COEFFICIENT AND ELIMINATION ALGEBRAS IN RESOLUTION OF SINGULARITIES*

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Dedicated to H. Hironaka on his 80th birthday

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Introduction. Given a variety X over a field k one wants to find a desingularization, which is a proper and birational morphism $X' \to X$, where X' is a regular variety and the morphism is an isomorphism over the regular points of X.

If X is embedded in a regular variety W, there is a notion of embedded desingularization and related to this is the notion of log-resolution of ideals in $\mathcal{O}_W$.

When the field k has characteristic zero it is well known that the problem of resolution is solved. The first proof of the existence of resolution of singularities is due to H. Hironaka in his monumental work [Hir64] (see also [Hir77]).

If characteristic of k is positive the problem of resolution in arbitrary dimension is still open. See [Hau10] for recent advances and obstructions (see also [Hau03]).

The proof by Hironaka is existential. There are constructive proofs, always in characteristic zero case, see for instance [VU89], [VU92], [BM97], we refer to [Hau03] for a complete list of references. Those constructive proofs give rise to algorithmic resolution of singularities, that allows to perform implementation at the computer [BS00], [FKP04].

Recently some techniques have appeared in order to try to prove the problem of resolution of singularities in the positive characteristic case. Rees algebras seem to be a useful tool in this context. Hironaka in [Hir03] and [Hir05] propose to use Rees algebras for proving log-resolution of ideals. The advantage of Rees algebras is that the algebra encodes in one object many ideals which are "equivalent" for the problem of log-resolution. Also Rees algebras have a good behavior with respect to integral closure, see for instance [VU08] and [VU07]. On the other hand Kawanoue and Matsuki [Kaw07], [KM06] use a different object, called idealistic filtration, which is similar to Rees algebras but with a grading over the real numbers.

In this paper we compare those structures and construct $\mathbb{Q}$-Rees algebras (1.7), which are algebras with grading over the rational numbers. We will see that Rees algebras, idealistic filtrations and $\mathbb{Q}$-Rees algebras encode (up to integral closure) the same information (1.15). One motivation to extend Rees algebras to a $\mathbb{Q}$-grading comes from the scaling operation (3.15), which is needed in the process of resolution of singularities. Since we restrict to rational numbers all properties related to integral closure and finiteness come easily, see 1.6.

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We will use $\mathbb{Q}$-Rees algebras in order to construct log-resolution of ideals in the characteristic zero case. Along the paper we have specified which constructions and results are valid in general or only in characteristic zero.

Section 1 is devoted to introduce the several algebras we have mentioned and to see that they are "equivalent" up to integral closure. In section 2 we extend the notion of $\mathbb{Q}$-Rees algebra to sheaves and define the order function.

In section 3 we define all operations on algebras: integral closure, radical saturation, differential saturation, scaling and the division by a hypersurface. This last operation together with the function ord, defined for any $\mathbb{Q}$-Rees algebra, are a key point in order to construct a resolution algorithm. Except integral closure, all operations may be expressed easily in terms of generators.

In section 4 we define coefficient algebra and see its properties. The stability by monoidal transformations 4.2.2 is valid in general, but the "maximal contact" (4.4) holds only if characteristic of k is zero. We also prove, in this setting, theorem 4.6 which was proved first by Włodarczyk [Wło05]. This theorem ensures that coefficient algebras do not depend on the choice of "maximal contact".

Section 5 defines elimination algebras in the most direct way, just to allow computations. See [VU07] for a more detailed discussion on elimination algebras and the behavior with integral closure. Elimination algebras fail to have a good stability by monoidal transformations, as illustrates example 5.3. On the other hand, if characteristic of k is zero, then both coefficient algebra and elimination algebra are isomorphic 5.5 for the étale topology.

Finally in section 6 we construct an algorithm of resolution of $\mathbb{Q}$-Rees algebras which gives a log-resolution algorithm. We see in 6.9 that we construct the same resolution for algebras with the same integral closure, see also [EV07].

1. $\mathbb{Q}$-Rees algebras. Fix R to be a noetherian ring over a field k. The characteristic of the field k is arbitrary, unless specified.

We consider Rees algebras and extend this notion to a suitable definition of Rees algebras over rational numbers.

DEFINITION 1.1. [VU08] Let R be a noetherian ring. Consider the graded algebra:

$$R[T] = \bigoplus_{n \in \mathbb{N}} RT^n.$$

A Rees algebra in R is a graded subalgebra

$$\mathcal{J} = \bigoplus_{n \in \mathbb{N}} J_n T^n \subset R[T]$$

such that $J_0 = R$ and $\mathcal{J}$ is finitely generated as R-algebra.

Equivalently we may define a Rees algebra as a collection of ideals $\{J_n\}_{n \in \mathbb{N}}$ such that:
  1. $J_0 = R$,
  2. $J_m J_n \subset J_{m+n}$ for any $m, n \in \mathbb{N}$, and

3. there exist elements $f_1, \ldots, f_r \in R$ and degrees $n_1, \ldots, n_r \in \mathbb{N}$, with $f_i \in J_{n_i}$, $i = 1, \ldots, r$, such that for any $n \in \mathbb{N}$ the ideal J_n is generated (as ideal) by the set

$$\{f_{i_1} \cdots f_{i_\ell} \mid n_{i_1} + \cdots + n_{i_\ell} = n\}.$$

We will be interested in considering the equivalence class of Rees algebras up to integral closure.

DEFINITION 1.2. Two Rees algebras (1.1) $\mathcal{J}_1, \mathcal{J}_2 \subset R[T]$ are *equivalent* if they have the same integral closure as subrings of $R[T]$.

1.3. Let $\mathcal{J} = \oplus_n J_n T^n$ be a Rees algebra. For any $n \in \mathbb{N}$ set the ideal

$$I_n = \sum_{m \geq n} J_m.$$

It can be checked that $\mathcal{I} = \oplus_n I_n T^n$ is a Rees algebra. In fact if $\{f_1 T^{n_1}, \ldots, f_r T^{n_r}\}$ is a set of generators of the algebra $\mathcal{J}$ then

$$\{f_i T^m \mid i = 1, \ldots, r, \ 0 \leq m \leq n_i\}$$

is a set of generators of $\mathcal{I}$. In fact it can be checked that for any $n \in \mathbb{N}$

$$I_n = \langle f_{i_1} \cdots f_{i_\ell} \mid n_{i_1} + \cdots + n_{i_\ell} \geq n \rangle.$$

Note that the Rees algebra $\mathcal{I}$ satisfies the following property

(1.3.1) $\qquad\qquad\qquad$ if $n_1 \geq n_2$ then $I_{n_1} \subset I_{n_2}$.

In fact $\mathcal{I}$ is the smallest Rees algebra satisfying (1.3.1) and such that $\mathcal{J} \subset \mathcal{I}$. On the other hand we have that $\mathcal{J} \subset \mathcal{I}$ is a finite extension (i.e. the algebras $\mathcal{J}$ and $\mathcal{I}$ are equivalent (1.2)). In order to prove the last sentence, it is enough to prove that if $f \in J_n$ and $m \leq n$, then fT^m is integral over $\mathcal{J}$. We have that

$$f \in J_n \Longrightarrow f^m \in J_{nm} \Longrightarrow f^n \in J_{nm}$$

so that the element fT^m fulfills the monic equation $Z^n - (f^n T^{nm}) = 0$.

So that, up to integral closure, condition (1.3.1) may be added to our definition of algebras.

DEFINITION 1.4. An $\mathbb{N}$-Rees algebra is a Rees algebra $\mathcal{J}$ (1.1) satisfying (1.3.1). Equivalently, an $\mathbb{N}$-Rees algebra is a collection of ideals $\{J_n\}_{n \in \mathbb{N}}$ such that
1. $J_0 = R$,
2. $J_m J_n \subset J_{m+n}$, for any $m, n \in \mathbb{N}$,
3. if $n \leq m$ then $J_m \subset J_n$, and
4. there exist elements $f_1, \ldots, f_r$ and degrees $n_1, \ldots, n_r$, with $f_i \in J_{n_i}$, $i = 1, \ldots, r$, such that for any $n \in \mathbb{N}$ the ideal J_n is generated (as ideal) by the set

$$\{f_{i_1} \cdots f_{i_\ell} \mid n_{i_1} + \cdots + n_{i_\ell} \geq n\}.$$

1.5. If R is a regular local ring there is a well-defined notion of order of ideals $J \subset R$, which can be extended to $\mathbb{N}$-Rees algebras (see 2.4). In general the order will be a rational number and we will be interested in algebras with order one. Algebras with order one will allow to define inductive procedures in the process of resolution of singularities. So that we will need to *normalize* algebras in order to have order one, this procedure will be called *scaling* and it will lead naturally to consider graduation over the rational numbers.

1.6. Set $\mathbb{Q}_{\geq 0} = \{a \in \mathbb{Q} \mid a \geq 0\}$. We will consider the $\mathbb{Q}$-graded algebra

$$\text{Gr-}\mathbb{Q}(R) = \bigoplus_{a \in \mathbb{Q}_{\geq 0}} RT^a.$$

Note that this graded algebra is the limit of the $\mathbb{N}$-graded algebras $R[T^{\frac{1}{N}}]$ for $N \in \mathbb{N}$

$$\text{Gr-}\mathbb{Q}(R) = \bigcup_{N \in \mathbb{N}} R[T^{\frac{1}{N}}].$$

So that every finite set of $\text{Gr-}\mathbb{Q}(R)$ is included in $R[T^{\frac{1}{N}}]$ for N big enough. This condition will allow us to use properties of Rees algebras for *finitely generated* subalgebras of $\text{Gr-}\mathbb{Q}(R)$.

