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MANIFOLDS ADMITTING GENERIC IMMERSIONS INTO $\mathbb{C}^N$*

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Dedicated to Salah Baouendi in friendship and respect on the occassion of his 70^{th} birthday

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The purpose of this paper is to examine bundle-theoretic conditions which are equivalent to a manifold admitting a generic immersion into $\mathbb{C}^N$. In the first section we state the main result and discuss its application to the most familiar case of totally real immersions. We establish the main result in the second section, by applying Gromov theory as presented by Eliashberg and Mishachev [7]. We discuss a technical point related to complex structures in section 3, which shows the limitations of the argument in the previous section. Section 4 is a summary of results, mostly established in the 1980's, on totally real immersions and embeddings; these are presented from a topological perspective. Open questions are collected in the final section.

1. Generic Immersions. Let n and k be integers with $n \geq 0$ and $k \geq 1$. Call an immersion $\pi : M^{2n+k} \to \mathbb{C}^{n+k}$ *generic* if it satisfies any of the following equivalent conditions (here J denotes the standard complex structure on the tangent bundle to $\mathbb{C}^{n+k}$):

1. At each point $p \in M$, the real vector space $\pi_* TM \cap J\pi_* TM$ has the smallest possible dimension (which is $2n$).
2. At each point $p \in M$, $\pi_* TM + J\pi_* TM = T\mathbb{C}^{n+k}|_{\pi(p)}$.
3. At each point $p \in M$, the complex vector space $\pi_* \mathbb{C}TM \cap T^{0,1}(\mathbb{C}^{n+k})$ has the smallest possible dimension (which is n).
4. At each point $p \in M$, $\pi^*(dz_1 \wedge \cdots \wedge dz_{n+k}) \neq 0$ where $(z_1, \ldots, z_{n+k})$ are the usual coordinates on $\mathbb{C}^{n+k}$.
5. At each point $p \in M$, the map $\Psi : \mathbb{C}T_pM \to T^{0,1}_{\pi(p)}(\mathbb{C}^{n+k})$ is surjective, where Ψ is given by projecting
$$\pi_* \zeta \in \mathbb{C}T_{\pi(p)}\mathbb{C}^{n+k} = T^{1,0}_{\pi(p)}(\mathbb{C}^{n+k}) \oplus T^{0,1}_{\pi(p)}(\mathbb{C}^{n+k})$$
into the second factor.
And for generic *embeddings*, we add one more equivalent condition:
6. $\pi(M)$ is given by equations
$$\rho_j(z, \bar{z}) = 0, \quad j = 1, \ldots, k$$
on $\mathbb{C}^{n+k}$ with
$$\partial\rho_1 \wedge \cdots \wedge \partial\rho_k \neq 0 \text{ at each point of } \pi(M).$$

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Note that this condition could have also been stated, in terms of local defining functions, for immersions.

A generic immersion $M^k \to \mathbb{C}^k$ (i.e., $n = 0$) is also called *totally real*. This usage is justified by the fact that with these dimensions $TM \cap JTM = \{0\}$ at all points of M and so TM contains no complex line. Some authors, for example [4], allow totally real to also apply when $\dim M \leq k$ and distinguish the case of equality by calling it *maximally totally real*. A real subspace $\mathcal{R}$ of $\mathbb{C}^{n+k}$ may, or may not, contain a nonzero complex subspace. If $d = \dim\ \mathcal{R} > n + k$ then $\mathcal{R}$ does contain a complex subspace. Indeed, $\mathcal{R}$ must contain a complex subspace of complex dimension at least $d - n - k$. For $\mathcal{R} = \pi_* TM|_p$, $d - n - k = n$. Thus $\pi_* TM|_p$ must contain a complex subspace of complex dimension n. Generic means that it contains no larger complex subspace. That is, the largest complex subspace in $\pi_* TM|_p$ is as small as possible.

The bundle $H \subset TM$ defined by

$$H = \{v \in TM : \pi_* v \in \pi_* TM \cap J\pi_* TM\} \tag{1}$$

is called the *CR bundle* of M. For a generic map the codimension of this bundle in TM (that is, the rank of TM/H) is the same as the codimension of $\pi(M)$ as an immersed submanifold of $\mathbb{C}^{n+k}$.

To be generic is a pointwise condition, so the equivalent formulations also apply, *mutatis mutandis*, for immersions of M^{2n+k} into any complex manifold X^{n+k}.

Proof that the conditions are equivalent. Conditions 1 and 2 are equivalent because of a basic result about the dimension of sums and intersections of subspaces of a vector space.

We have $\zeta \in \{\pi^*(dz_1), \ldots, \pi^*(dz_{n+k})\}^\perp$ if and only if $\pi_*(\zeta) \in T^{0,1}(\mathbb{C}^{n+k})$. So $\pi^*(dz_1 \wedge \cdots \wedge dz_{n+k}) \neq 0$ if and only if $\operatorname{rank}(\pi_* \mathbb{C}TM \cap T^{0,1}(\mathbb{C}^{n+k})) = n$. So 3 and 4 are also equivalent.

