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DISSIPATIVE HYPERBOLIC GEOMETRIC FLOW*

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Abstract. In this paper we introduce a new kind of hyperbolic geometric flows — dissipative hyperbolic geometric flow. This kind of flow is defined by a system of quasilinear wave equations with dissipative terms. Some interesting exact solutions are given, in particular, a new concept — hyperbolic Ricci soliton is introduced and some of its geometric properties are described. We also establish the short-time existence and uniqueness theorem for the dissipative hyperbolic geometric flow, and prove the nonlinear stability of the flow defined on the Euclidean space of dimension larger than 2. Wave character of the evolving metrics and curvatures is illustrated and the nonlinear wave equations satisfied by the curvatures are derived.

Key words. Dissipative hyperbolic geometric flow, quasilinear wave equation, hyperbolic Ricci soliton, short-time existence, nonlinear stability.

AMS subject classifications. 58J45, 58J47

1. Introduction. Let $\mathscr{M}$ be an n-dimensional complete Riemannian manifold with Riemannian metric g_{ij}. The following evolution equation for the metric g_{ij}

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \mathscr{F}_{ij}\left(g, \frac{\partial g}{\partial t}\right) = 0 \tag{1.1}$$

has been recently introduced and named as *general version of hyperbolic geometric flow* by Kong and Liu [11], where R_{ij} is the corresponding Ricci curvature tensor and $\mathscr{F}_{ij}$ is a given smooth symmetric tensor on the Riemannian metric g and its first order derivative with respect to t. A special but important case is

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}. \tag{1.2}$$

Usually, we call (1.2) the *standard hyperbolic geometric flow* or simply *hyperbolic geometric flow*. (1.1) and (1.2) are two nonlinear systems of second order partial differential equations on the metric g_{ij}.

For the hyperbolic geometric flow (1.2), some interesting exact solutions have been constructed by Kong and Liu [11]. Recently, Kong, Liu and Xu [13] have investigated the evolution of Riemann surfaces under the flow (1.2) and given some results on the global existence and blowup phenomenon of smooth solutions to the flow equation (1.2). In our paper [2], we prove the short-time existence for the hyperbolic geometric flow (1.2) and the nonlinear stability of the Euclidean space with dimension larger than 4. Moreover, we also study the wave character of the curvatures for the flow (1.2) and derive the equations satisfied by curvatures including the Riemannian curvature tensor R_{ijkl}, the Ricci curvature tensor R_{ij} and the scalar curvature R. However, these evolution equations are quite complicated. In general, the solution of the hyperbolic geometric flow (1.2) may blowup in a finite time even for smooth initial data.

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Motivated by the well-developed theory of the dissipative hyperbolic equations, we introduce a new geometric analytical tool — dissipative hyperbolic geometric flow:

$$\begin{aligned}\frac{\partial^2 g_{ij}}{\partial t^2} &= -2R_{ij} + 2g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} - \left(d + 2g^{pq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial g_{ij}}{\partial t} + \\ &\quad \frac{1}{n-1}\left[\left(g^{pq}\frac{\partial g_{pq}}{\partial t}\right)^2 + \frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right] g_{ij}\end{aligned} \tag{1.3}$$

where $g_{ij}(t)$ stands for a family of Riemannian metrics defined on $\mathscr{M}$, and d is a positive constant. The derivation of (1.3) is given in Section 6. Here we would like to point out that the reason that we choose (1.3) as the equation form of dissipative hyperbolic geometric flow is that, in the case it possesses a simpler equation satisfied by the scalar curvature. Noting the dissipative property of (1.3), we expect that the dissipative hyperbolic geometric flow admits a global smooth solution (i.e., a family of Riemannian metrics) for all $t \geq 0$, and the solution (metrics) has some good or anticipant geometric properties for relatively general initial data in the case that the dissipative coefficient d is chosen to be suitably large.

In the present paper we will focus on some basic properties enjoyed by the dissipative hyperbolic geometric flow. The first basic property is on the *hyperbolic Ricci soliton*. The hyperbolic Ricci soliton is a new concept which we introduce in this paper. We will prove that there does not exist steady gradient hyperbolic Ricci soliton with initial metric of positive average scalar curvature on n-dimensional compact manifold (where $n \geq 3$). Comparing with the traditional Ricci flow, here we need the assumption that the initial metric has non-negative average scalar curvature. If this assumption does not hold, then the question whether there exist steady gradient hyperbolic Ricci solitons still remains open. See Theorem 3.1 for the detail.

The second fundamental property is the short-time existence and uniqueness theorem for the dissipative hyperbolic geometric flow. For compact manifolds, we can prove that the dissipative hyperbolic geometric flow always admits a unique smooth solution (a family of Riemannian metrics) for smooth initial data. See Theorem 4.1. Notice that the dissipative hyperbolic geometric flow (1.3) is only weakly hyperbolic, since the symbol of the derivative of $E = E(g_{ij}) \stackrel{\triangle}{=} -2R_{ij}$ has zero eigenvectors in the natural coordinates. In order to reduce the nonlinear weakly hyperbolic partial differential equation (1.3) to a nonlinear symmetric system of strictly hyperbolic partial differential equations, we use harmonic coordinates introduced by DeTurck and Kazdan [4]. Then by the standard theory of symmetric hyperbolic system, we can prove the short-time existence and uniqueness theorem 4.1.

The third property is the nonlinear stability. By the global existence theory of dissipative wave equations, we can prove the global nonlinear stability of the Euclidean space $\mathbb{R}^n$ with $n \geq 3$. See Theorem 5.1 for the details. In the proof of nonlinear stability, the dissipative property of the flow (1.3) play an important role.

The fourth fundamental property is the wave character of the curvatures. Since the dissipative hyperbolic geometric flow is described by a system of quasilinear wave equations on the metrics $g_{ij}(t, x)$, the wave property of the metric implies the wave character of the curvatures. The equations will play an important role in the future study. See Section 6 for the details.

By the way, we would like to point out that, a hyperbolic version of mean curvature flow has been developed in [10] and [12], and some physical discussions of hyperbolic geometric flow governed by mean curvature can be found in [5] and [8].

The paper is organized as follows. In Section 2, we introduce the dissipative hyperbolic geometric flow equation and give a useful lemma. In order to understand the basics of the dissipative hyperbolic geometric flow, we construct some exact solutions. These solutions may be useful in physics. In Section 3, we introduce the steady gradient hyperbolic Ricci soliton, and prove Theorem 3.1 — one of the main results in this paper. Section 4 is devoted to the short-time existence and uniqueness of the flow, while Section 5 is devoted to the global nonlinear stability of the Euclidean space $\mathbb{R}^n$ with $n \geq 3$. The wave character of the curvatures is discussed in Section 6, and the nonlinear wave equations satisfied by the curvatures are also derived in this section.

2. Dissipative hyperbolic geometric flow. The dissipative hyperbolic geometric flow considered here is defined by the equation (1.3), namely,

$$\begin{aligned} \frac{\partial^2 g_{ij}}{\partial t^2} &= -2R_{ij} + 2g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} - \left(d + 2g^{pq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial g_{ij}}{\partial t} + \\ &\quad \frac{1}{n-1}\left[\left(g^{pq}\frac{\partial g_{pq}}{\partial t}\right)^2 + \frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right] g_{ij} \end{aligned} \tag{2.1}$$

where $g_{ij}(t)$ stands for a family of Riemannian metrics defined on $\mathscr{M}$, and d is a positive constant. The reason that we choose (2.1) as the equation form of dissipative hyperbolic geometric flow is as follows: in this case the flow possesses a simpler equation satisfied by the scalar curvature. See the derivation of (2.1) in Section 6.

We first establish some useful equations from the flow equation (2.1). Let

$$u(x,t) = g^{ij}\frac{\partial g_{ij}}{\partial t}, \tag{2.2}$$

$$v(x,t) = \left|\frac{\partial g}{\partial t}\right|^2 = g^{ik}g^{jl}\frac{\partial g_{ij}}{\partial t}\frac{\partial g_{kl}}{\partial t}, \tag{2.3}$$

$$w(x,t) = g^{ik}g^{jl}g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t}\frac{\partial g_{kl}}{\partial t} \tag{2.4}$$

and denote the matrix

$$G(x,t) = \left(\frac{\partial g_{ij}}{\partial t}g^{jk}\right). \tag{2.5}$$

Then we have

$$u(x,t) = trG(x,t), \quad v(x,t) = trG^2(x,t), \quad w(x,t) = trG^3(x,t), \tag{2.6}$$

where trG stands for the trace of the matrix G. Thus by (2.1) we obtain

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \frac{\partial}{\partial t}\left(g^{ij}\frac{\partial g_{ij}}{\partial t}\right) \\
&= \frac{\partial g^{ij}}{\partial t}\frac{\partial g_{ij}}{\partial t} + g^{ij}\frac{\partial^2 g_{ij}}{\partial t^2} \\
&= -g^{ik}g^{jl}\frac{\partial g_{ij}}{\partial t}\frac{\partial g_{kl}}{\partial t} + g^{ij}\left[-2R_{ij} + 2g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} - 2g^{pq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{ij}}{\partial t}\right. \\
&\quad \left. -d\cdot\frac{\partial g_{ij}}{\partial t} + \frac{1}{n-1}\left(\left(g^{pq}\frac{\partial g_{pq}}{\partial t}\right)^2 + \left(\frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right)\right)g_{ij}\right] \\
&= -2R - \frac{n-2}{n-1}u^2 - du - \frac{1}{n-1}v
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&\frac{\partial v(x,t)}{\partial t} \\
= \;& 2\frac{\partial g^{ij}}{\partial t}g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} + 2g^{ij}g^{pq}\frac{\partial^2 g_{ip}}{\partial t^2}\frac{\partial g_{jq}}{\partial t} \\
= \;& -2g^{ir}g^{js}g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2g^{ij}g^{pq}\frac{\partial g_{jq}}{\partial t}\left[-2R_{ip} + 2g^{rs}\frac{\partial g_{ir}}{\partial t}\frac{\partial g_{ps}}{\partial t}\right. \\
& -2\left(g^{rs}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial g_{ip}}{\partial t} - d\frac{\partial g_{ip}}{\partial t} + \frac{1}{n-1}\left(g^{rs}\frac{\partial g_{rs}}{\partial t}\right)^2 g_{ip} \\
& \left. +\frac{1}{n-1}\left(\frac{\partial g^{rs}}{\partial t}\frac{\partial g_{rs}}{\partial t}\right)g_{ip}\right] \\
= \;& 2w - 4g^{ik}g^{jl}\frac{\partial g_{ij}}{\partial t}R_{kl} - \left(4 + \frac{2}{n-1}\right)uv - 2dv + \frac{2}{n-1}u^3.
\end{aligned} \tag{2.8}$$