DEFINITION 1.7. A $\mathbb{Q}$-Rees algebra over R is a graded subalgebra

$$\mathcal{J} = \bigoplus_{a \in \mathbb{Q}_{\geq 0}} J_a T^a \subset \text{Gr-}\mathbb{Q}(R)$$

such that the collection of ideals $\{J_a\}_{a \in \mathbb{Q}_{\geq 0}}$ satisfies the following properties:
1. $J_0 = R$,
2. $J_a J_b \subset J_{a+b}$ for all $a, b \in \mathbb{Q}_{\geq 0}$,
3. $J_b \subset J_a$ if $a \leq b$ and
4. $\mathcal{J}$ is finitely generated, which means that there are elements $f_1, \ldots, f_r \in R$ and degrees $a_1, \ldots, a_r \in \mathbb{Q}_{\geq 0}$, with $f_i \in J_{a_i}$, $i = 1, \ldots, r$, such that for any $a \in \mathbb{Q}_{\geq 0}$ the ideal J_a is generated (as ideal) by the set

$$\{f_{i_1} \cdots f_{i_\ell} \mid a_{i_1} + \cdots + a_{i_\ell} \geq a\}.$$

1.8. With the notation of 1.7, we say that the $\mathbb{Q}$-Rees algebra is generated by $f_1 T^{a_1}, \ldots, f_r T^{a_r}$.

Note that the $\mathbb{Q}$-Rees algebra $\mathcal{J}$ is the smallest algebra satisfying properties (1), (2) and (3) of 1.7 and containing the elements $f_1 T^{a_1}, \ldots, f_r T^{a_r}$.

1.9. Note that the condition of being finitely generated may be expressed as follows: There are homogeneous elements $f_1 T^{a_1}, \ldots, f_r T^{a_r} \in \mathcal{J}$ and there is an integer N such that
1. $a_i N$ is an integer for $i = 1, \ldots, r$,
2. the finite set

$$\{f_i T^{\frac{m}{N}} \mid m \leq a_i N, \ i = 1, \ldots, r\}$$

generates $\mathcal{J} \cap R[T^{\frac{1}{N}}]$ as an R-algebra, and

3. for any $a \in \mathbb{Q}_{\geq 0}$ we have that $J_a = J_{\frac{\lceil aN \rceil}{N}}$.

Where $\lceil a \rceil$ denotes the smallest integer bigger than or equal to a.

In other words, a $\mathbb{Q}$-Rees algebra $\mathcal{J} \subset \text{Gr-}\mathbb{Q}(R)$ is equivalent to an $\mathbb{N}$-Rees algebra in $R[T^{\frac{1}{N}}]$, for big enough N, where we fill rational levels according to the rule $J_a = J_{\frac{\lceil aN \rceil}{N}}$, $a \in \mathbb{Q}_{\geq 0}$.

Note also that if $\mathcal{J}$ is a $\mathbb{Q}$-Rees algebra and $b \in \mathbb{Q}_{\geq 0}$ then $\mathcal{J} \cap R[T^b]$ is an $\mathbb{N}$-Rees algebra in $R[T^b]$.

Integral closure will be the criterion to be equivalent for $\mathbb{Q}$-Rees algebras.

DEFINITION 1.10. Two $\mathbb{Q}$-Rees algebras $\mathcal{J}_1$ and $\mathcal{J}_2$ are equivalent if the Rees algebras $\mathcal{J}_1 \cap R[T]$ and $\mathcal{J}_2 \cap R[T]$ have the same integral closure in $R[T]$ (i.e. they are equivalent (1.2)).

PROPOSITION 1.11. *Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two $\mathbb{Q}$-Rees algebras in $\text{Gr-}\mathbb{Q}(R)$. The following statements are equivalent:*
   1. $\mathcal{J}_1$ and $\mathcal{J}_2$ are equivalent $\mathbb{Q}$-Rees algebras (1.10).
   2. For some $b \in \mathbb{Q}_{>0}$, the algebras $\mathcal{J}_1 \cap R[T^b]$ and $\mathcal{J}_2 \cap R[T^b]$ have the same integral closure in $R[T^b]$.
   3. For any $b \in \mathbb{Q}_{>0}$, the algebras $\mathcal{J}_1 \cap R[T^b]$ and $\mathcal{J}_2 \cap R[T^b]$ have the same integral closure in $R[T^b]$.

Proof. It follows from the fact that all the extensions

$$R[T^M] \subset R[T], \quad R[T] \subset R[T^{\frac{1}{N}}], \quad R[T^{\frac{M}{N}}] \subset R[T^{\frac{1}{N}}]$$

are finite for $M, N \in \mathbb{N}$. □

1.12. These algebraic structures, Rees algebras (1.1), $\mathbb{N}$-Rees algebras (1.4) and $\mathbb{Q}$-Rees algebras (1.7) are "equivalent" up to integral closure.

First of all, it follows from 1.3 that Rees algebras (1.1) and $\mathbb{N}$-Rees algebras (1.4) are equivalent up to integral closure.

For the equivalence of $\mathbb{N}$-Rees algebras and $\mathbb{Q}$-Rees algebras, we check that we can associate a $\mathbb{Q}$-Rees algebra to every $\mathbb{N}$-Rees algebra and vice-versa.

Fix a ring R, let $\mathcal{I} = \oplus_n I_n T^n \subset R[T]$ be an $\mathbb{N}$-Rees algebra. Set $\mathcal{I} \text{Gr-}\mathbb{Q}(R) = \mathcal{J} = \oplus_a J_a T^a \subset \text{Gr-}\mathbb{Q}(R)$ as follows:

$$J_a = I_{\lceil a \rceil}, \qquad \forall a \in \mathbb{Q}_{\geq 0}.$$

It is clear that $\mathcal{I} \text{Gr-}\mathbb{Q}(R)$ is a $\mathbb{Q}$-Rees algebra.

Reciprocally, if $\mathcal{J}$ is a $\mathbb{Q}$-Rees algebra then set $\mathcal{I} = \mathcal{J} \cap R[T]$, which is an $\mathbb{N}$-Rees algebra.

Now for an $\mathbb{N}$-Rees algebra $\mathcal{I}$, we have that $\mathcal{I} = (\mathcal{I} \text{Gr-}\mathbb{Q}(R)) \cap R[T]$. Reciprocally if $\mathcal{J}$ is a $\mathbb{Q}$-Rees algebra then $\mathcal{J}$ and $(\mathcal{J} \cap R[T]) \text{Gr-}\mathbb{Q}(R)$ are equivalent $\mathbb{Q}$-Rees algebras (1.10).

Recently Rees algebras have been used in new approaches to resolution of singularities in positive characteristic, see [Hir03], [Hir05], [VU07], [VU08]. On the other hand, in [Kaw07] and [KM06] the authors define *idealistic filtrations*, which are collections of ideals indexed by real numbers with the same properties as a $\mathbb{Q}$-Rees algebra (1.7).

DEFINITION 1.13. [Kaw07, 2.1.3.1] An *idealistic filtration* is a collection of ideals $\mathcal{J} = \{J_a\}_{a \in \mathbb{R}_{\geq 0}}$ such that

1. $J_0 = R$,
2. $J_a J_b \subset J_{a+b}$ for all $a, b \in \mathbb{R}_{\geq 0}$ and
3. $J_b \subset J_a$ if $a \leq b$.

We could also denote $\mathcal{J} = \oplus_{a \in \mathbb{R}_{\geq 0}} J_a T^a$.

1.14. The idealistic filtration generated by a subset $L \subset R \times \mathbb{R}_{\geq 0}$ is the minimal idealistic filtration containing the set L. An idealistic filtration is *rationally and finitely generated* (r.f.g) if it is generated by a finite set in $R \times \mathbb{Q}_{\geq 0}$:

$$\{f_1 T^{a_1}, \ldots, f_r T^{a_r}\}, \qquad a_1, \ldots, a_r \in \mathbb{Q}_{\geq 0}\,.$$

1.15. The three notions: Rees algebras, $\mathbb{Q}$-Rees algebras and rationally and finitely generated idealistic filtrations are "equivalent" up to integral closure.

This fact can be proved by constructing the Rees algebra corresponding to a rationally and finitely generated idealistic filtration, see [Kaw07, 2.3.2.1] and [Kaw07, 2.3.2.2].

Alternatively, by 1.12 and 1.3 it is enough to see the equivalence between $\mathbb{Q}$-Rees algebras and rationally and finitely generated idealistic filtrations. And this follows by considering $\mathbb{Q}$-Rees algebras and rationally and finitely generated idealistic filtrations generated by $f_1 T^{a_1}, \ldots, f_r T^{a_r}$ (1.8 and 1.14).

2. Sheaves. Let W be a smooth scheme over a (perfect) field k.

2.1. A sequence of coherent sheaves of ideals $\{I_n\}_{n \in \mathbb{N}}$, $I_n \subset \mathcal{O}_W$, defines a sheaf of graded algebras $\mathcal{I} = \oplus_n I_n T^n$ if $I_0 = \mathcal{O}_W$ and $I_m I_n \subset I_{m+n}$ for any $m, n \in \mathbb{N}$.

We say that $\mathcal{I} = \oplus_n I_n T^n$ is a Rees algebra over W if there is an affine open covering $\{U_i\}_{i \in \Lambda}$ of W such that

$$\mathcal{I}(U_i) = \bigoplus_{n \in \mathbb{N}} I_n(U_i) T^n \qquad \text{is a finitely generated } \mathcal{O}_W(U_i)\text{-algebra}$$

for any $i \in \Lambda$. Or equivalently, $\mathcal{I}(U_i)$ is a Rees algebra in the ring $\mathcal{O}_W(U_i)[T]$.

Analogously we could define, $\mathbb{N}$-Rees algebras and $\mathbb{Q}$-Rees algebras over W.

DEFINITION 2.2. A $\mathbb{Q}$-Rees algebra over W is denoted by

$$\mathcal{J} = \bigoplus_{a \in \mathbb{Q}_{\geq 0}} J_a T^a$$

where $\{J_a\}_{a \in \mathbb{Q}_{\geq 0}}$ is a collection of coherent sheaves of ideals $J_a \subset \mathcal{O}_W$ such that:
1. $J_0 = \mathcal{O}_W$,
2. $J_a J_b \subset J_{a+b}$ for all $a, b \in \mathbb{Q}_{\geq 0}$,
3. $J_b \subset J_a$ if $a \leq b$ and
4. There exist an open covering of W, by affine open sets $\{U_i\}_{i \in \Lambda}$ such that $\mathcal{J}(U_i)$ is a $\mathbb{Q}$-Rees algebra in Gr-$\mathbb{Q}(\mathcal{O}_W(U_i))$.

