a \in B(N, k, a)$ using the previous lemma, again, and so

$$U(\frac{r^p b_1}{E_{p-1}}) \in \left(r^{-2}(b'_0 + \frac{r^p b'_1}{E_{p-1}})\right) + r^{2p-4}S(N, k, r) \cap r^{-2}S(N, k, r^p).$$

Now both $U(r^p b_1/E_{p-1})$ and elements of $r^{2p-4}S(N, k, r)$ have q-expansions divisible by r^p since $2p - 4 > p$. By the q-expansion principle [Ka] [Cor. 1.6.2, 1.9.1] there exists a weight $k + p - 1$ integral form b''_1 bounded by one such that

$$r^{-2}(b'_0 E_{p-1} + r^p b'_1) = r^p b''_1.$$

Thus $U(r^p b_1/E_{p-1}) \equiv r^p b_1''/E_{p-1} \mod r^{2p-4}S(N,k,r) \cap r^{-2}S(N,k,r^p)$ and so $U(f) \in T(r)$.

In particular, $||U^n||_{S(K,N,k,r^p)} \leq |r|^{-2}$, for $n \geq 0$. This improves Lemma 3.11.7 of [Ka] and Proposition II.3.9 of [Go]. Although this result is valid for $p = 5$, it is not enough to extend Dwork's bound on the dimension of the unit root subspace Lemma 3.12.4 of [Ka].

Corollary 4. *Suppose K is a finitely ramified extension of $\mathbf{Q}_p$. If $s \in K$, $v(s) < \min\{p/(p+1), p/2e(K/\mathbf{Q}_p)\}$ (e.g., if $p > 2$ and $e(K/\mathbf{Q}_p) = [p/2]+1$ take $v(s) = \dfrac{[p/2]}{[p/2]+1}$), $U(S(R_K,N,k,s)) \subseteq S(R_K,N,k,s)$.*

Proof. If $r \in \overline{K}$ and $r^p = s$, the previous theorem and the fact that U is defined over $\mathbf{Q}_p$ implies $U(S(R_K,N,k,s)) \subset S(K,N,k,s) \cap r^{-2}S(R_{K[r]},N,k,s)$. However, this equals $S(R_K,N,k,s)$ since $v(r^2) < 1/e(K/\mathbf{Q}_p)$).

Corollary 5. *Suppose K and L are finitely ramified extensions of $\mathbf{Q}_p$ and $K \subseteq L$, $s \in K$ and $t \in L$. Then $U(S(R_K,N,k,s)) \subseteq S(R_L,N,k,t)$ if $v(t) \leq v(s)/p < \frac{1}{p+1}$.*

Proof.

$$\begin{aligned} U(S(R_K,N,k,s)) &\subseteq U(S(R_L,N,k,t^p)) \\ &\subseteq S(R_L,N,k,t^p) + t^{-2}S(R_L,N,k,t^p) \cap t^{2p-4}S(R_L,N,k,t) \\ &\subseteq S(R_L,N,k,t). \end{aligned}$$

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