THEOREM 2.1. *For the dissipative hyperbolic geometric flow (2.1), the quantities $u(x,t)$, $v(x,t)$ and $w(x,t)$ satisfy the following equations*

$$\frac{\partial u(x,t)}{\partial t} = -2R - \frac{n-2}{n-1}u^2 - du - \frac{1}{n-1}v \tag{2.9}$$

and

$$\frac{\partial v(x,t)}{\partial t} = 2w - 4g^{ik}g^{jl}\frac{\partial g^{ij}}{\partial t}R_{kl} - (4 + \frac{2}{n-1})uv - 2dv + \frac{2}{n-1}u^3. \tag{2.10}$$

In order to understand basically the dissipative hyperbolic geometric flow, in what follows we construct some exact solutions.

Consider the following Cauchy problem

$$\begin{cases}
\frac{\partial^2 g_{ij}}{\partial t^2} = & -2R_{ij} + 2g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} - \left(d + 2g^{pq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial g_{ij}}{\partial t} \\
& +\frac{1}{n-1}\left[\left(g^{pq}\frac{\partial g_{pq}}{\partial t}\right)^2 + \frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right]g_{ij}, \\
g_{ij}(x,0) & = g^0_{ij}(x), \quad \frac{\partial g_{ij}}{\partial t}(x,0) = k^0_{ij}(x),
\end{cases} \tag{2.11}$$

where $g^0_{ij}(x)$ is a Riemannian metric on the manifold $\mathscr{M}$, and $k^0_{ij}(x)$ is a symmetric tensor on $\mathscr{M}$.

If we assume that the initial metric $g^0_{ij}(x)$ is Ricci flat, and the initial velocity $k^0_{ij}(x)$ vanishes, then easily see that $g_{ij}(x,t) = g^0_{ij}(x)$ is the unique smooth solution to the Cauchy problem (2.11).

If we assume that the initial Riemannian metric is Einstein, that is to say,

$$R_{ij}(x,0) = \lambda g_{ij}(x,0), \quad \forall\, x \in \mathscr{M}, \tag{2.12}$$

where λ is a constant. Furthermore, we suppose that

$$\frac{\partial g_{ij}}{\partial t}(x,0) = \mu g_{ij}(x,0), \tag{2.13}$$

where μ is an another constant. Let

$$g_{ij}(x,t) = \rho(t) g_{ij}(x,0). \tag{2.14}$$

By the definition of the Ricci tensor, we have

$$R_{ij}(x,t) = R_{ij}(x,0) = \lambda g_{ij}(x,0), \quad \forall\, x \in \mathscr{M}. \tag{2.15}$$

It follows from (2.13) and (2.14) that

$$\rho(0) = 1, \quad \rho'(0) = \mu. \tag{2.16}$$

Substituting (2.14) into the evolution equation (2.1) gives the following ODE

$$\rho''(t) = -d\rho'(t) - 2\lambda. \tag{2.17}$$

The solution of (2.17) with the initial data (2.16) reads

$$\rho(t) = 1 - \frac{2\lambda}{d}t - \left(\frac{\mu}{d} + \frac{2\lambda}{d^2}\right)(e^{-dt} - 1). \tag{2.18}$$

It follows from (2.18) that

$$\rho'(t) = -\frac{2\lambda}{d} + \left(\mu + \frac{2\lambda}{d}\right)e^{-dt}. \tag{2.19}$$

Noting that $d > 0$, we distinguish the following three cases to discuss:

Case I. $\lambda > 0$.

In this case, it follows from (2.18) that

$$\lim_{t\to+\infty} \rho(t) = -\infty.$$

Thus the evolving metric $g_{ij}(x,t)$ shrinks homothetically to a point as t approaches some finite time T.

Case II. $\lambda = 0$.

In the present situation, $\rho(t) = 1 - \frac{\mu}{d}(e^{-dt} - 1)$. If $\frac{\mu}{d} < -1$, then the evolving metric $g_{ij}(x,t)$ shrinks homothetically to a point as t approaches the time $T \triangleq -\frac{1}{d}\ln(1 + \frac{d}{\mu})$; If $\frac{\mu}{d} > -1$, then the metric $g_{ij}(x,t)$ evolves smoothly and is positive defined for all time; If $\frac{\mu}{d} = -1$, the metric $g_{ij}(x,t)$ evolves smoothly and is positive defined for all time, but it shrinks homothetically to a point as $t \to +\infty$.

Case III. $\lambda < 0$.

In this case, if $\mu < 0$ and $\rho(T_0) \le 0$, where $T_0 \triangleq -\frac{1}{d}\ln\left(\frac{2\lambda}{2\lambda+d\mu}\right)$, then the evolving metric $g_{ij}(x,t)$ shrinks homothetically to a point as t approaches some finite time not later than T. Otherwise, $g_{ij}(x,t)$ is smooth and positive defined for all time.

Summarizing the above argument leads to the following theorem.

THEOREM 2.2. *For the Cauchy problem (2.11) of the dissipative hyperbolic geometric flow, suppose that the assumptions (2.12)-(2.13) are satisfied. Then, if one of the following conditions is satisfied, then the evolving metric $g_{ij}(x,t)$ shrinks homothetically to a point as t approaches some finite time:*

(a) $\lambda > 0$;

(b) $\lambda = 0$ *and* $\mu < -d$;

(c) $\lambda < 0$, $\mu < 0$ *and* $\rho\left(\frac{1}{d}\ln\left(\frac{2\lambda}{2\lambda+d\mu}\right)\right) \ge 0$.

For the other instances, $g_{ij}(x,t)$ are smooth and positive defined for all time. In addition, if $\lambda = 0$ and $\mu = -d < 0$, the metric $g_{ij}(x,t)$ evolves smoothly and is positively defined for all time, but it shrinks homothetically to a point as $t \to +\infty$.

3. Hyperbolic Ricci soliton. The theory of soliton solutions plays an important role in the study of geometric analysis, in particular in the study of Ricci flow. In this section we first introduce a new concept — steady hyperbolic Ricci soliton for the flow (2.1), and then describe its properties.

DEFINITION 3.1. *A solution to an evolution equation is called a steady soliton, if it evolves under a one-parameter subgroup of the symmetry group of the equation; A solution to the dissipative hyperbolic geometric flow (2.1) is called a steady hyperbolic Ricci soliton, if it moves by a one-parameter subgroup of the symmetry group of the equation (2.1).*

If φ_t is a one-parameter group of diffeomorphisms generated by a vector field V on $\mathscr{M}$, then the hyperbolic Ricci soliton is given by

$$g_{ij}(x,t) = \varphi_t^* g_{ij}(x,0) = g_{ij}(\varphi_t(x),0). \tag{3.1}$$

It implies that

$$\frac{\partial}{\partial t} g_{ij}(x,t) = \mathfrak{L}_V g_{ij} = g_{ik}\nabla_j V^k + g_{jk}\nabla_i V^k \triangleq T_{ij} \tag{3.2}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t^2} g_{ij}(x,t) &= \mathfrak{L}_V \mathfrak{L}_V g_{ij} = \mathfrak{L}_V T_{ij} \\
&= T_{ij;k}V^k + T_{kj}V^k_{;i} + T_{ki}V^k_{;j} \\
&= (g_{ip}\nabla_j V^p + g_{jp}\nabla_i V^p)_{;k}V^k + (g_{kp}\nabla_j V^p + g_{jp}\nabla_k V^p)V^k_{;i} \\
&\quad + (g_{kp}\nabla_i V^p + g_{ip}\nabla_k V^p)V^k_{;j} \\
&= (g_{ip}\nabla_k\nabla_j V^p + g_{jp}\nabla_k\nabla_i V^p)\, V^k + g_{kp}(\nabla_i V^k \cdot \nabla_j V^p + \nabla_j V^k \cdot \nabla_i V^p) \\
&\quad + g_{ip}\nabla_j V^k \cdot \nabla_k V^p + g_{jp}\nabla_i V^k \cdot \nabla_k V^p,
\end{aligned} \tag{3.3}$$