2.3. Let $\mathcal{J}$ be a $\mathbb{Q}$-Rees algebra over W and consider an affine open set $U \subset W$ such that $\mathcal{J}(U)$ is a $\mathbb{Q}$-Rees algebra in Gr-$\mathbb{Q}(\mathcal{O}_W(U))$. For any open set $V \subset U$ and for any point $\xi \in U$ there are natural maps:

$$\mathcal{J}(U) \to \mathcal{J}(V), \qquad \mathcal{J}(U) \to \mathcal{J}_\xi.$$

If $f_1 T^{a_1}, \ldots, f_r T^{a_r} \in \mathcal{J}(U)$ are generators of $\mathcal{J}(U)$ as $\mathbb{Q}$-Rees algebra (1.8) then it is clear that the images of $f_1 T^{a_1}, \ldots, f_r T^{a_r}$ in $\mathcal{J}(V)$ (resp. $\mathcal{J}_\xi$) are also generators of the $\mathbb{Q}$-Rees algebra $\mathcal{J}(V)$ (resp. $\mathcal{J}_\xi$).

We say that $\mathcal{J}$ is the zero algebra at a point ξ if $(J_a)_\xi = 0$ for all $a \in \mathbb{Q}_{>0}$. It follows from definition 2.2 that if $(J_b)_\xi = 0$ for some $b \in \mathbb{Q}_{>0}$ then $(J_a)_\xi = 0$ for all $a \in \mathbb{Q}_{>0}$.

DEFINITION 2.4. Let $\mathcal{J} = \oplus_a J_a T^a$ be a $\mathbb{Q}$-Rees algebra over W and let $\xi \in W$ be a point. Assume that $\mathcal{J}$ is not the zero algebra at ξ. Define the order of $\mathcal{J}$ at ξ:

$$\mathrm{ord}(\mathcal{J})(\xi) = \inf_{a \in \mathbb{Q}_{>0}} \frac{\mathrm{ord}(J_a)(\xi)}{a}$$

where $\mathrm{ord}(J_a)(\xi)$ denotes the order of the sheaf of ideals J_a at ξ

$$\mathrm{ord}(J_a)(\xi) = \max\{n \in \mathbb{N} \mid (J_a)_\xi \subset \mathfrak{m}_\xi^n\}$$

and $\mathfrak{m}_\xi \subset \mathcal{O}_{W,\xi}$ is the maximal ideal of the local ring $\mathcal{O}_{W,\xi}$.

If $\mathcal{J}$ is the zero algebra at ξ we may set $\mathrm{ord}(\mathcal{J})(\xi) = \infty$.

We set the singular locus of $\mathcal{J}$ as the set

$$\mathrm{Sing}(\mathcal{J}) = \{\xi \in W \mid \mathrm{ord}(\mathcal{J})(\xi) \geq 1\}.$$

2.5. The order of $\mathcal{J}$ at any point ξ is always a rational number and it can be computed from a finite set of generators. If $f_1 T^{a_1}, \ldots, f_r T^{a_r} \in \mathcal{J}_\xi$ are generators of $\mathcal{J}$ at ξ then

$$\mathrm{ord}(\mathcal{J})(\xi) = \min\left\{\frac{\mathrm{ord}(f_1)}{a_1}, \ldots, \frac{\mathrm{ord}(f_r)}{a_r}\right\}.$$

It follows that $\mathrm{ord}(\mathcal{J}) : W \to \mathbb{Q}_{\geq 0}$ is an upper semicontinuous function. In particular the singular locus $\mathrm{Sing}(\mathcal{J})$ is a closed set.

2.6. The upper semicontinuous function $\mathrm{ord}(\mathcal{J})$ will stratify $\mathrm{Sing}(\mathcal{J})$ by locally closed sets and one may focus to the maximum stratum, which is closed.

We say that a $\mathbb{Q}$-Rees algebra $\mathcal{J}$ is *simple* if $\mathrm{ord}(\mathcal{J})(\xi) = 1$ for any $\xi \in \mathrm{Sing}(\mathcal{J})$.

We will see that, after a scaling operation (3.15) we may reduce to the simple case.

3. Operations. A very important concept with Rees algebras is integral closure. Given a Rees algebra $\mathcal{J} \subset \mathcal{O}_W[T]$, the integral closure $\bar{\mathcal{J}} \subset \mathcal{O}_W[T]$ of $\mathcal{J}$ in $\mathcal{O}_W[T]$ is the Rees algebra generated by all the elements of $\mathcal{O}_W[T]$ that are integral over $\mathcal{J}$. Note that it is well-known that the integral closure is a finitely generated $\mathcal{O}_W$-algebra.

There is an open covering of W, such that for any open set U of the covering, an element $f T^n \in \mathcal{O}_W(U)[T]$ belongs to $\bar{\mathcal{J}}$ if and only if there exist $m \in \mathbb{N}$ and a monic polynomial

$$p(Z) = Z^m + a_1 Z^{m-1} + \cdots + a_m, \qquad \text{with } a_i \in J_{in}, \quad i = 1, \ldots, m$$

such that $p(f) = 0$.

In fact we have that $\mathcal{J}$ is equivalent to another Rees algebra $\mathcal{J}_1$ if and only if $\bar{\mathcal{J}}_1 = \bar{\mathcal{J}}$.

We may define the analogous notion for $\mathbb{Q}$-Rees algebras.

DEFINITION 3.1. Let $\mathcal{J} = \oplus_a J_a T^a$ be a $\mathbb{Q}$-Rees algebra and let U be an open set of W. An element $fT^a \in \mathcal{O}_W(U)[T]$ is integral over $\mathcal{J}$ if there exist $m \in \mathbb{N}$ and a monic polynomial

$$p(Z) = Z^m + a_1 Z^{m-1} + \cdots + a_m, \qquad \text{with } a_i \in J_{ia}, \quad i = 1, \ldots, m$$

such that $p(f) = 0 \in \mathcal{O}_W(U)$.

We set the integral closure $\bar{\mathcal{J}}$ of $\mathcal{J}$ as the $\mathcal{O}_W$-algebra generated by all fT^a integral over $\mathcal{J}$.

From the definition we can not say that $\bar{\mathcal{J}}$ is a $\mathbb{Q}$-Rees algebra, since the property of being finitely generated is not clear. However, this fact will follow from radical saturation. This concept was defined in [Kaw07, 2.1.3.1] and we will see the connection with [LJT74].

DEFINITION 3.2. [LJT74] Let $\mathcal{J} = \oplus_a J_a T^a$ be a $\mathbb{Q}$-Rees algebra over W. Fix a point $\xi \in W$. For an element $f \in \mathcal{O}_{W,\xi}$ we set

$$\nu_{\mathcal{J}}(f) = \sup\{a \mid fT^a \in \mathcal{J}\}.$$

Note that $\nu_{\mathcal{J}}(f)$ may be infinite if $f = 0$.

3.3. With the previous notation, for any $f \in \mathcal{O}_{W,\xi}$ consider the sequence

$$\left\{ \frac{\nu_{\mathcal{J}}(f^n)}{n} \right\}_{n=1}^{\infty}.$$

Using the same arguments as in [LJT74, 0.2.1] we may prove that this sequence converges to some value in $\mathbb{R} \cup \{\infty\}$. So that it makes sense to set

$$\bar{\nu}_{\mathcal{J}}(f) = \lim_{n \to \infty} \frac{\nu_{\mathcal{J}}(f^n)}{n}.$$

DEFINITION 3.4. Let $\mathcal{J} = \oplus_a J_a T^a$ be a $\mathbb{Q}$-Rees algebra over W. An homogeneous element $fT^a \in \mathcal{O}_W[T]$ is radical over $\mathcal{J}$ if $\bar{\nu}_{\mathcal{J}}(f) \geq a$.

The radical saturation of $\mathcal{J}$ is the $\mathcal{O}_W$-algebra generated by all fT^a radical over $\mathcal{J}$. We denote the radical saturation of $\mathcal{J}$ as $\mathfrak{R}(\mathcal{J})$.

From the definition, the radical saturation may not be a $\mathbb{Q}$-Rees algebra, since the condition of being finitely generated is not immediate.

3.5. It can be proved that fT^a is radical over $\mathcal{J}$ (3.4) if and only if there are sequences $\{a_\ell\}_{\ell=1}^{\infty}$ and $\{n_\ell\}_{\ell=1}^{\infty}$ such that
  • $a_\ell \in \mathbb{Q}_{\geq 0}$, $n_\ell \in \mathbb{N}$ for all ℓ,
  • $\lim_{\ell \to \infty} a_\ell = a \in \mathbb{Q}_{\geq 0}$, and
  • $f^{n_\ell} T^{a_\ell n_\ell} \in \mathcal{J}$.

We recall the definition of radical saturation for idealistic filtrations [Kaw07, 2.1.3.1]. An idealistic filtration is said to be radical saturated if:

$$\text{(radical)} \qquad f^m \in J_{am}, \ f \in R, m \in \mathbb{N} \Longrightarrow f \in J_a$$
$$\text{(continuity)} \qquad f \in J_{a_\ell}, \ \lim_{\ell \to \infty} a_\ell = a \Longrightarrow f \in J_a.$$

It follows that the two concepts: radical saturation of $\mathbb{Q}$-Rees algebras (3.4) and radical saturation of idealistic filtrations [Kaw07, 2.1.3.1] coincide.

PROPOSITION 3.6. *Let $\mathcal{J}$ be a $\mathbb{Q}$-Rees algebra over W. The radical saturation $\mathfrak{R}(\mathcal{J})$ is a $\mathbb{Q}$-Rees algebra.*

Proof. It follows from 3.5 and [Kaw07, 2.3.2.4] where it is proved that radical saturation of a r. f. g. idealistic filtration is also a r. f. g. idealistic filtration. In fact this result also appears in [LJT74, §4]. Both proofs use similar arguments inspired in [Nag57].