where $\mathfrak{L}_V$ stands for the Lie derivative with respect to the vector field V. Thus, the equation (2.1) can be reduced to

$$\begin{aligned}
&(g_{ip}\nabla_k\nabla_j V^p + g_{jp}\nabla_k\nabla_i V^p)\, V^k + g_{kp}(\nabla_i V^k \cdot \nabla_j V^p + \nabla_j V^k \cdot \nabla_i V^p)\\
&+g_{ip}\nabla_j V^k \cdot \nabla_k V^p + g_{jp}\nabla_i V^k \cdot \nabla_k V^p\\
=&-2R_{ij} + 2g^{pq}(g_{ik}\nabla_p V^k + g_{pk}\nabla_i V^k)(g_{jl}\nabla_q V^l + g_{ql}\nabla_j V^l)\\
&-2g^{pq}(g_{pk}\nabla_q V^k + g_{qk}\nabla_p V^k)(g_{il}\nabla_j V^l + g_{jl}\nabla_i V^l) - d(g_{ik}\nabla_j V^k + g_{jk}\nabla_i V^k)\\
&+\frac{1}{n-1}\left[g^{pq}(g_{pk}\nabla_q V^k + g_{qk}\nabla_p V^k)\right]^2 g_{ij}\\
&-\frac{1}{n-1}\left[g^{pr}g^{qs}(g_{pk}\nabla_q V^k + g_{qk}\nabla_p V^k)(g_{rl}\nabla_s V^l + g_{sl}\nabla_r V^l)\right] g_{ij}.
\end{aligned} \tag{3.4}$$

We predigest it into the following

$$\begin{aligned}
&2R_{ij} + (g_{ip}\nabla_k\nabla_j V^p + g_{jp}\nabla_k\nabla_i V^p)\, V^k\\
=&\, 2g^{pq}g_{ik}g_{jl}\nabla_p V^k\nabla_q V^l + g_{ik}\nabla_j V^l\nabla_l V^k + g_{jk}\nabla_i V^l\nabla_l V^k\\
&-(d+4\nabla_k V^k)(g_{il}\nabla_j V^l + g_{jl}\nabla_i V^l) + \frac{4}{n-1}(\nabla_q V^q)^2 g_{ij}\\
&-\frac{2}{n-1}\left(g_{kl}g^{pq}\nabla_p V^k\nabla_q V^l + \nabla_p V^q\nabla_q V^p\right) g_{ij}.
\end{aligned} \tag{3.5}$$

If the vector field V is the gradient of a function f on $\mathscr{M}$, then the soliton is called **a steady gradient hyperbolic Ricci soliton**. In what follows, we consider the steady gradient hyperbolic Ricci soliton.

For the steady gradient hyperbolic Ricci soliton, the equation (3.5) becomes

$$\begin{aligned}
&2R_{ij} + (g_{ip}\nabla_k\nabla_j\nabla^p f + g_{jp}\nabla_k\nabla_i\nabla^p f)\,\nabla^k f\\
=&\, 2g^{pq}g_{ik}g_{jl}\nabla_p\nabla^k f\nabla_q\nabla^l f + g_{ik}\nabla_j\nabla^l f\nabla_l\nabla^k f + g_{jk}\nabla_i\nabla^l f\nabla_l\nabla^k f\\
&-(d+4\nabla_k\nabla^k f)(g_{il}\nabla_j\nabla^l f + g_{jl}\nabla_i\nabla^l f) + \frac{4}{n-1}(\nabla_q\nabla^q f)^2 g_{ij}\\
&-\frac{2}{n-1}\left(g_{kl}g^{pq}\nabla_p\nabla^k f\nabla_q\nabla^l f + \nabla_p\nabla^q f\nabla_q\nabla^p f\right) g_{ij}.
\end{aligned}$$

That is to say,

$$\begin{aligned}
&R_{ij} + \nabla_k(\nabla_i\nabla_j f)\nabla^k f\\
=&\, 2g^{pq}\nabla_p\nabla_i f\nabla_q\nabla_j f - (d+4\triangle f)\nabla_i\nabla_j f\\
&+\frac{2}{n-1}(\triangle f)^2 g_{ij} - \frac{2}{n-1}(g^{pq}g^{kl}\nabla_p\nabla_k f\nabla_q\nabla_l f)g_{ij}.
\end{aligned} \tag{3.6}$$

Taking the trace on i and j yields

$$R + \nabla_k(\triangle f \cdot \nabla^k f) = -\frac{2}{n-1}|\nabla^2 f|^2 - \frac{n-3}{n-1}(\triangle f)^2 - d\cdot\triangle f. \tag{3.7}$$

Thus, the following theorem comes easily from (3.5)-(3.7).

Theorem 3.1. *For the dissipative hyperbolic geometric flow, (3.5) and (3.6) are the evolution equations satisfied by the steady hyperbolic Ricci soliton and the steady gradient hyperbolic Ricci soliton, respectively. Furthermore, for an n-dimensional* *compact manifold with $n \geq 3$, if the average scalar curvature of the initial metric is non-negative, i.e.,*

$$r(0) \triangleq \frac{\int_{\mathscr{M}} R(x,0)dV}{\int_{\mathscr{M}} dV} \geq 0, \tag{3.8}$$

then for the steady gradient hyperbolic Ricci soliton, the generating function f must satisfy the condition $Hess(f) \equiv 0$ on $\mathscr{M}$, i.e., f is a constant and the solution metric $g_{ij}(x,t) \equiv g_{ij}(x,0)$ is Ricci flat for all time t. In reverse, if the initial metric $g_{ij}(x,0)$ is Ricci flat and the function $f \equiv$ constant, then it is obvious that the steady gradient hyperbolic Ricci soliton generated by f is a solution to the dissipative hyperbolic geometric flow.

4. Short-time existence and uniqueness. In this section, we reduce the dissipative hyperbolic geometric flow (2.1) to a symmetric hyperbolic system in the so-called harmonic coordinates (see [4]), then based on this, we prove the short-time existence and uniqueness theorem for the flow equation (2.1).

Let $g_{ij}(x,t)$ be a family of metrics on an $n > 1$ dimensional manifold $\mathscr{M}$. We consider the space-time $\mathbb{R} \times \mathscr{M}$ equipped with the following Lorentzian metric

$$ds^2 = -dt^2 + g_{ij}(x,t)dx^i dx^j. \tag{4.1}$$

It follows from (3.4) in Dai, Kong and Liu [2] that

$$\begin{aligned}\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} = & \frac{\partial^2 g_{ij}}{\partial t^2} - g^{kl}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \left(g_{ik}\frac{\partial \Gamma^k}{\partial x^j} + g_{jk}\frac{\partial \Gamma^k}{\partial x^i}\right) \\ & + 2g^{kl}g_{pq}\Gamma^p_{ik}\Gamma^q_{jl} + \frac{\partial g_{ij}}{\partial x_k}\Gamma^k \\ & + \left(g_{ik}\Gamma^k_{rs}g^{pr}g^{qs}\frac{\partial g_{pq}}{\partial x^j} + g_{jk}\Gamma^k_{rs}g^{pr}g^{qs}\frac{\partial g_{pq}}{\partial x^i}\right),\end{aligned} \tag{4.2}$$

where

$$\Gamma^k \triangleq g^{ij}\Gamma^k_{ij}. \tag{4.3}$$

Then the evolution equation (2.1) for the dissipative hyperbolic geometric flow can be reduced to the following

$$\begin{aligned}\frac{\partial^2 g_{ij}}{\partial t^2} - g^{kl}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} = & -\left(g_{ik}\frac{\partial \Gamma^k}{\partial x^j} + g_{jk}\frac{\partial \Gamma^k}{\partial x^i}\right) - 2g^{kl}g_{pq}\Gamma^p_{ik}\Gamma^q_{jl} - \frac{\partial g_{ij}}{\partial x_k}\Gamma^k \\ & - \left(g_{ik}\Gamma^k_{rs}g^{pr}g^{qs}\frac{\partial g_{pq}}{\partial x^j} + g_{jk}\Gamma^k_{rs}g^{pr}g^{qs}\frac{\partial g_{pq}}{\partial x^i}\right) \\ & + 2g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} - 2g^{pq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{ij}}{\partial t} - d\frac{\partial g_{ij}}{\partial t} \\ & + \frac{1}{n-1}(g^{pq}\frac{\partial g_{pq}}{\partial t})^2 g_{ij} + \frac{1}{n-1}(\frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t})g_{ij}.\end{aligned} \tag{4.4}$$

Similar to [4], we make use of the harmonic coordinates such that, for fixed time t, it holds that

$$\Gamma^k(x,t) \triangleq g^{ij}\Gamma^k_{ij} \equiv 0, \text{ when } x \text{ is in an open neighborhood of point } p \in \mathscr{M}. \tag{4.5}$$

Then the equation (4.4) can be written as

$$\frac{\partial^2 g_{ij}}{\partial t^2} = g^{kl}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \widetilde{H_{ij}}(g_{kl}, \frac{\partial g_{kl}}{\partial t}, \frac{\partial g_{kl}}{\partial x^p}), \tag{4.6}$$

where

$$\begin{aligned}\widetilde{H_{ij}}(g_{kl}, \frac{\partial g_{kl}}{\partial t}, \frac{\partial g_{kl}}{\partial x^p}) = & -2g^{kl}g_{pq}\Gamma^p_{ik}\Gamma^q_{jl} - \left(g_{ik}\Gamma^k_{rs}g^{pr}g^{qs}\frac{\partial g_{pq}}{\partial x^j} + g_{jk}\Gamma^k_{rs}g^{pr}g^{qs}\frac{\partial g_{pq}}{\partial x^i}\right)\\ & +2g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} - 2g^{pq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{ij}}{\partial t} - d\frac{\partial g_{ij}}{\partial t}\\ & +\frac{1}{n-1}(g^{pq}\frac{\partial g_{pq}}{\partial t})^2 g_{ij} + \frac{1}{n-1}(\frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t})g_{ij}\end{aligned} \tag{4.7}$$

are homogenous quadratic with respect to $\frac{\partial g_{kl}}{\partial x^p}$ and $\frac{\partial g_{kl}}{\partial t}$ except the dissipative term $d\frac{\partial g_{ij}}{\partial t}$ and rational with respect to g_{kl} with non-zero denominator $det(g_{ij}) \neq 0$. By introducing the new unknowns g_{ij}, $h_{ij} = \frac{\partial g_{ij}}{\partial t}$, $g_{ij,k} = \frac{\partial g_{kl}}{\partial x^k}$, the system (4.6) can be transformed into a system of partial differential equations of first order

$$\begin{cases}\dfrac{\partial g_{ij}}{\partial t} = h_{ij},\\ g^{kl}\dfrac{\partial g_{ij,k}}{\partial t} = g^{kl}\dfrac{\partial h_{ij}}{\partial x^k},\\ \dfrac{\partial h_{ij}}{\partial t} = g^{kl}\dfrac{\partial g_{ij,k}}{\partial x^l} + \widetilde{H}_{ij}.\end{cases} \tag{4.8}$$

In the C^2 class, the system (4.8) is equivalent to (4.6). It is easy to see that (4.8) is a quasilinear symmetric hyperbolic system, which can be rewritten as

$$A^0(u)\frac{\partial u}{\partial t} = A^j(u)\frac{\partial u}{\partial x^j} + B(u), \tag{4.9}$$

where $u = (g_{ij}, g_{ij,k}, h_{ij})^T$ is the $\frac{1}{2}n(n+1)(n+2)$-dimensional unknown vector function and the coefficient matrices A^0, A^j, B are given by

$$A^0(u) = A^0(g_{ij}, g_{ij,k}, h_{ij}) = \begin{pmatrix} I & 0 & 0 & \cdots & 0 & 0\\ 0 & g^{11}I & g^{12}I & \cdots & g^{1n}I & 0\\ 0 & g^{21}I & g^{22}I & \cdots & g^{2n}I & 0\\ \vdots & \cdots & & & & \\ 0 & g^{n1}I & g^{n2}I & \cdots & g^{nn}I & 0\\ 0 & 0 & 0 & \cdots & 0 & I\end{pmatrix},$$

$$A^j(u) = A^j(g_{kl}, g_{kl,p}, h_{kl}) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & g^{j1}I\\ 0 & 0 & 0 & \cdots & 0 & g^{j2}I\\ \cdots & \cdots & \cdots & & & \\ 0 & 0 & 0 & \cdots & 0 & g^{jn}I\\ 0 & g^{1j}I & g^{2j}I & \cdots & g^{nj}I & 0\end{pmatrix},$$

where 0 is the $\left(\frac{1}{2}n(n+1)\right) \times \left(\frac{1}{2}n(n+1)\right)$ zero matrix, I is the $\left(\frac{1}{2}n(n+1)\right) \times \left(\frac{1}{2}n(n+1)\right)$ identity matrix,

$$B(u) = B(g_{ij}, g_{ij,p}, h_{ij}) = \begin{pmatrix} h_{ij} \\ 0 \\ \widetilde{H}_{ij} \end{pmatrix},$$

in which 0 is the $\frac{1}{2}n^2(n+1)$-dimensional zero vector.

By the theory of the symmetric hyperbolic system ([6], [7]), we can obtain the following theorem.

THEOREM 4.1. *Let $(\mathscr{M}, g^0_{ij}(x))$ be an n-dimensional compact Riemannian manifold. Then there exists a constant $\eta > 0$ such that the Cauchy problem (2.11) has a unique smooth solution $g_{ij}(x,t)$ on $\mathscr{M} \times [0, \eta]$.*

REMARK 4.1. *Theorem 4.1 can also be proved in a manner similar to that in DeTurck (see [3], [1], [2]).*

5. Nonlinear stability of Euclidean metrics. This section is devoted to the nonlinear stability of the dissipative hyperbolic geometric flow (2.1) defined on the Euclidean space with the dimension larger than two.

We consider the following Cauchy problem for the dissipative hyperbolic geometric flow (2.1),

$$\begin{cases} \dfrac{\partial^2 g_{ij}}{\partial t^2} & = -2R_{ij} + 2g^{pq}\dfrac{\partial g_{ip}}{\partial t}\dfrac{\partial g_{jq}}{\partial t} - \left(d + 2g^{pq}\dfrac{\partial g_{pq}}{\partial t}\right)\dfrac{\partial g_{ij}}{\partial t} \\ & \quad + \dfrac{1}{n-1}\left[\left(g^{pq}\dfrac{\partial g_{pq}}{\partial t}\right)^2 + \dfrac{\partial g^{pq}}{\partial t}\dfrac{\partial g_{pq}}{\partial t}\right] g_{ij}, \\ g_{ij}(x,0) & = \delta_{ij} + \epsilon g^0_{ij}(x), \ \dfrac{\partial g_{ij}}{\partial t}(x,0) = \epsilon g^1_{ij}(x), \end{cases} \tag{5.1}$$

where $g^0_{ij}(x)$ and $g^1_{ij}(x)$ are given symmetric tensors defined on the Euclidean space $\mathbb{R}^n$.

THEOREM 5.1. *The flat metric $g_{ij} = \delta_{ij}$ on the Euclidean space $\mathbb{R}^n$ with $n \geq 3$ is globally nonlinearly stable with respect to the given tensor $\left(g^0_{ij}(x), g^1_{ij}(x)\right) \in C^\infty_0(\mathbb{R}^n)$, i.e., there exists a positive constant $\epsilon_0 = \epsilon_0\left(g^0_{ij}(x), g^1_{ij}(x)\right) > 0$ such that, for any $\epsilon \in (0, \epsilon_0]$, the initial value problem (4.1) admits a unique smooth solution $g_{ij}(x,t)$ for all time $t \geq 0$.*

REMARK 5.1. *For the standard hyperbolic geometric flow (1.2), we can only obtain the nonlinear stability of the Euclidean space $\mathbb{R}^n$ with $n \geq 5$ (see [2]). Under suitable assumptions, similar results are true for general hyperbolic geometric flow (1.1).*

Proof of Theorem 5.1. Let the symmetric tensor h_{ij} on $\mathbb{R}^n$ defined by

$$h_{ij}(x,t) = g_{ij}(x,t) - \delta_{ij} \tag{5.2}$$

and δ^{ij} be the inverse of δ_{ij}. Then for small h,

$$H^{ij} \triangleq g^{ij} - \delta^{ij} = -h^{ij} + O^{ij}(h^2), \tag{5.3}$$

where $h^{ij} = \delta^{ik}\delta^{jl}h_{kl}$ and $O^{ij}(h^2)$ vanishes to the second order at $h = 0$. Then the Cauchy problem (5.1) for the metric $g_{ij}(x,t)$ is equivalent to the following initial value problem for the tensor $h_{ij}(x,t)$ in the harmonic coordinates x^i around the origin in $\mathbb{R}^n$

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}h_{ij}(x,t) = (\delta^{kl} + H^{kl})\dfrac{\partial^2 h_{ij}}{\partial x^k x^l} + \widetilde{H_{ij}}(\delta_{kl} + h_{kl}, \dfrac{\partial h_{kl}}{\partial t}, \dfrac{\partial h_{kl}}{\partial x^p}), \\ t = 0 : h_{ij}(x,0) = \epsilon g^0_{ij}(x), \ \dfrac{\partial h_{ij}}{\partial t}(x,0) = \epsilon g^1_{ij}(x), \end{cases} \tag{5.4}$$

where $\widetilde{H_{ij}}(\delta_{kl} + h_{kl}, \frac{\partial h_{kl}}{\partial t}, \frac{\partial h_{kl}}{\partial x^p})$ is defined in (4.7). Thus, the Cauchy problem (5.4) can be reduced to the following

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}h_{ij}(x,t) - \delta^{kl}\dfrac{\partial^2 h_{ij}}{\partial x^k x^l} + d\dfrac{\partial h_{ij}}{\partial t} = \bar{H}_{ij}(\delta_{kl} + h_{kl}, \dfrac{\partial h_{kl}}{\partial t}, \dfrac{\partial h_{kl}}{\partial x^p}), \\ t = 0 : h_{ij}(x,0) = \epsilon g^0_{ij}(x), \ \dfrac{\partial g_{ij}}{\partial t}(x,0) = \epsilon g^1_{ij}(x), \end{cases} \tag{5.5}$$

where

$$\bar{H}_{ij}(\delta_{kl} + h_{kl}, \frac{\partial h_{kl}}{\partial t}, \frac{\partial h_{kl}}{\partial x^p}) = H^{kl}\frac{\partial^2 h_{ij}}{\partial x^k x^l} + d\frac{\partial h_{ij}}{\partial t} + \widetilde{H_{ij}}(\delta_{kl} + h_{kl}, \frac{\partial h_{kl}}{\partial t}, \frac{\partial h_{kl}}{\partial x^p}). \tag{5.6}$$