By the equivalence 1.15 we conclude then that $\mathfrak{R}(\mathcal{J})$ is finitely generated and then it is a $\mathbb{Q}$-Rees algebra. $\square$

PROPOSITION 3.7. *The integral closure $\bar{\mathcal{J}}$ is a $\mathbb{Q}$-Rees algebra. In fact $\bar{\mathcal{J}} = \mathfrak{R}(\mathcal{J})$.*

Proof. It follows from 3.6 and [Kaw07, 2.3.2.7]. See also [LJT74, §4]. $\square$

The order function is well defined up to equivalence of $\mathbb{Q}$-Rees algebras.

PROPOSITION 3.8. *Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two equivalent $\mathbb{Q}$-Rees algebras (1.10). Then for any $\xi \in W$ we have $\mathrm{ord}(\mathcal{J}_1)(\xi) = \mathrm{ord}(\mathcal{J}_2)(\xi)$.*

Proof. It follows from the fact that $\mathrm{ord}(\mathcal{J})(\xi) = \mathrm{ord}(\mathfrak{R}(\mathcal{J}))(\xi)$, for any $\xi \in W$, and 3.7. $\square$

Another important operation on $\mathbb{Q}$-Rees algebras is differential saturation. This notion appears also for Rees algebras (see [VU08] and [VU07]) and for idealistic filtrations [Kaw07]. See also [Hir03].

Set Diff_W^m to be the sheaf of differentials operators of order $\leq m$. We will say that a $\mathbb{Q}$-Rees algebra is differentially saturated if it is stable by the action of differentials, to be more precise:

DEFINITION 3.9. Let $\mathcal{J} = \oplus_a J_a T^a$ be a $\mathbb{Q}$-Rees algebra. We say that $\mathcal{J}$ is differentially saturated if

$$\mathrm{Diff}_W^m(J_a) \subset J_{a-m}, \qquad \forall a \in \mathbb{Q}_{\geq 0}, \ \forall m \in \mathbb{N}, \quad a \geq m.$$

If $\mathcal{J}$ is any $\mathbb{Q}$-Rees algebra, we denote $\mathrm{Diff}(\mathcal{J})$ to be the minimal $\mathbb{Q}$-Rees algebra, differentially saturated and containing $\mathcal{J}$.

3.10. The $\mathbb{Q}$-Rees algebra $\mathrm{Diff}(\mathcal{J})$ always exists and it may be computed from a set of generators of $\mathcal{J}$. If $f_1 T^{a_1}, \ldots, f_r T^{a_r}$ is a set of generators of $\mathcal{J}(U)$ (for some suitable open set U), then a set of generators of $\mathrm{Diff}(\mathcal{J})(U)$ is

$$\{ D(f_i) T^{a_i - \ell_i} \mid i = 1, \ldots, r, \ D \in \mathrm{Diff}_W^{\ell_i}(U), \ell_i \in \mathbb{Z}_{\leq a_i} \}.$$

Note that, since Diff_W^m is a locally free sheaf of finite rank, we could consider a finite set of generators for $\mathrm{Diff}(\mathcal{J})(U)$.

PROPOSITION 3.11. *Let $\mathcal{J}$ be a $\mathbb{Q}$-Rees algebra. If $\xi \in \mathrm{Sing}(\mathcal{J})$ then $\mathrm{ord}(\mathcal{J})(\xi) = \mathrm{ord}(\mathrm{Diff}(\mathcal{J}))(\xi)$.*

If $\xi \notin \mathrm{Sing}(\mathcal{J})$ then $\mathrm{ord}(\mathrm{Diff}(\mathcal{J}))(\xi) = 0$, or equivalently $\mathrm{Diff}(\mathcal{J})_\xi = \mathrm{Gr}\text{-}\mathbb{Q}(\mathcal{O}_W)$.

Proof. See also [VU08]. For the second fact you can see also [KM06, 1.1.2.1].

First of all assume $\xi \notin \mathrm{Sing}(\mathcal{J})$, then there exist $a \in \mathbb{Q}_{>0}$ and $f \in (J_a)_\xi$ such that $\mathrm{ord}(f) < a$. We may assume that $a \in \mathbb{N}$, since for any positive integer m, we have $f^m \in (J_{ma})_\xi$ and $\mathrm{ord}(f^m) < ma$. Then $\mathrm{ord}(f) \leq a - 1$ and there exist a differential operator $D \in \mathrm{Diff}^{a-1}_{\mathcal{O}_{W,\xi}}$ such that $D(f) \in \mathcal{O}_{W,\xi}$ is a unit (or equivalently $\mathrm{ord}(D(f)) = 0$). By definition 3.9 $D(f)T \in \mathrm{Diff}(\mathcal{J})_\xi$ and this implies that $\mathrm{Diff}(\mathcal{J})_\xi = \mathrm{Gr}\text{-}\mathbb{Q}(\mathcal{O}_W)$.

Now assume that $\xi \in \mathrm{Sing}(\mathcal{J})$. Set $\alpha = \mathrm{ord}(\mathcal{J})(\xi)$. For any $f \in \mathcal{O}_{W,\xi}$ with $fT^a \in \mathcal{J}$, any integer $0 \leq k \leq a - 1$ and any differential operator D of order k, we have

$$\frac{\mathrm{ord}(D(f))}{a - k} \geq \frac{\mathrm{ord}(f) - k}{a - k} \geq \frac{\mathrm{ord}(f)}{a} \geq 1$$

and we conclude that $\mathrm{ord}(\mathcal{J})(\xi) = \mathrm{ord}(\mathrm{Diff}(\mathcal{J}))(\xi)$. Note that the hypothesis $\xi \in \mathrm{Sing}(\mathcal{J})$ is necessary in order to have $\mathrm{ord}(f) \geq a$ in the chain of inequalities above. $\square$

COROLLARY 3.12. $\mathrm{Sing}(\mathcal{J}) = \mathrm{Sing}(\mathrm{Diff}(\mathcal{J}))$.

DEFINITION 3.13. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two $\mathbb{Q}$-Rees algebras over W. We define $\mathcal{J}_1 \odot \mathcal{J}_2$ as the $\mathbb{Q}$-Rees algebra generated by $\mathcal{J}_1 \cup \mathcal{J}_2$.

3.14. If $f_1 T^{a_1}, \ldots, f_r T^{a_r}$ are generators of $\mathcal{J}_1$ and $g_1 T^{b_1}, \ldots, g_s T^{b_s}$ are generators of $\mathcal{J}_2$ it is clear that

$$f_1 T^{a_1}, \ldots, f_r T^{a_r}, g_1 T^{b_1}, \ldots, g_s T^{b_s}$$

are generators of $\mathcal{J}_1 \odot \mathcal{J}_2$.

The notation $\odot$ appeared in [EV07]. The use of $\odot$ instead of $\oplus$ was motivated since the notation $\mathcal{J}_1 \oplus \mathcal{J}_2$ could be ambiguous.

DEFINITION 3.15. Given $\mathcal{J}$ and $b \in \mathbb{Q}_{>0}$ we could do a scaling on the levels of $\mathcal{J}$ as follows:

$$\mathcal{J}^b = \bigoplus_{a \in \mathbb{Q}_{\geq 0}} J_a T^{\frac{a}{b}}.$$

3.16. Note that $fT^a \in \mathcal{J}$ if and only if $fT^{\frac{a}{b}} \in \mathcal{J}^b$.

If $f_1 T^{a_1}, \ldots, f_r T^{a_r}$ are generators of $\mathcal{J}$ then $f_1 T^{\frac{a_1}{b}}, \ldots, f_r T^{\frac{a_r}{b}}$ are generators of $\mathcal{J}^b$.

Note also that the order of $\mathcal{J}^b$ is multiplied by b by definition of the order function.

$$\mathrm{ord}(\mathcal{J}^b)(\xi) = b\,\mathrm{ord}(\mathcal{J})(\xi), \qquad \forall \xi \in W.$$

DEFINITION 3.17. Let $\mathcal{J}$ be a $\mathbb{Q}$-Rees algebra over W and let $\ell \in \mathbb{Q}_{\geq 0}$. Fix a (smooth) hypersurface $H \subset W$.

We say that $I(H)^\ell$ divides $\mathcal{J}$ if $J_a \subset I(H)^{\lceil a\ell \rceil}$ for any $a \in \mathbb{Q}_{\geq 0}$.

Note that $J_a = I(H)^{\lceil a\ell \rceil} I_a$ for some ideal I_a. We set $I(H)^{-\ell} \mathcal{J}$ to be the $\mathbb{Q}$-Rees algebra generated by $\{I_a T^a \mid a \in \mathbb{Q}_{\geq 0}\}$.

3.18. Note that in general $\bigoplus_a I_a T^a$ is not a $\mathbb{Q}$-Rees algebra since condition (3) in 1.7 is not always satisfied.

3.19. Let $H \subset W$ be a (smooth) hypersurface, denote ν_H the valuation associated to the hypersurface H.

If $f_1 T^{a_1}, \ldots, f_r T^{a_r}$ are generators of $\mathcal{J}$ then $I(H)^\ell$ divides $\mathcal{J}$ if and only if $\nu_H(f_i) \geq a_i \ell$ for $i = 1, \ldots, r$.

Set ℓ_H to be the supremum of all $\ell \in \mathbb{Q}_{\geq 0}$ such that $I(H)^\ell$ divides $\mathcal{J}$. It follows from the above characterization that

$$\ell_H = \min \left\{ \frac{\nu_H(f_i)}{a_i} \mid i = 1, \ldots, r \right\}.$$

3.20. Let $\mathcal{J}$ be a $\mathbb{Q}$-Rees algebra, and let $C \subset \mathrm{Sing}(\mathcal{J})$ be a smooth and closed set. Let $\Pi : W' \to W$ be the monoidal transformation with center C. The total transform of $\mathcal{J}$ is $\mathcal{J}^* = \oplus_a J_a^* T^a$, where J_a^* is the total transform of J_a. Note that $\mathcal{J}^*$ is a $\mathbb{Q}$-Rees algebra. Set $H \subset W'$ to be the exceptional divisor of Π.