By the definition (4.7) and (5.2)-(5.3), we have

$$
\begin{aligned}
&\bar{H}_{ij}(\delta_{kl}+h_{kl},\frac{\partial h_{kl}}{\partial t},\frac{\partial h_{kl}}{\partial x^p})\\
=&H^{kl}\frac{\partial^2 h_{ij}}{\partial x^k x^l}\\
&-\frac{1}{2}(\delta^{kl}+H^{kl})(\delta_{pq}+h_{pq})(\delta^{pa}+H^{pa})(\delta^{qb}+H^{qb})\\
&\cdot\left(\frac{\partial h_{ai}}{\partial x^k}+\frac{\partial h_{ak}}{\partial x^i}-\frac{\partial h_{ik}}{\partial x^a}\right)\left(\frac{\partial h_{bj}}{\partial x^l}+\frac{\partial h_{bl}}{\partial x^j}-\frac{\partial h_{jl}}{\partial x^b}\right)\\
&-\frac{1}{2}(\delta_{ik}+h_{ik})(\delta^{pr}+H^{pr})(\delta^{qs}+H^{qs})(\delta^{ka}+H^{ka})\\
&\cdot\left(\frac{\partial h_{ar}}{\partial x^s}+\frac{\partial h_{as}}{\partial x^r}-\frac{\partial h_{rs}}{\partial x^a}\right)\left(\frac{\partial h_{pq}}{\partial x^j}\right)\\
&-\frac{1}{2}(\delta_{jk}+h_{jk})(\delta^{pr}+H^{pr})(\delta^{qs}+H^{qs})(\delta^{ka}+H^{ka})\\
&\cdot\left(\frac{\partial h_{ar}}{\partial x^s}+\frac{\partial h_{as}}{\partial x^r}-\frac{\partial h_{rs}}{\partial x^a}\right)\left(\frac{\partial h_{pq}}{\partial x^i}\right)\\
&+2(\delta^{pq}+H^{pq})\frac{\partial h_{ip}}{\partial t}\frac{\partial h_{jq}}{\partial t}-2(\delta^{pq}+H^{pq})\frac{\partial h_{pq}}{\partial t}\frac{\partial h_{ij}}{\partial t}\\
&+\frac{1}{n-1}\left((\delta^{pq}+H^{pq})\frac{\partial h_{pq}}{\partial t}\right)^2(\delta_{ij}+h_{ij})-\frac{1}{n-1}(\delta^{pa}+H^{pa})\\
&\cdot(\delta^{qb}+H^{qb})\frac{\partial h_{pq}}{\partial t}\frac{\partial h_{ab}}{\partial t}(\delta_{ij}+h_{ij})\\
=&-\frac{1}{2}\delta^{kl}\delta^{ab}\left(\frac{\partial h_{ai}}{\partial x^k}+\frac{\partial h_{ak}}{\partial x^i}-\frac{\partial h_{ik}}{\partial x^a}\right)\left(\frac{\partial h_{bj}}{\partial x^l}+\frac{\partial h_{bl}}{\partial x^j}-\frac{\partial h_{jl}}{\partial x^b}\right)\\
&-\frac{1}{2}\delta^{pr}\delta^{qs}\left(\frac{\partial h_{ir}}{\partial x^s}+\frac{\partial h_{is}}{\partial x^r}-\frac{\partial h_{rs}}{\partial x^i}\right)\left(\frac{\partial h_{pq}}{\partial x^j}\right)\\
&-\frac{1}{2}\delta^{pr}\delta^{qs}\left(\frac{\partial h_{jr}}{\partial x^s}+\frac{\partial h_{js}}{\partial x^r}-\frac{\partial h_{rs}}{\partial x^j}\right)\left(\frac{\partial h_{pq}}{\partial x^i}\right)\\
&+2\delta^{pq}\frac{\partial h_{ip}}{\partial t}\frac{\partial h_{jq}}{\partial t}-2\delta^{pq}\frac{\partial h_{pq}}{\partial t}\frac{\partial h_{ij}}{\partial t}\\
&+\frac{1}{n-1}\left(\delta^{pq}\frac{\partial h_{pq}}{\partial t}\right)^2\delta_{ij}-\frac{1}{n-1}\left(\delta^{pa}\delta^{qb}\frac{\partial h_{pq}}{\partial t}\frac{\partial h_{ab}}{\partial t}\right)\delta_{ij}\\
&-h^{kl}\frac{\partial^2 h_{ij}}{\partial x^k x^l}+O(||h_{kl}||+||Dh_{kl}||)^3\\
=&O(||h_{kl}||+||Dh_{kl}||+||\frac{\partial^2 h_{ij}}{\partial x^k x^l}||)^2+O(||h_{kl}||+||Dh_{kl}||)^3,
\end{aligned}
\tag{5.7}
$$

where

$$
Dh_{kl}\triangleq(\frac{\partial h_{kl}}{\partial t},\frac{\partial h_{kl}}{\partial x^p})
$$

and $||\cdot||$ stands for the norm with respect to the flat metric δ_{ij}.

By the theory of dissipative wave equations (see [14], [15]), we know that, for sufficiently small $\epsilon>0$, the Cauchy problem (5.5), i.e., (5.1), admits a unique smooth solution for all $t\geq 0$ on $\mathbb{R}^n$ with $n\geq 3$. The proof of Theorem 5.1 is completed. □

6. Wave character of curvatures — Derivation of dissipative hyperbolic geometric flow. In this section, we will illustrate why we choose (2.1) as the equation of the dissipative hyperbolic geometric flow. Based on this we derive the nonlinear wave equations satisfied by the curvatures. The results presented in this section show the wave character of curvatures.

We first assume that the metrics on a manifold $\mathscr{M}$ evolve by the following equation

$$\frac{\partial^2 g_{ij}}{\partial t^2}(x,t) = -2R_{ij}(x,t) + G_{ij}(x,t), \tag{6.1}$$

where

$$\begin{aligned} G_{ij}(x,t) = a g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} + b g^{pq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{ij}}{\partial t} + d\frac{\partial g_{ij}}{\partial t} + e g^{pq}\frac{\partial g_{pq}}{\partial t} g_{ij} + \\ f(g^{pq}\frac{\partial g_{pq}}{\partial t})^2 g_{ij} + h\left(\frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right) g_{ij} \end{aligned} \tag{6.2}$$

and the terms $a,\ b,\ d,\ e,\ f,\ h$ are all constants determined below. Then direct calculation gives

$$\begin{aligned} \frac{\partial^2 R}{\partial t^2} &= g^{ik}\left(g^{jl}\frac{\partial^2 R_{ijkl}}{\partial t^2} + 2\frac{\partial g^{jl}}{\partial t}\frac{\partial R_{ijkl}}{\partial t} + R_{ijkl}\frac{\partial^2 g^{jl}}{\partial t^2}\right) + 2\frac{\partial g^{ik}}{\partial t}\frac{\partial R_{ik}}{\partial t} + R_{ik}\frac{\partial^2 g^{ik}}{\partial t^2} \\ &= g^{ik}g^{jl}\frac{\partial^2 R_{ijkl}}{\partial t^2} + 2\frac{\partial g^{jl}}{\partial t}(\frac{\partial R_{jl}}{\partial t} - \frac{\partial g^{ik}}{\partial t}R_{ijkl}) + 2\frac{\partial g^{ik}}{\partial t}\frac{\partial R_{ik}}{\partial t} + 2R_{ik}\frac{\partial^2 g^{ik}}{\partial t^2} \\ &= g^{ik}g^{jl}\frac{\partial^2 R_{ijkl}}{\partial t^2} + 4\frac{\partial g^{ik}}{\partial t}\frac{\partial R_{ik}}{\partial t} - 2\frac{\partial g^{jl}}{\partial t}\frac{\partial g^{ik}}{\partial t}R_{ijkl} \\ &\quad +2R_{ik}\left(-g^{ir}g^{ks}\frac{\partial^2 g_{rs}}{\partial t^2} + 2g^{ir}g^{ks}g^{pq}\frac{\partial g_{pr}}{\partial t}\frac{\partial g_{qs}}{\partial t}\right) \\ &= g^{ik}g^{jl}\frac{\partial^2 R_{ijkl}}{\partial t^2} + 4\frac{\partial g^{ik}}{\partial t}\frac{\partial R_{ik}}{\partial t} - 2\frac{\partial g^{jl}}{\partial t}\frac{\partial g^{ik}}{\partial t}R_{ijkl} + \\ &\quad 4g^{ir}g^{ks}g^{pq}\frac{\partial g_{pr}}{\partial t}\frac{\partial g_{qs}}{\partial t}R_{ik} + 4|Ric|^2 - 2g^{ir}g^{ks}R_{ik}G_{rs}, \end{aligned} \tag{6.3}$$

where $|Ric|^2 = g^{ik}g^{jl}R_{ij}R_{kl}$ is the norm of Ricci curvature tensor $Ric = R_{ik}$. In (6.3), we have made use of the evolution equation (6.1).