Note also that $I(H)$ divides $\mathcal{J}^*$. The $\mathbb{Q}$-Rees algebra $\mathcal{J}' = I(H)^{-1} \mathcal{J}^*$ is called the transform of $\mathcal{J}$.

Let $f_1 T^{a_1}, \ldots, f_r T^{a_r}$ be generators of $\mathcal{J}$. Denote f_i^* the total transform of f_i, via the morphism $\mathcal{O}_W \to \mathcal{O}_{W'}$. From the fact $C \subset \mathrm{Sing}(\mathcal{J})$ it follows that $f_i^* = x^{\lceil a_i \rceil} g_i$ for some $g_i \in \mathcal{O}_{W'}$ and where $I(H) = (x)$. The transform $\mathcal{J}'$ is generated by $g_1 T^{a_1}, \ldots, g_r T^r$.

3.21. Let $E = \{H_1, \ldots, H_N\}$ be a set of smooth hypersurfaces in W having only normal crossings and let $\mathcal{J} = \oplus_a J_a T^a$ be a $\mathbb{Q}$-Rees algebra over W.

For $i = 1, \ldots, r$, set ℓ_{H_i} to be the maximum such that $I(H_i)^{\ell_{H_i}}$ divides $\mathcal{J}$ (3.19). Note that for any $a \in \mathbb{Q}_{\geq 0}$

$$J_a = I(H_1)^{\lceil a\ell_{H_1} \rceil} \cdots I(H_N)^{\lceil a\ell_{H_N} \rceil} I_a$$

where $I_a \subset \mathcal{O}_W$ is a sheaf of ideals.

Set $\mathcal{I} = E^{-1} \mathcal{J}$ to be the $\mathbb{Q}$-Rees algebra generated by $\{I_a T^a \mid a \in \mathbb{Q}_{\geq 0}\}$. We will call $E^{-1} \mathcal{J}$ the non monomial part of $\mathcal{J}$ with respect to E.

3.22. Consider some point or some affine open set of W. Suppose that $f_1 T^{a_1}, \ldots, f_r T^{a_r}$ are generators of $\mathcal{J}$ and $I(H_j) = (x_j)$, $j = 1, \ldots, N$. Set $c_{i,j} = \nu_{H_j}(f_i)$, for $i = 1, \ldots, r$ and $j = 1, \ldots, N$. We have

$$f_i = x_1^{c_{i,1}} \cdots x_N^{c_{i,N}} g_i$$

where the equation $g_i \notin (x_j)$ for any $j = 1, \ldots, N$. Set $c'_{i,j} = c_{i,j} - \lceil a_i \ell_{H_j} \rceil$. Note that $c'_{i,j} \geq 0$ since $\ell_{H_j} = \min\{\frac{c_{i,j}}{a_i} \mid i = 1, \ldots, r\}$. The $\mathbb{Q}$-Rees algebra $E^{-1} \mathcal{J}$ is generated by

$$h_1 T^{a_1}, \ldots, h_r T^{a_r}$$

where

$$h_i = x_1^{c'_{i,1}} \cdots x_N^{c'_{i,N}} g_i, \qquad i = 1, \ldots, r.$$

4. Coefficient algebras. For any morphism $V \to W$ of smooth varieties we have the natural morphism of sheaves $\mathcal{O}_W \to \mathcal{O}_V$. We may also consider a morphism $\text{Gr-}\mathbb{Q}(\mathcal{O}_W) \to \text{Gr-}\mathbb{Q}(\mathcal{O}_V)$.

DEFINITION 4.1. Fix W a smooth variety of pure dimension and a $\mathbb{Q}$-Rees algebra $\mathcal{J}$ over W. If $V \subset W$ is a smooth subvariety of pure dimension, we set the coefficient algebra of $\mathcal{J}$ with respect to V as follows

$$\text{Coeff}_V(\mathcal{J}) = \text{Diff}(\mathcal{J})\,\text{Gr-}\mathbb{Q}(\mathcal{O}_V).$$

4.2. It is well known that the coefficient algebra describes the same singular locus as $\mathcal{J}$. If $V \subset W$ is a smooth subvariety of W then

$$(4.2.1) \qquad\qquad \text{Sing}(\mathcal{J}) \cap V = \text{Sing}(\text{Coeff}_V(\mathcal{J})).$$

Moreover, equality 4.2.1 is preserved by transformations. To be more precise, set $\mathcal{A} = \text{Coeff}_V(\mathcal{J})$ and consider a sequence of transformations (3.20):

$$(4.2.2) \qquad
\begin{array}{ccccccccc}
W = W_0 & \longleftarrow & W_1 & \longleftarrow & W_2 & \longleftarrow & \cdots & \longleftarrow & W_N \\
 & & H_1 & & H_2 & & & & H_N \\
\mathcal{J} = \mathcal{J}_0 & & \mathcal{J}_1 & & \mathcal{J}_2 & & & & \mathcal{J}_N \\
V = V_0 & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_N \\
\mathcal{A} = \mathcal{A}_0 & & \mathcal{A}_1 & & \mathcal{A}_2 & & & & \mathcal{A}_N
\end{array}$$

where for any $i = 1, \ldots, N$,
- $\Pi_i : W_i \to W_{i-1}$ is a monoidal transformation with center $C_{i-1} \subset \text{Sing}(\mathcal{J}_{i-1}) \cap V_{i-1} \subset W_{i-1}$,
- $H_i \subset W_i$ is the exceptional divisor of Π_i,
- $\mathcal{J}_i = I(H_i)^{-1}\mathcal{J}_{i-1}^*$ is the transform of $\mathcal{J}_{i-1}$,
- V_i is the strict transform of V_{i-1} and
- $A_i = I(H_i \cap V_i)^{-1}\mathcal{A}_{i-1}^*$ is the transform of $\mathcal{A}_{i-1}$.

Then we have the equality

$$\text{Sing}(\mathcal{J}_N) \cap V_N = \text{Sing}(\mathcal{A}_N)$$

in fact inductively we have

$$\text{Sing}(\mathcal{J}_i) \cap V_i = \text{Sing}(\mathcal{A}_i), \qquad i = 0, \ldots, N.$$

This fact follows from 4.3.

THEOREM 4.3. *(Giraud) Let $\mathcal{J}$ be a $\mathbb{Q}$-Rees algebra over W and $W' \to W$ be a monoidal transformation with center $C \subset \text{Sing}(\mathcal{J})$.*

Set $\mathcal{D} = \text{Diff}(\mathcal{J})$ the differential saturation of $\mathcal{J}$. Recall that $\text{Sing}(\mathcal{J}) = \text{Sing}(\mathcal{D})$ by 3.12. Denote by $\mathcal{J}' = I(H)^{-1}\mathcal{J}^*$ and $\mathcal{D}' = I(H)^{-1}\mathcal{D}^*$ the transforms of $\mathcal{J}$ and $\mathcal{D}$, respectively. Then

$$\mathcal{J}' \subset \mathcal{D}' \subset \text{Diff}(\mathcal{J}').$$

Proof. See [EV07]. □

A key fact for proving resolution of singularities in characteristic zero is the existence (locally) of hypersurfaces $V \subset W$ such that the singular locus of a simple $\mathbb{Q}$-Rees algebra $\mathcal{J}$ (2.6) is included in V.

THEOREM 4.4. *Assume that characteristic of the ground field k is zero.*

Let $\mathcal{J}$ be a simple $\mathbb{Q}$-Rees algebra (2.6) over W and $\xi \in \mathrm{Sing}(\mathcal{J})$. There is an open set $\xi \in U \subset W$ and a smooth hypersurface $\xi \in V \subset U$ such that

(4.4.1) $$I_U(V)T \subset \mathrm{Diff}(\mathcal{J})|_U$$

where $I_U(V)$ denotes the ideal sheaf defined by V in U.

Proof. See [EV07]. The algebra $\mathcal{J}$ is simple, so that there is an equation $fT^b \in \mathcal{J}$ with $\mathrm{ord}(f) = b \in \mathbb{Z}$. Since we are in characteristic zero, there is a differential operator D of order $b-1$ such that $D(f)$ has order one. Note that $D(f)T \in \mathrm{Diff}(\mathcal{J})$. At a suitable neighborhood the equation $D(f)$ defines a smooth hypersurface V. $\square$

Theorem 4.4 does not hold in positive characteristic case and this is one of the main obstructions to find a proof in the general case.

4.5. Let $\mathcal{J}$ be a simple $\mathbb{Q}$-Rees algebra over W. Assume that there exist a smooth hypersurface $V \subset W$ such that $I(V)T \subset \mathrm{Diff}(\mathcal{J})$. It follows from 4.4 that this assumption may be always satisfied in all open sets of a suitable open covering of W, over a field of characteristic zero.

Note that $I(V)T \subset \mathrm{Diff}(\mathcal{J})$ implies that $\mathrm{Sing}(\mathcal{J}) \subset V$. Set $\mathcal{A} = \mathrm{Coeff}_V(\mathcal{J})$, by 4.2.1 we have $\mathrm{Sing}(\mathcal{J}) = \mathrm{Sing}(\mathcal{A})$. In fact for a sequence of transformations as in 4.2.2 we have for any $i = 1, \ldots, N$:

- $I(V_i)T \subset \mathrm{Diff}(\mathcal{J}_i)$,
- $\mathrm{Sing}(\mathcal{J}_i) \subset V_i$ and
- $\mathrm{Sing}(\mathcal{J}_i) = \mathrm{Sing}(\mathcal{A}_i)$.

Theorem 4.4 allows to choose hypersurfaces V with the property 4.4.1. This is the inductive argument in the proof of resolution of singularities. We replace the simple algebra $\mathcal{J}$ in W with the algebra $\mathrm{Coeff}_V(\mathcal{J})$ in V. But we have to prove that the total procedure will be independent of the choice of V. This problem was originally solved by Hironaka, by considering an equivalence relation of objects and proving that the procedure of resolution depends only on the equivalence class of the algebra $\mathcal{J}$.

An alternative path is the solution given by Włodarczyk [Wło05].

THEOREM 4.6. *Let $\mathcal{J}$ be a simple $\mathbb{Q}$-Rees algebra. Fix a point $\xi \in \mathrm{Sing}(\mathcal{J})$. Assume that V_1 and V_2 are two smooth hypersurfaces of W, $\xi \in V_1 \cap V_2$. Then there is an étale neighborhood U of ξ in W and an automorphism $\varphi : U \to U$ such that*

$$\varphi(\mathrm{Diff}(\mathcal{J})|_U) = \mathrm{Diff}(\mathcal{J})|_U, \qquad \varphi(V_1) = V_2.$$

Proof. We repeat the proof of [Wło05] in terms of $\mathbb{Q}$-Rees algebras.

Set $I(V_i) = (u_i)$ for some $u_i \in \mathcal{O}_{W,\xi}$, $i = 1, 2$. There are $x_2, \ldots, x_d \in \mathcal{O}_{W,\xi}$ such that $u_1, x_2, \ldots, x_d$ and $u_2, x_2, \ldots, x_d$ are both regular systems of parameters at $\mathcal{O}_{W,\xi}$.

Consider the automorphism $\varphi^\sharp$ of $\hat{\mathcal{O}}_{W,\xi}$ sending u_1 to u_2 and fixing $x_2, \ldots, x_d$. Note that this automorphism can be lifted to a suitable étale neighborhood. But, for simplification, we will consider all ideals in the completion $\hat{\mathcal{O}}_{W,\xi}$. Denote $\hat{\mathcal{J}} = \mathcal{J}\,\mathrm{Gr}\text{-}\mathbb{Q}(\hat{\mathcal{O}}_{W,\xi})$.

Let $fT^a \in \mathrm{Diff}(\mathcal{J})_\xi$. Consider the image of f in the completion $f \in \hat{\mathcal{O}}_{W,\xi}$, note that f is a power series $f = F(u_1, x_2, \ldots, x_d)$, then $\varphi^\sharp(F(u_1, x_2, \ldots, x_d)) =$

$F(u_2, x_2, \ldots, x_d)$. By assumption $u_i T \in \mathrm{Diff}(\hat{\mathcal{J}})$, $i = 1, 2$, set $h = u_2 - u_1$. We have $F(u_2, x_2, \ldots, x_d) = F(u_1 + h, x_2, \ldots, x_d)$ and

$$F(u_1 + h, x_2, \ldots, x_d) = \sum_{i=0}^{\infty} h^i \frac{1}{i!} \frac{\partial^i F}{\partial u_1^i}(u_1, x_2, \ldots, x_d).$$

Now note that $h^i T^i \in \mathrm{Diff}(\hat{\mathcal{J}})$ for all $i \geq 0$, and $FT^a \in \mathrm{Diff}(\hat{\mathcal{J}})$ implies that $\frac{\partial^i F}{\partial u_1^i} T^{a-i} \in \mathrm{Diff}(\hat{\mathcal{J}})$ for $0 \leq i < a$. We conclude $\varphi^\sharp(f) T^a \in \mathrm{Diff}(\hat{\mathcal{J}})$ and then $\varphi^\sharp(f) T^a \in \mathrm{Diff}(\mathcal{J})_\xi$ at a suitable étale neighborhood. $\square$

5. Elimination algebras. Villamayor has introduced in the paper [VU07] the concept of elimination algebra. The coefficient algebra is defined for an inclusion $V \subset W$ and the elimination algebra will be defined for a smooth morphism $W \to V$. In characteristic zero both algebras will encode the same information.

DEFINITION 5.1. Let $\mathcal{J}$ be a $\mathbb{Q}$-Rees algebra over W, with pure dimension $d = \dim(W)$. Let V a regular algebraic variety of pure dimension $d-1$ and $\beta : W \to V$ be a smooth morphism.

Consider the natural sheaf homomorphism $\mathcal{O}_V \to \mathcal{O}_W$, which induces an homomorphism $\mathrm{Gr}\text{-}\mathbb{Q}(\mathcal{O}_V) \to \mathrm{Gr}\text{-}\mathbb{Q}(\mathcal{O}_W)$. The elimination algebra of $\mathcal{J}$ in V is

$$\mathcal{R}_V(\mathcal{J}) = \mathcal{J} \cap \mathrm{Gr}\text{-}\mathbb{Q}(\mathcal{O}_V).$$

The elimination algebra has good properties when the algebra $\mathcal{J}$ is simple and differentially saturated. In fact in [VU07] the elimination algebra is only defined for simple algebras saturated by the relative differentials with respect to the morphism β. Note also that in [VU07] the elimination algebra is constructed using a universal algebra in terms of symmetric polynomials and after it is proved the equivalence with definition 5.1. Definition as in [VU07] allows to prove properties related to integral closure.

The following theorem proves that singular locus of elimination algebra and the original algebra may be identified.

THEOREM 5.2. [VU07] *Let $\mathcal{J}$ be a simple $\mathbb{Q}$-Rees algebra over W, and let $\beta : W \to V$ be a smooth morphism, $\dim V = \dim W - 1$. Assume that $\mathcal{J} = \mathrm{Diff}(\mathcal{J})$ then*

$$\mathrm{Sing}(\mathcal{R}_V(\mathcal{J})) = \beta(\mathrm{Sing}(\mathcal{J})).$$

Moreover, β is 1-1 between the points of $\mathrm{Sing}(\mathcal{R}_V(\mathcal{J}))$ and $\beta(\mathrm{Sing}(\mathcal{J}))$.

Unfortunately the equality in 5.2 does not hold, in general, after monoidal transformation.

EXAMPLE 5.3. Consider a field k of characteristic two. Set $W = \mathrm{Spec}(k[x, y, z])$, $V = \mathrm{Spec}(k[y, z])$ and $\beta : W \to V$ the usual projection. Consider the $\mathbb{Q}$-Rees algebra $\mathcal{J}$ generated by $(x^2 + y^2 z)T^2$. The differential saturation is generated by

$$(x^2 + y^2 z)T^2, \ y^2 T.$$

The elimination algebra of the differential saturation $\mathcal{A} = \mathcal{R}_V(\mathrm{Diff}(\mathcal{J}))$ is generated by $y^2 T$.

Now consider the blowing-up with center at the origin of W and consider the affine chart where the ideal of the exceptional divisor is z. The transform $\mathcal{J}_1$ of $\mathcal{J}$ is generated by $(x^2 + y^2 z)T^2$. The transform $\mathcal{A}_1$ of the elimination algebra $\mathcal{A}$ is generated by $y^2 z T$. Note that $\mathrm{Sing}(\mathcal{J}_1)$ is a line and $\mathrm{Sing}(\mathcal{A}_1)$ is a union of two lines with

$$\mathrm{Sing}(\mathcal{J}_1) \subset \mathrm{Sing}(\mathcal{A}_1).$$

If the characteristic of k is zero then equality in theorem 5.2 is stable by monoidal transformation. In fact, after an étale extension, the coefficient and elimination algebra are isomorphic.

REMARK 5.4. Given X a topological space and Y a subspace of X. Remember that a *retraction* $r : X \to Y$ is a continuous map such that the restriction of r to Y is the identity map on Y.

THEOREM 5.5. *Assume that the characteristic of the ground field k is zero.*

Let $\mathcal{J}$ be a simple $\mathbb{Q}$-Rees algebra. Fix a closed point $\xi \in \mathrm{Sing}(\mathcal{J})$. There is an étale neighborhood of ξ, U in W, a smooth hypersurface $V \subset U$ and a retraction $\beta : U \to V$ such that

$$\mathrm{Coeff}_V(\mathcal{J}|_U) = \mathcal{R}_V(\mathrm{Diff}(\mathcal{J}|_U)).$$

Proof. By theorem 4.4 there is an equation $z \in \mathcal{O}_{W,\xi}$ of order one and with $zT \in \mathrm{Diff}(\mathcal{J})$. Consider $x_2, \ldots, x_d \in \mathcal{O}_{W,\xi}$ such that $z, x_2, \ldots, x_d$ is a regular system of parameters. Set $\hat{\mathcal{J}} = \mathcal{J} \, \mathrm{Gr}\text{-}\mathbb{Q}(\hat{\mathcal{O}}_{W,\xi})$. We will prove that

$$\mathrm{Coeff}_{\hat{V}}(\hat{\mathcal{J}}) = \mathcal{R}_{\hat{V}}(\mathrm{Diff}(\hat{\mathcal{J}}))$$

where V is the hypersurface defined by $z = 0$. Note that we are considering $\hat{\mathcal{O}}_{W,\xi}$ as the power series ring $k'[[z, x_2, \ldots, x_d]]$ and $\mathcal{O}_{\hat{V},\xi}$ as the power series ring $k'[[x_2, \ldots, x_d]]$, where $k' \supset k$ is the residue field at ξ. The retraction to $\hat{V}$ corresponds to the inclusion of those power series rings.

The inclusion $\mathcal{R}_{\hat{V}}(\mathrm{Diff}(\hat{\mathcal{J}})) \subset \mathrm{Coeff}_{\hat{V}}(\hat{\mathcal{J}})$ is now clear.

Assume that $fT^b \in \hat{\mathcal{J}}$. We can express f as

$$f = a_0(x) + a_1(x)z + a_2(x)z^2 + \cdots + a_{b-1}(x)z^{b-1} + a_b(x, z)z^b$$

where $a_i(x) \in k'[[x_2, \ldots, x_d]]$, for $i = 0, \ldots, b - 1$ and $a_b(x, z) \in k'[[z, x_2, \ldots, x_d]]$. Note that $a_0(x)T^b \in \mathrm{Coeff}_{\hat{V}}(\hat{\mathcal{J}})$.