We choose the normal coordinates around a fixed point p on the manifold $\mathscr{M}$ such that $\Gamma^k_{ij}(p) = 0$. By the computations (5.2)-(5.5) in [2], we get

$$\begin{aligned} \frac{\partial^2}{\partial t^2}R_{ijkl} = &\frac{1}{2}\left[\frac{\partial^2}{\partial x^i\partial x^l}\left(\frac{\partial^2 g_{kj}}{\partial t^2}\right) + \frac{\partial^2}{\partial x^i\partial x^j}\left(\frac{\partial^2 g_{kl}}{\partial t^2}\right) - \frac{\partial^2}{\partial x^i\partial x^k}\left(\frac{\partial^2 g_{jl}}{\partial t^2}\right)\right] \\ &-\frac{1}{2}\left[\frac{\partial^2}{\partial x^j\partial x^l}\left(\frac{\partial^2 g_{ki}}{\partial t^2}\right) + \frac{\partial^2}{\partial x^j\partial x^i}\left(\frac{\partial^2 g_{kl}}{\partial t^2}\right) - \frac{\partial^2}{\partial x^j\partial x^k}\left(\frac{\partial^2 g_{il}}{\partial t^2}\right)\right] \\ &+2g_{pq}\left(\frac{\partial}{\partial t}\Gamma^p_{il}\cdot\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\cdot\frac{\partial}{\partial t}\Gamma^q_{ik}\right). \end{aligned} \tag{6.4}$$

Then it follows from (6.1) and (6.4) that

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}R_{ijkl} = & \frac{1}{2}\left[\frac{\partial^2}{\partial x^i\partial x^l}(-2R_{kj}) + \frac{\partial^2}{\partial x^i\partial x^j}(-2R_{kl}) - \frac{\partial^2}{\partial x^i\partial x^k}(-2R_{jl})\right] \\
& -\frac{1}{2}\left[\frac{\partial^2}{\partial x^j\partial x^l}(-2R_{ki}) + \frac{\partial^2}{\partial x^j\partial x^i}(-2R_{kl}) - \frac{\partial^2}{\partial x^j\partial x^k}(-2R_{il})\right] \\
& +2g_{pq}\left(\frac{\partial}{\partial t}\Gamma^p_{il}\cdot\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\cdot\frac{\partial}{\partial t}\Gamma^q_{ik}\right) \\
& +\frac{1}{2}\left[\frac{\partial^2}{\partial x^i\partial x^l}G_{kj} + \frac{\partial^2}{\partial x^i\partial x^j}G_{kl} - \frac{\partial^2}{\partial x^i\partial x^k}G_{jl}\right] \\
& -\frac{1}{2}\left[\frac{\partial^2}{\partial x^j\partial x^l}G_{ki} + \frac{\partial^2}{\partial x^j\partial x^i}G_{kl} - \frac{\partial^2}{\partial x^j\partial x^k}G_{il}\right].
\end{aligned}
\tag{6.5}
$$

Similar to Hamilton [9], by Theorem 5.1 in [2] we have

$$
\begin{aligned}
& \frac{1}{2}\left[\frac{\partial^2}{\partial x^i\partial x^l}(-2R_{kj}) + \frac{\partial^2}{\partial x^i\partial x^j}(-2R_{kl}) - \frac{\partial^2}{\partial x^i\partial x^k}(-2R_{jl})\right] \\
& -\frac{1}{2}\left[\frac{\partial^2}{\partial x^j\partial x^l}(-2R_{ki}) + \frac{\partial^2}{\partial x^j\partial x^i}(-2R_{kl}) - \frac{\partial^2}{\partial x^j\partial x^k}(-2R_{il})\right] \\
& +2g_{pq}\left(\frac{\partial}{\partial t}\Gamma^p_{il}\cdot\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\cdot\frac{\partial}{\partial t}\Gamma^q_{ik}\right) \\
= & \triangle R_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}\right) \\
& -g^{pq}\left(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}\right) \\
& +2g_{pq}\left(\frac{\partial}{\partial t}\Gamma^p_{il}\cdot\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\cdot\frac{\partial}{\partial t}\Gamma^q_{ik}\right),
\end{aligned}
\tag{6.6}
$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$ and $\triangle$ is the Laplacian with respect to the evolving metric.

Combining (6.3), (6.5) and (6.6) and referring to the computations in Theorem

5.3 in [2] leads to

$$\begin{aligned}
\frac{\partial^2 R}{\partial t^2} &= \Delta R + 2|\mathrm{Ric}|^2 \\
&\quad +2g^{ik}g^{jl}g_{pq}\left(\frac{\partial}{\partial t}\Gamma^p_{il}\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\frac{\partial}{\partial t}\Gamma^q_{ik}\right) \\
&\quad -2g^{ik}g^{jp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial}{\partial t}R_{ijkl} \\
&\quad -2g^{ip}g^{kq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ik}}{\partial t} + 4R_{ik}g^{ip}g^{rq}g^{sk}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} \\
&\quad +\frac{1}{2}g^{ik}g^{jl}\left[\frac{\partial^2}{\partial x^i\partial x^l}G_{kj} + \frac{\partial^2}{\partial x^i\partial x^j}G_{kl} - \frac{\partial^2}{\partial x^i\partial x^k}G_{jl}\right] \\
&\quad -\frac{1}{2}g^{ik}g^{jl}\left[\frac{\partial^2}{\partial x^j\partial x^l}G_{ki} + \frac{\partial^2}{\partial x^j\partial x^i}G_{kl} - \frac{\partial^2}{\partial x^j\partial x^k}G_{il}\right] \\
&\quad -2g^{ir}g^{ks}R_{ik}G_{rs} \\
&= \Delta R + 2|\mathrm{Ric}|^2 \\
&\quad +2g^{ik}g^{jl}g_{pq}\left(\frac{\partial}{\partial t}\Gamma^p_{il}\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\frac{\partial}{\partial t}\Gamma^q_{ik}\right) \\
&\quad -2g^{ik}g^{jp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial}{\partial t}R_{ijkl} \\
&\quad -2g^{ip}g^{kq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ik}}{\partial t} + 4R_{ik}g^{ip}g^{rq}g^{sk}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} \\
&\quad +g^{ik}g^{jl}\left(\frac{\partial^2}{\partial x^i\partial x^l}G_{kj} - \frac{\partial^2}{\partial x^i\partial x^k}G_{jl}\right) - 2g^{ir}g^{ks}R_{ik}G_{rs}. \qquad (6.7)
\end{aligned}$$

In the normal coordinates, we have

$$\begin{aligned}
& g^{ik}g^{jl}\left(\frac{\partial^2}{\partial x^i\partial x^l}G_{kj} - \frac{\partial^2}{\partial x^i\partial x^k}G_{jl}\right) - 2g^{ir}g^{ks}R_{ik}G_{rs} \\
&= g^{ik}g^{jl}\left(\nabla_i\nabla_l G_{kj} + \nabla_i\Gamma^p_{lk}G_{pj} + \nabla_i\Gamma^p_{lj}G_{pk}\right) \\
&\quad -g^{ik}g^{jl}\left(\nabla_j\nabla_l G_{ki} + \nabla_j\Gamma^p_{lk}G_{pi} + \nabla_j\Gamma^p_{li}G_{pk}\right) - 2g^{ir}g^{ks}R_{ik}G_{rs} \\
&= g^{ik}g^{jl}(\nabla_i\nabla_l G_{kj} - \nabla_j\nabla_l G_{ki}) + g^{ik}g^{jl}\nabla_i\Gamma^p_{lk}G_{pj} - g^{ik}g^{jl}\nabla_j\Gamma^p_{lk}G_{pi} \\
&\quad +g^{ik}g^{jl}(\nabla_i\Gamma^p_{lj} - \nabla_j\Gamma^p_{li})G_{pk} - 2g^{ir}g^{ks}R_{ik}G_{rs} \\
&= g^{ik}g^{jl}(\nabla_i\nabla_l G_{kj} - \nabla_j\nabla_l G_{ki}) + g^{ik}g^{jl}R^p_{ijl}G_{pk} - 2g^{ir}g^{ks}R_{ik}G_{rs} \\
&= g^{ik}g^{jl}(\nabla_i\nabla_l G_{kj} - \nabla_j\nabla_l G_{ki}) - g^{ir}g^{ks}R_{ik}G_{rs}, \qquad (6.8)
\end{aligned}$$

where we have made use of the following equality in the normal coordinates

$$R^p_{ijl} = \frac{\partial\Gamma^p_{lj}}{\partial x^i} - \frac{\partial\Gamma^p_{li}}{\partial x^j} = \nabla_i\Gamma^p_{lj} - \nabla_j\Gamma^p_{li}$$

and ∇_l means the covariant derivative in the direction $\frac{\partial}{\partial x^l}$. In the normal coordinates, we easily obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 g_{ij}}{\partial x^k\partial x^l}\right) = \frac{\partial g_{ip}}{\partial t}\frac{\partial\Gamma^p_{lj}}{\partial x^k} + \frac{\partial g_{jp}}{\partial t}\frac{\partial\Gamma^p_{li}}{\partial x^k} + g_{ip}\frac{\partial}{\partial t}\frac{\partial\Gamma^p_{lj}}{\partial x^k} + g_{jp}\frac{\partial}{\partial t}\frac{\partial\Gamma^p_{li}}{\partial x^k}.$$