It is enough to prove that $a_0(x)T^b \in \mathrm{Diff}(\hat{\mathcal{J}})$. In fact we will see by descending induction that $a_i(x)T^{b-i} \in \mathrm{Diff}(\hat{\mathcal{J}})$, for $i = 0, \ldots, b - 1$.

Recall that $zT \in \mathrm{Diff}(\hat{\mathcal{J}})$ and then

$$\frac{\partial^{b-1} f}{\partial z^{b-1}} = (b-1)! \left(a_{b-1}(x) + z \sum_{j=0}^{b-1} \binom{b}{j+1} \frac{\partial^j a_b(x, z)}{\partial z^j} \frac{z^j}{j!} \right) \implies a_{b-1}(x)T \in \mathrm{Diff}(\hat{\mathcal{J}}).$$

Fix $i < b-1$, assume that $a_j(x)T^{b-j} \in \mathrm{Diff}(\hat{\mathcal{J}})$ for $j = i+1, \dots, b-1$. It follows from the expression

$$\frac{\partial^i f}{\partial z^i} = i! \left(a_i(x) + \sum_{j=i+1}^{b-1} \binom{j}{i} a_j(x) z^{j-i} + z^{b-i} \sum_{j=0}^{i} \binom{b}{i-j} \frac{\partial^j a_b(x,z)}{\partial z^j} \frac{z^j}{j!} \right)$$

that $a_i(x)T^{b-i} \in \mathrm{Diff}(\hat{\mathcal{J}})$, as required. $\square$

6. Algorithm of resolution in characteristic zero case. Along this section we assume that the characteristic of the field k is zero.

We will describe here an algorithm of resolution for $\mathbb{Q}$-Rees algebras. It is inspired in [EH02].

DEFINITION 6.1. Let $\mathcal{J}$ be a $\mathbb{Q}$-Rees algebra over W. Assume that E is a set of hypersurfaces of W having only normal crossings.

A resolution of $\mathcal{J}$ over (W, E) is a sequence of monoidal transformations:

(6.1.1) $(W, E) = (W_0, E_0) \longleftarrow (W_1, E_1) \longleftarrow \cdots \longleftarrow (W_N, E_N)$

such that, for $i = 0, \dots, N-1$
  - If $\mathcal{J}_i$ is the transform of $\mathcal{J}$ in W_i then $W_{i+1} \to W_i$ is the monoidal transformation with center $C_i \subset \mathrm{Sing}(\mathcal{J}_i)$,
  - C_i has normal crossings with E_i,
  - E_{i+1} consists of all strict transforms of E_i and the exceptional divisor of $W_{i+1} \to W_i$ and
  - $\mathrm{Sing}(\mathcal{J}_N) = \emptyset$.

A log-resolution of an ideal $J \subset \mathcal{O}_W$ can be achieved by a resolution of the $\mathbb{Q}$-Rees algebra generated by JT.

We will construct the sequence 6.1.1 inductively on the number of blowing-ups and the dimension d of W.

6.2. At every step we will define an upper semicontinuous function $Fc_i^{(d)} : W_i \to \Lambda$ where $\Lambda = ((\mathbb{Q}_{\geq 0} \cup \{\infty\}) \times \mathbb{Z}_{\geq 0})^{\mathbb{N}}$ is ordered lexicographically. The function $Fc_i^{(d)}$ will depend on the previous steps, say the functions $Fc_0^{(d)}, \dots, Fc_{i-1}^{(d)}$.

In fact the situation is local. If $\xi_i \in W_i$, the definition of the value $Fc_i(\xi_i)$ is local on the sequence 6.1.1. It depends only on the values of $Fc_0(\xi_0), \dots, Fc_{i-1}(\xi_{i-1})$, where $\xi_0, \dots, \xi_{i-1}$ are the images of ξ_i at $W_0, \dots, W_{i-1}$, respectively, and on the stalks $\mathcal{J}_{0,\xi_0}, \dots, \mathcal{J}_{i-1,\xi_{i-1}}$.

Each center $C_i \subset W_i$ will be the set of points where the function $Fc_i^{(d)}$ is maximum:

$$C_i = \underline{\mathbf{Max}}\, Fc_i^{(d)} = \{\xi \in W_i \mid Fc_i^{(d)}(\xi) = \max Fc_i^{(d)}\}.$$

Moreover, it can be proved that sequence 6.1.1 together with functions Fc have the following properties:
  1. $C_i = \underline{\mathbf{Max}}\, Fc_i^{(d)}$,
  2. $\max Fc_0^{(d)} > \max Fc_1^{(d)} > \cdots \max Fc_{N-1}^{(d)}$ and
  3. if $\xi_i \in W_i \setminus C_i$ then ξ_i identifies with a point $\xi_{i+1} \in W_{i+1}$ and $Fc_i^{(d)}(\xi_i) = Fc_{i+1}^{(d)}(\xi_{i+1})$.

6.3. We will require also a stability property with smooth morphisms.

Let $\mathcal{J}$ be a $\mathbb{Q}$-Rees algebra over (W, E), let $W' \to W$ be a smooth morphism. Set E' to be the set of hypersurfaces of W' obtained by the pullback of E. Set $\mathcal{J}'$ to be the $\mathbb{Q}$-Rees algebra obtained also by pullback.

In this situation we have two sequences, say the resolutions of $\mathcal{J}$ and $\mathcal{J}'$.

Denote the resolution of $\mathcal{J}$:

$$(6.3.1) \quad \begin{array}{ccccccc} (W_0, E_0) & \longleftarrow & (W_1, E_1) & \longleftarrow & \cdots & \longleftarrow & (W_N, E_N) \\ \mathcal{J}_0 & & \mathcal{J}_1 & & & & \mathcal{J}_N \\ F\mathrm{c}_0^{(d)}(\mathcal{J}) & & F\mathrm{c}_1^{(d)}(\mathcal{J}) \end{array}$$

where $F\mathrm{c}_i^{(d)}(\mathcal{J}) : W_i \to \Lambda$, $i = 0, \dots, N-1$, are the functions associated to the resolution of $\mathcal{J}$.

On the other hand denote the resolution of $\mathcal{J}'$:

$$(6.3.2) \quad \begin{array}{ccccccc} (W'_0, E'_0) & \longleftarrow & (W'_1, E'_1) & \longleftarrow & \cdots & \longleftarrow & (W'_{N'}, E'_{N'}) \\ \mathcal{J}'_0 & & \mathcal{J}'_1 & & & & \mathcal{J}'_{N'} \\ F\mathrm{c}_0^{(d')}(\mathcal{J}') & & F\mathrm{c}_1^{(d')}(\mathcal{J}') \end{array}$$

where $F\mathrm{c}_i^{(d')}(\mathcal{J}') : W'_i \to \Lambda$, $i = 0, \dots, N'-1$, are the functions associated to the resolution of $\mathcal{J}'$.

The stability property says that the sequence 6.3.2 is the pullback, via $W' \to W$, of the sequence 6.3.1; and functions take the same values

$$F\mathrm{c}_i^{(d')}(\mathcal{J}')(\xi') = F\mathrm{c}_i^{(d)}(\mathcal{J})(\xi)$$

for any $\xi' \in W'_i$ and any $i = 0, 1, \dots, N'-1$. Where $\xi' \in W'_i$ maps to $\xi \in W_i$.

6.4. Dimension one case. If $\dim W = 1$, the singular locus $\mathcal{J}$ is a finite number of points in W. We set $F\mathrm{c}_0^{(1)} = ((\mathrm{ord}(\mathcal{J}), 0), (0, 0), \dots)$. The blowing-up with center $C_0 = \underline{\mathbf{Max}}\, F\mathrm{c}_0^{(1)}$, $W_1 \to W_0 = W$, is an isomorphism, but the transform $\mathcal{J}_1$ is a different $\mathbb{Q}$-Rees algebra. Note that $\mathcal{J}_1 = I(C_0)^{-1} \mathcal{J}^*$ and if $\xi \in C_0$ then $\mathrm{ord}(\mathcal{J}_1)(\xi) = \mathrm{ord}(\mathcal{J}_0)(\xi) - 1$.

If $\mathrm{Sing}(\mathcal{J}_1) \neq \emptyset$ then $\max F\mathrm{c}_1^{(1)} \geq ((1, 0), (0, 0), \dots)$. Set $F\mathrm{c}_1^{(1)} = ((\mathrm{ord}(\mathcal{J}_1), 1), (0, 0), \dots)$ and continue with this procedure. It is easy to prove that we obtain a sequence as in 6.1.1 with the required properties in 6.2.

6.5. In what follows we will fix a dimension $d > 1$ and assume that we have constructed resolution of $\mathbb{Q}$-Rees algebras over varieties of dimension $d - 1$. The constructed procedure satisfies properties in 6.1 and 6.2, and also stability property 6.3 for smooth morphisms of relative dimension zero.

6.6. Initial step. Fix a point $\xi_0 \in \mathrm{Sing}(\mathcal{J}_0)$. We construct the value $F\mathrm{c}_0^{(d)}(\xi_0)$ as follows:

Set $\omega = \omega_{\xi_0,0}^{(d)} = \mathrm{ord}(\mathcal{J}_0)(\xi_0)$ and set

$$\mathcal{I}_{\xi_0,0} = \mathcal{J}_0, \qquad \mathcal{P}_{\xi_0,0} = \mathcal{I}_{\xi_0,0}^{\frac{1}{\omega}}, \qquad \mathcal{T}_{\xi_0,0} = \mathcal{P}_{\xi_0,0} \odot \mathcal{E}_{\xi_0,0}$$

where $\mathcal{E}_{\xi_0,0}$ is the $\mathbb{Q}$-Rees algebra generated by $\{I(H)T \mid H \in E_0,\ \xi_0 \in H\}$.