This implies that

$$
\begin{aligned}
\nabla_l(\frac{\partial g_{jp}}{\partial t}) &= \frac{\partial^2 g_{jp}}{\partial x^l\partial t} - \Gamma^q_{lj}\frac{\partial g_{pq}}{\partial t} - \Gamma^q_{lp}\frac{\partial g_{jq}}{\partial t} = \frac{\partial^2 g_{ip}}{\partial x^l\partial t},\\
\nabla_i\nabla_l(\frac{\partial g_{jp}}{\partial t}) &= \frac{\partial}{\partial x^i}(\frac{\partial^2 g_{jp}}{\partial x^l\partial t} - \Gamma^q_{lj}\frac{\partial g_{pq}}{\partial t} - \Gamma^q_{lp}\frac{\partial g_{jq}}{\partial t}) - \Gamma^r_{il}(\frac{\partial^2 g_{jp}}{\partial x^r\partial t} - \Gamma^q_{rj}\frac{\partial g_{pq}}{\partial t} - \Gamma^q_{rp}\frac{\partial g_{jq}}{\partial t})\\
&\quad -\Gamma^r_{ij}(\frac{\partial^2 g_{rp}}{\partial x^l\partial t} - \Gamma^q_{lr}\frac{\partial g_{pq}}{\partial t} - \Gamma^q_{lp}\frac{\partial g_{rq}}{\partial t}) - \Gamma^r_{ip}(\frac{\partial^2 g_{jr}}{\partial x^l\partial t} - \Gamma^q_{lj}\frac{\partial g_{rq}}{\partial t} - \Gamma^q_{rl}\frac{\partial g_{jq}}{\partial t})\\
&= \frac{\partial}{\partial t}(\frac{\partial^2 g_{ij}}{\partial x^k\partial x^l}) - \frac{\partial \Gamma^q_{lj}}{\partial x^i}\frac{\partial g_{pq}}{\partial t} - \frac{\partial \Gamma^q_{lp}}{\partial x^i}\frac{\partial g_{jq}}{\partial t}\\
&= g_{jr}\frac{\partial}{\partial t}\frac{\partial \Gamma^r_{lp}}{\partial x^i} + g_{pr}\frac{\partial}{\partial t}\frac{\partial \Gamma^r_{lj}}{\partial x^i}.
\end{aligned}
$$

By the direct computations, we have

$$
\begin{aligned}
& g^{ik}g^{jl}\nabla_i\nabla_l(g^{pq}\frac{\partial g_{jp}}{\partial t}\frac{\partial g_{kq}}{\partial t}) - g^{ik}g^{jl}\nabla_j\nabla_l(g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{kq}}{\partial t})\\
&= g^{ik}g^{jl}g^{pq}\nabla_i\nabla_l(\frac{\partial g_{jp}}{\partial t}\frac{\partial g_{kq}}{\partial t}) - g^{ik}g^{jl}g^{pq}\nabla_j\nabla_l(\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{kq}}{\partial t})\\
&= \Bigg(-2\frac{\partial g^{ij}}{\partial t}\frac{\partial R_{ij}}{\partial t} - 2\frac{\partial g^{ij}}{\partial t}\frac{\partial}{\partial t}\frac{\partial \Gamma^p_{jp}}{\partial x^i} - g_{kl}\frac{\partial g^{ik}}{\partial t}\frac{\partial g^{jl}}{\partial t}R_{ij}\\
&\quad +\frac{\partial g^{ik}}{\partial t}\frac{\partial g^{jl}}{\partial t}R_{ijkl} - 2g^{ik}g^{jl}\frac{\partial g_{ip}}{\partial t}\frac{\partial}{\partial t}\frac{\partial \Gamma^p_{kl}}{\partial x^j}\Bigg)\\
&\quad +g^{ik}g^{jl}g^{pq}\left(\frac{\partial^2 g_{jp}}{\partial t\partial x^i}\frac{\partial^2 g_{kq}}{\partial t\partial x^l} + \frac{\partial^2 g_{jp}}{\partial t\partial x^l}\frac{\partial^2 g_{kq}}{\partial t\partial x^i} - 2\frac{\partial^2 g_{ip}}{\partial t\partial x^j}\frac{\partial^2 g_{kq}}{\partial t\partial x^l}\right). \qquad (6.9)
\end{aligned}
$$

Analogously, we obtain

$$
\begin{aligned}
& g^{ik}g^{jl}\nabla_i\nabla_l(g^{pq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{jk}}{\partial t}) - g^{ik}g^{jl}\nabla_j\nabla_l(g^{pq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{ik}}{\partial t})\\
&= g^{ik}g^{jl}g^{pq}\nabla_i\nabla_l(\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{jk}}{\partial t}) - g^{ik}g^{jl}g^{pq}\nabla_j\nabla_l(\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{ik}}{\partial t})\\
&= (g^{pq}\frac{\partial g_{pq}}{\partial t})(\frac{\partial R}{\partial t} - \frac{\partial g^{jl}}{\partial t}R_{jl}) - 2(g^{pq}\frac{\partial g_{pq}}{\partial t})g^{ik}\frac{\partial}{\partial t}\frac{\partial \Gamma^r_{kr}}{\partial x^i} - 2\frac{\partial g^{ik}}{\partial t}\frac{\partial}{\partial t}\frac{\partial \Gamma^r_{kr}}{\partial x^i}\\
&\quad +g^{ik}g^{jl}g^{pq}\left(\frac{\partial^2 g_{pq}}{\partial t\partial x^i}\frac{\partial^2 g_{jk}}{\partial t\partial x^l} + \frac{\partial^2 g_{pq}}{\partial t\partial x^l}\frac{\partial^2 g_{jk}}{\partial t\partial x^i} - 2\frac{\partial^2 g_{pq}}{\partial t\partial x^j}\frac{\partial^2 g_{ik}}{\partial t\partial x^l}\right). \qquad (6.10)
\end{aligned}
$$

On the other hand, we can easily derive the following equations.

$$
\begin{aligned}
& g^{ik}g^{jl}\nabla_i\nabla_l(\frac{\partial g_{jk}}{\partial t}) - g^{ik}g^{jl}\nabla_j\nabla_l(\frac{\partial g_{ik}}{\partial t}) \\
= {} & g^{ik}g^{jl}\left(g_{jp}\frac{\partial}{\partial t}\frac{\partial \Gamma^p_{kl}}{\partial x^i} + g_{kp}\frac{\partial}{\partial t}\frac{\partial \Gamma^p_{jl}}{\partial x^i}\right) - g^{ik}g^{jl}\left(g_{ip}\frac{\partial}{\partial t}\frac{\partial \Gamma^p_{kl}}{\partial x^j} + g_{kp}\frac{\partial}{\partial t}\frac{\partial \Gamma^p_{il}}{\partial x^j}\right) \\
= {} & g^{jl}\frac{\partial}{\partial t}R_{jl}, \qquad (6.11) \\
& g^{ik}g^{jl}\nabla_i\nabla_l\left(g^{pq}\frac{\partial g_{pq}}{\partial t}g_{jk}\right) - g^{ik}g^{jl}\nabla_j\nabla_l\left(g^{pq}\frac{\partial g_{pq}}{\partial t}g_{ik}\right) \\
= {} & g^{ik}g^{jl}g^{pq}g_{kj}\nabla_i\nabla_l(\frac{\partial g_{pq}}{\partial t}) - g^{ik}g^{jl}g^{pq}g_{ik}\nabla_j\nabla_l(\frac{\partial g_{pq}}{\partial t}) \\
= {} & -2(n-1)g^{ik}\frac{\partial}{\partial t}\frac{\partial \Gamma^l_{kl}}{\partial x^i}, \qquad (6.12) \\
& g^{ik}g^{jl}\nabla_i\nabla_l\left((g^{pq}\frac{\partial g_{pq}}{\partial t})^2 g_{jk}\right) - g^{ik}g^{jl}\nabla_j\nabla_l\left((g^{pq}\frac{\partial g_{pq}}{\partial t})^2 g_{ik}\right) \\
= {} & (1-n)g^{jl}g^{pq}g^{rs}\nabla_j\nabla_l\left(\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}\right) \\
= {} & -(n-1)\left[4(g^{pq}\frac{\partial g_{pq}}{\partial t})g^{jl}\frac{\partial}{\partial t}\frac{\partial \Gamma^s_{ls}}{\partial x^j} + 2g^{jl}g^{pq}g^{rs}\frac{\partial^2 g_{pq}}{\partial t\partial x^j}\frac{\partial^2 g_{rs}}{\partial t\partial x^l}\right] \qquad (6.13)
\end{aligned}
$$

and

$$
\begin{aligned}
& g^{ik}g^{jl}\nabla_i\nabla_l(\frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}g_{jk}) - g^{ik}g^{jl}\nabla_j\nabla_l(\frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}g_{ik}) \\
= {} & (n-1)g^{jl}g^{pr}g^{qs}\nabla_j\nabla_l\left(\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}\right) \\
= {} & 4(n-1)g^{ik}g^{jl}\frac{\partial g_{ip}}{\partial t}\frac{\partial}{\partial t}\frac{\partial \Gamma^p_{kl}}{\partial x^j} + 2(n-1)g^{jl}g^{pq}g^{rs}\frac{\partial^2 g_{pr}}{\partial t\partial x^j}\frac{\partial^2 g_{qs}}{\partial t\partial x^l}. \qquad (6.14)
\end{aligned}
$$