The $\mathbb{Q}$-Rees algebra $\mathcal{P}_{\xi_0,0}$ has order one at ξ_0. We may consider an open neighborhood of ξ_0 such that the algebra $\mathcal{P}_{\xi_0,0}$ is simple. By 4.4 we could consider a more suitable neighborhood U in order to choose a smooth hypersurface $W_0^{(d-1)} \subset U$ satisfying 4.4.1. Set $\mathcal{J}_{\xi_0,0}^{(d-1)} = \mathrm{Coeff}_{W_0^{(d-1)}}(\mathcal{T}_{\xi_0,0})$.

If $\mathcal{J}_{\xi_0,0}^{(d-1)} = 0$ then we set $F\mathrm{c}_0^{(d-1)} = ((\infty,0),(0,0),\dots)$.

If $\mathcal{J}_{\xi_0,0}^{(d-1)} \neq 0$ then, by induction (6.5), the value $F\mathrm{c}_0^{(d-1)}(\xi_0)$ associated to $\mathcal{J}_{\xi_0,0}^{(d-1)}$ and $(W_0^{(d-1)}, E_0^{(d-1)})$ is already defined, where we set $E_0^{(d-1)} = \emptyset$.

In any case, we may set

$$F\mathrm{c}_0^{(d)}(\xi_0) = ((\omega_{\xi_0,0}^{(d)},0), F\mathrm{c}_0^{(d-1)}(\xi_0))$$

where the second component is 0 because at the beginning there are no exceptional divisors. We will define this component in 6.8.1.

The value $F\mathrm{c}_0^{(d)}(\xi_0)$ is well-defined and does not depend on the choice of the hypersurface V. This follows from 4.6 and property 6.3 applied to V and smooth morphisms of relative dimension zero.

With this procedure we define a function $F\mathrm{c}_0^{(d)} : W_0 \to \Lambda$. The upper semicontinuity follows by the upper semicontinuity of functions ord and $F\mathrm{c}_0^{(d-1)}$.

The closed set $C_0 = \underline{\mathbf{Max}} \, F\mathrm{c}_0^{(d)}$ is smooth by the inductive assumption on the dimension $d-1$.

On the other hand, C_0 has only normal crossings with E_0 by the definition of $\mathcal{T}_0$.

6.7. Step i. Assume now that $d = \dim W > 1$. And suppose that we have already constructed the first i steps of the sequence 6.1.1.

(6.7.1) $$(W_0, E_0) \longleftarrow (W_1, E_1) \longleftarrow \cdots \longleftarrow (W_i, E_i)$$

and the functions

$$F\mathrm{c}_0^{(d)}, \; F\mathrm{c}_1^{(d)}, \dots, \; F\mathrm{c}_{i-1}^{(d)}$$

satisfying properties in 6.1 and 6.2, and also stability property 6.3 for smooth morphisms of relative dimension zero.

For any $j = 0, \dots, i-1$, the algebra $\mathcal{J}_j$ is the transform of $\mathcal{J}_0$ in W_j. We denote by $E_{j,0}$ to be the set of strict transforms of E_0 in W_j. Note that $E_{j,0} \subset E_j$. We set the non-monomial part of $\mathcal{J}_j$

(6.7.2) $$\mathcal{I}_j = (E_j \setminus E_{j,0})^{-1} \mathcal{J}_j.$$

For any point $\xi_i \in W_i$ we denote $\xi_j \in W_j$, $j = 0, \dots, i-1$ to be the image of ξ_i in W_j.

By construction the first coordinate of the $F\mathrm{c}_j^{(d)}(\xi_j)$ is $\omega_{\xi_j,j}^{(d)} = \mathrm{ord}(\mathcal{I}_j)(\xi_j)$. We have the chain of inequalities

$$\omega_{\xi_0,0}^{(d)} \geq \omega_{\xi_1,1}^{(d)} \geq \cdots \geq \omega_{\xi_{i-1},i-1}^{(d)}.$$

6.8. With the situation as in 6.7 we want to define the function $F\mathrm{c}_i^{(d)} : W_i \to \Lambda$.

Fix a point $\xi_i \in W_i$. Set $I_i = (E_i \setminus E_{i,0})^{-1} \mathcal{J}_i$ to be the non-monomial part and set $\omega_{\xi_i,i}^{(d)} = \mathrm{ord}(\mathcal{I}_i)(\xi_i)$. We have the inequalities

$$\omega_{\xi_0,0}^{(d)} \geq \omega_{\xi_1,1}^{(d)} \geq \cdots \geq \omega_{\xi_{i-1},i-1}^{(d)} \geq \omega_{\xi_i,i}^{(d)}.$$

First note that if $\omega_{\xi_i,i}^{(d)} = 0$ then the algebra $\mathcal{J}_i$ reduce to a monomial and it is easy to define $Fc_i^{(d)}(\xi_i)$ in order to enlarge 6.7.1 (locally at ξ_i) to a resolution.

So that we may assume $\omega_{\xi_i,i}^{(d)} > 0$. Set j_0 the minimum index such that

$$\omega_{\xi_{j_0},j_0}^{(d)} = \cdots = \omega_{\xi_{i-1},i-1}^{(d)} = \omega_{\xi_i,i}^{(d)}.$$

Denote by E_{i,j_0} to be the set of strict transforms of E_{j_0} in W_i, $E_{i,j_0} \subset E_i$. Define

$$(6.8.1) \qquad n_{\xi_i,i} = \#\{H \in E_{i,j_0} \mid \xi_i \in E_{i,j_0}\}.$$

If $\omega = \omega_{\xi_i,i}^{(d)}$ we set

$$\mathcal{P}_{\xi_i,i} = \mathcal{I}_i^{\frac{1}{\omega}} \odot \mathcal{J}_i, \qquad \mathcal{T}_{\xi_i,i} = \mathcal{P}_{x_i,i} \odot \mathcal{E}_{\xi_i,i}$$

where $\mathcal{E}_{\xi_i,i}$ is the $\mathbb{Q}$-Rees algebra generated by $\{I(H)T \mid H \in E_{i,j_0}, \xi_i \in H\}$.

The order of $\mathcal{P}_{\xi_i,i}$ at ξ_i is one, so that there is an open neighborhood U where $\mathcal{P}_{\xi_i,i}$ is simple. By 4.4 we could shrink U and choose a smooth hypersurface $W_i^{(d-1)} \subset U$ such that $I(W_i^{(d-1)})T \subset \mathrm{Diff}(\mathcal{P}_{\xi_i,i})|_U$.

Set $E_i^{(d-1)} = E_i \setminus E_{i,j_0}$ and $\mathcal{J}_i^{(d-1)} = \mathrm{Coeff}_{W_i^{(d-1)}}(\mathcal{T}_{\xi_i,i})$.

If $\mathcal{J}_i^{(d-1)} = 0$ then we set $Fc_i^{(d-1)}(\xi_i) = ((\infty,0),(0,0),\ldots)$.

If $\mathcal{J}_i^{(d-1)} \neq 0$ then by induction hypothesis we may consider the resolution of $\mathcal{J}_{j_0}^{(d-1)}$ in $W_{j_0}^{(d-1)}$. And we set

$$Fc_i^{(d)}(\xi_i) = ((w_{\xi_i,i}^{(d-1)}, n_{\xi_i,i}), Fc_i^{(d-1)}(\xi_i)).$$

The function $Fc_i^{(d)}$ is upper semicontinuous by construction and the upper semicontinuity of $Fc_i^{(d-1)}$.

In order to prove that the center $C_i = \underline{\mathbf{Max}} Fc_i^{(d)}$ is smooth, note that if $Fc_i^{(d-1)}(C_i) = ((\infty,0),(0,0),\ldots)$ then $C_i = W_i^{(d-1)}$. If $Fc_i^{(d-1)}(C_i) \neq ((\infty,0),(0,0),\ldots)$ then $C_i = \underline{\mathbf{Max}} Fc_i^{(d-1)}$ and it is smooth by induction hypothesis.

6.9. Given a $\mathbb{Q}$-Rees algebra $\mathcal{J}$ we have constructed a sequence 6.1.1 and functions $Fc_i^{(d)}$, $i = 0, \ldots, N-1$ as in 6.2.

Assume that we have two equivalent $\mathbb{Q}$-Rees algebras $\mathcal{J}$ and $\mathcal{J}'$ (1.10). Then associated to $\mathcal{J}$ and $\mathcal{J}'$ are two sequences as in 6.1.1, say:

$$(6.9.1) \qquad (W,E) = (W_0, E_0) \longleftarrow (W_1, E_1) \longleftarrow \cdots \longleftarrow (W_N, E_N).$$
$$(6.9.2) \qquad (W,E) = (W_0', E_0') \longleftarrow (W_1', E_1') \longleftarrow \cdots \longleftarrow (W_{N'}', E_{N'}').$$

We also have functions $Fc_0^{(d)}, \ldots, Fc_N^{(d)}$, associated to $\mathcal{J}$, and $Fc_0'^{(d)}, \ldots, Fc_{N'}'^{(d)}$, associated to $\mathcal{J}'$.

We claim that both sequences are equal and moreover the functions are equal: $N = N'$ and $Fc_i^{(d)} = Fc_i'^{(d)}$, $i = 0, \dots, N - 1$.

To prove our claim, it is enough to see that all operations involved are stable by integral closure:

We may proceed by induction on $i = 0, \dots, N - 1$. If $\mathcal{J}_j$ and $\mathcal{J}_j'$, $j = 0, \dots, i$ are equivalent and the sequences in 6.9.1 and 6.9.2 coincide for the first i steps, then

- the algebras $\mathcal{I}_j$ and $\mathcal{I}_j'$ defined as in 6.7.2 are equivalent,
- the algebras $\mathcal{P}_{\xi_i,i}$, $\mathcal{T}_{\xi_i,i}$ and $\mathcal{P}_{\xi_i,i}'$, $\mathcal{T}_{\xi_i,i}'$ defined as in 6.8 are, respectively, equivalent.

Finally the centers $C_i = C_i'$ coincide and the transforms $\mathcal{J}_{i+1}$ and $\mathcal{J}_{i+1}'$ are equivalent.

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