It follows from (6.7)-(6.14) that

$$\begin{aligned}
\frac{\partial^2 R}{\partial t^2} &= \Delta R + 2|Ric|^2 \\
&+ \left[4\frac{\partial g^{ik}}{\partial t}\frac{\partial R_{ik}}{\partial t} - 2\frac{\partial g^{ik}}{\partial t}\frac{\partial g^{jl}}{\partial t}R_{ijkl} + 4g^{ir}g^{ks}g^{pq}\frac{\partial g_{pr}}{\partial t}\frac{\partial g_{qs}}{\partial t}R_{ik}\right. \\
&-2a\frac{\partial g^{ik}}{\partial t}\frac{\partial R_{ik}}{\partial t} + a\frac{\partial g^{ik}}{\partial t}\frac{\partial g^{jl}}{\partial t}R_{ijkl} - 2ag^{ir}g^{ks}g^{pq}\frac{\partial g_{pr}}{\partial t}\frac{\partial g_{qs}}{\partial t}R_{ik} \\
&\left.+(bg^{pq}\frac{\partial g_{pq}}{\partial t} + d)\frac{\partial R}{\partial t} - \left(eg^{pq}\frac{\partial g_{pq}}{\partial t} + f(g^{pq}\frac{\partial g_{pq}}{\partial t})^2 + h\frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right)R\right] \\
&+ \left[(4(n-1)h - 2a)g^{ik}g^{jl}\frac{\partial g_{ip}}{\partial t}\frac{\partial}{\partial t}\frac{\partial \Gamma^p_{kl}}{\partial x^j} - (2a+2b)\frac{\partial g^{ik}}{\partial t}\frac{\partial}{\partial t}\frac{\partial \Gamma^p_{kp}}{\partial x^i}\right. \\
&\left.- \left(2bg^{pq}\frac{\partial g_{pq}}{\partial t} + 2(n-1)e + 4(n-1)fg^{pq}\frac{\partial g_{pq}}{\partial t}\right)g^{ik}\frac{\partial}{\partial t}\frac{\partial \Gamma^p_{kp}}{\partial x^i}\right] \\
&+ \left(-2b - 2(n-1)f - 2\right)g^{ik}g^{jl}g^{pq}\frac{\partial^2 g_{ik}}{\partial t\partial x^p}\frac{\partial^2 g_{jl}}{\partial t\partial x^q} \\
&+(2b+8)g^{ik}g^{jl}g^{pq}\frac{\partial^2 g_{ik}}{\partial t\partial x^j}\frac{\partial^2 g_{pl}}{\partial t\partial x^q} + \left(2(n-1)h - 2a + 6\right)g^{ik}g^{jl}g^{pq}\frac{\partial^2 g_{ij}}{\partial t\partial x^p}\frac{\partial^2 g_{kl}}{\partial t\partial x^q} \\
&+(a-8)g^{ik}g^{jl}g^{pq}\frac{\partial^2 g_{ij}}{\partial t\partial x^l}\frac{\partial^2 g_{kp}}{\partial t\partial x^q} + (a-4)g^{ik}g^{jl}g^{pq}\frac{\partial^2 g_{ip}}{\partial t\partial x^j}\frac{\partial^2 g_{kl}}{\partial t\partial x^q}. \qquad (6.15)
\end{aligned}$$

In (6.15), if we take

$$a = 2,\ b = -2,\ e = 0,\ f = -\frac{2b}{4(n-1)} = \frac{1}{n-1},\ h = \frac{2a}{4(n-1)} = \frac{1}{n-1}, \qquad (6.16)$$

then we have

$$\begin{aligned}
\frac{\partial^2 R}{\partial t^2} &= \Delta R + 2|Ric|^2 + (d - 2g^{pq}\frac{\partial g_{pq}}{\partial t})\frac{\partial R}{\partial t} - \frac{1}{n-1}\left[(g^{pq}\frac{\partial g_{pq}}{\partial t})^2 + \frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right]R \\
&+ \left[4g^{ik}g^{jl}g^{pq}\frac{\partial^2 g_{ik}}{\partial t\partial x^j}\frac{\partial^2 g_{pl}}{\partial t\partial x^q} + 4g^{ik}g^{jl}g^{pq}\frac{\partial^2 g_{ij}}{\partial t\partial x^p}\frac{\partial^2 g_{kl}}{\partial t\partial x^q}\right. \\
&\left.-6g^{ik}g^{jl}g^{pq}\frac{\partial^2 g_{ij}}{\partial t\partial x^l}\frac{\partial^2 g_{kp}}{\partial t\partial x^q} - 2g^{ik}g^{jl}g^{pq}\frac{\partial^2 g_{ip}}{\partial t\partial x^j}\frac{\partial^2 g_{kl}}{\partial t\partial x^q}\right]. \qquad (6.17)
\end{aligned}$$

In this case, the corresponding evolution equation reads

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} + G_{ij}, \qquad (6.18)$$

where

$$\begin{aligned}
G_{ij} \quad = \quad & 2g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} - 2g^{pq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{ij}}{\partial t} + d\frac{\partial g_{ij}}{\partial t} + \\
& \frac{1}{n-1}\left[(g^{pq}\frac{\partial g_{pq}}{\partial t})^2 + (\frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t})\right]g_{ij}.
\end{aligned} \qquad (6.19)$$

Taking $d = -\tilde{d}$, we obtain from (6.18) and (6.19) that

$$\begin{aligned}\frac{\partial^2 g_{ij}}{\partial t^2} &= -2R_{ij} + 2g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} - \left(\tilde{d} + 2g^{pq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial g_{ij}}{\partial t} + \\ & \frac{1}{n-1}\left[\left(g^{pq}\frac{\partial g_{pq}}{\partial t}\right)^2 + \frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right]g_{ij},\end{aligned} \tag{6.20}$$

where $\tilde{d}$ is a positive constant. Denoting $\tilde{d}$ by d in (6.20) gives the evolution equation (2.1) for the dissipative hyperbolic geometric flow.

THEOREM 6.1. *If we suppose that the evolution equation of the hyperbolic geometric flow is defined by (6.18)-(6.19) on a manifold $\mathscr{M}$, then the scalar curvature of the evolving metrics satisfies the nonlinear wave equation (6.17) in the normal coordinates.*

REMARK 6.1. *If we take $a = b = d = e = f = h = 0$, i.e., $G_{ij} \equiv 0$ in (6.1), then (6.1) is nothing but the standard hyperbolic geometric flow (1.2) (see [11]).*

REMARK 6.2. *For the evolution equation (6.17) of the scalar curvature, the last term can be written in the covariant form as follow*

$$g^{ik}g^{jl}g^{pq}\left(4\nabla_j\frac{\partial g_{ik}}{\partial t}\nabla_q\frac{\partial g_{pl}}{\partial t} + 4\nabla_p\frac{\partial g_{ij}}{\partial t}\nabla_q\frac{\partial g_{kl}}{\partial t} - 6\nabla_l\frac{\partial g_{ij}}{\partial t}\nabla_q\frac{\partial g_{kp}}{\partial t} - 2\nabla_j\frac{\partial g_{ip}}{\partial t}\nabla_q\frac{\partial g_{kl}}{\partial t}\right).$$

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REFERENCES

[1] H.-D. CAO AND X.-P. ZHU, *A complete proof of the Poincaré and geometrization conjectures - application of the Hamilton-Perelman theory of the Ricci flow*, Asian J. Math., 10 (2006), pp. 165–492.

[2] W.-R. DAI, D.-X. KONG AND K. LIU, *Hyperbolic geometric flow (I): short-time existence and nonlinear stability*, to appear in Pure and Applied Mathematics Quarterly: special issue dedicated to M. Atiyah's 80th birthday.

[3] D. DETURCK, *Deforming metrics in the direction of their Ricci tensors*, J. Differential Geometry, 18 (1983), pp. 157–162.

[4] D. DETURCK AND J. KAZDAN, *Some regularity theorems in Riemannian geometry*, Ann. scient. Ec. Norm. Sup., 14 (1981), pp. 249–260.

[5] M. FELDMAN, S.-Y. HA AND M. SLEMROD, *A geometric level-set formulation of a plasma-sheath interface*, Arch. Rational Mech. Anal., 178 (2005), pp. 81–123.

[6] A. E. FISCHER AND J. E. MARSDEN, *The Einstein evolution equations as a first-order quasilinear symmetric system hyperbolic system I*, Commun. Math. Phys., 28 (1972), pp. 1–38.

[7] K. O. FRIEDRICHS, *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math., 7 (1954), pp. 345–392.

[8] M. E. GURTIN AND P. PODIO-GUIDUGLI, *A hyperbolic theory for the evolution of plane curves*, SIAM. J. Math. Anal., 22 (1991), pp. 575–586.

[9] R. HAMILTON, *Three-manifolds with positive Ricci curvature*, J. Differential Geom., 17 (1982), pp. 255–306.

[10] C.-L. HE, D.-X. KONG AND K. LIU, *Hyperbolic mean curvature flow*, to appear in J. Differential Equations.

[11] D.-X. Kong and K. Liu, *Wave character of metrics and hyperbolic geometric flow*, J. Math. Phys., 48 (2007), 103508-1-103508-14.

[12] D.-X. Kong, K. Liu and Z.-G. Wang, *Hyperbolic mean curvature flow: Evolution of plane curves*, to appear in Acta Mathematica Scientia (Series B).

[13] D.-X. Kong, K. Liu and D.-L. Xu, *The hyperbolic geometric flow on Riemann surfaces*, arXiv:0709.1607.

[14] A. Matsumura, *Global existence and asymptotics of the solution of the second-order quasi-linear hyperbolic equations with the first-order dissipation*, Publ. RIMS, Kyoto Univ., 131 (1977), pp. 349–379.

[15] T. Nishida, *Global smooth solutions for the second-order quasilinear wave equations with the first-order dissipation*, Publications Mathématiques D'ORSAY, 1978.