To see that 1 and 3 are equivalent, note that for any subspace $V \subset \mathbb{C}T(\mathbb{C}^{n+k})$, we have that J maps V to itself if and only if

$$V = V \cap T^{1,0}(\mathbb{C}^{n+k}) \oplus V \cap T^{0,1}(\mathbb{C}^{n+k}).$$

Further, if $V = \overline{V}$ then $\operatorname{rank} V \cap T^{1,0}(\mathbb{C}^{n+k}) = \operatorname{rank} V \cap T^{0,1}(\mathbb{C}^{n+k})$. Thus for $V = \mathbb{C} \otimes (\pi_* TM \cap J\pi_* TM)$ we have

$$\operatorname{rank}_{\mathbb{R}}(\pi_* TM \cap J\pi_* TM) = 2 \operatorname{rank}_{\mathbb{C}} \mathbb{C} \otimes (\pi_* TM \cap J\pi_* TM \cap T^{0,1}(\mathbb{C}^{n+k})).$$

Now note that for any subspace $V \subset \mathbb{C}T(\mathbb{C}^{n+k})$,

$$V \cap T^{1,0}(\mathbb{C}^{n+k}) = V \cap JV \cap T^{0,1}(\mathbb{C}^{n+k}),$$

because if $\zeta \in V \cap T^{1,0}(\mathbb{C}^{n+k})$, then $J\zeta = -i\zeta \in V \cap JV \cap T^{0,1}(\mathbb{C}^{n+k})$. So

$$\operatorname{rank}_{\mathbb{R}}(\pi_* TM \cap J\pi_* TM) = 2 \operatorname{rank}_{\mathbb{C}}(\pi_* CTM \cap T^{0,1}(\mathbb{C}^{n+k})).$$

In particular, the left hand side is minimal if and only if the right hand side is minimal.

We now show 3 and 5 are equivalent. Note that Ψ is surjective if and only if $\ker \Psi$ has rank n. Since

$$\ker \Psi = \{\zeta \in \mathbb{C}TM : \pi_* \zeta \in T^{1,0}(\mathbb{C}^{n+k})\},$$

we have

$$\pi_* \ker \Psi = \pi_* \mathbb{C}TM \cap T^{1,0}(\mathbb{C}^{n+k}).$$

Since the right hand side has the same rank as $\pi_*\mathbb{C}TM \cap T^{0,1}(\mathbb{C}^{n+k})$,

$$\operatorname{rank}\ker\Psi = \operatorname{rank}\pi_*\mathbb{C}TM \cap T^{0,1}(\mathbb{C}^{n+k}).$$

Thus Ψ is surjective if and only if $\operatorname{rank}\pi_*\mathbb{C}TM \cap T^{0,1}(\mathbb{C}^{n+k}) = n$.

Finally, we show that condition 6 holds at a point if and only if condition 5 holds at that point. For convenience, we identify M with its image $\pi(M)$ in $\mathbb{C}^{n+k}$. Let $\eta \in T^{0,1}(\mathbb{C}^{n+k})$. Note that $\xi \in \mathbb{C}TM$ if and only if $d\rho_j(\xi) = 0$ for $j = 1, \ldots, k$. So η is in the range of Ψ if and only if there exists some $\xi \in T^{0,1}(\mathbb{C}^{n+k})$ with $d\rho_j(\xi + \eta) = 0$. Since $d = \partial + \overline{\partial}$ and the one-form ∂f is zero acting on $T^{0,1}(\mathbb{C}^{n+k})$, for all functions f, and similarly for $\overline{\partial}$ acting on $T^{1,0}(\mathbb{C}^{n+k})$, we see that Ψ is surjective if and only if

$$\partial\rho(\xi) = a_j, \qquad j = 1, \ldots, k$$

is solvable for all $a \in \mathbb{C}^k$. This happens precisely when condition 6 holds.

Theorem 1.1. *If $\pi : M^{2n+k} \to \mathbb{C}^{n+k}$ is a generic immersion, then*

$$\mathbb{C}TM = A \oplus B$$

where A is a trivial bundle of complex rank $n+k$ isomorphic to $T^{0,1}(\mathbb{C}^{n+k})|_{\pi(M)}$ and B is of complex rank n with $B \cap \overline{B} = \{0\}$.

Remark 1. *As is well-known, $\mathbb{C}TM^k$ is trivial if M^k has a totally real immersion into $\mathbb{C}^k$.*

Proof of the Theorem. We know that

$$\Psi : \mathbb{C}TM \to T^{0,1}(\mathbb{C}^{n+k})|_{\pi(M)}$$

is surjective. Thus $\mathbb{C}TM = A \oplus B$ with

$$A \cong T^{0,1}(\mathbb{C}^{n+k})|_{\pi(M)}$$

and $B = \ker\Psi$. If ζ and $\overline{\zeta}$ are both in B, then $\pi_*(\zeta) \in T^{1,0}(\mathbb{C}^{n+k}) \cap T^{0,1}(\mathbb{C}^{n+k})$. Thus $\pi_*(\zeta) = 0$. This shows that $B \cap \overline{B} = \{0\}$.

Remark 2. *Replacing $\mathbb{C}^{n+k}$ by any complex manifold X of the same dimension, we see that if $\pi : M \to X$ is a generic immersion, we have that*

$$\mathbb{C}TM/B \cong T^{0,1}(X)|_{\pi(M)}.$$

At first glance, this does not appear to be useful, since $\pi(M)$ is not "known", but it does suggest an interesting analogy (for which, see the end of this section).

It is convenient to have a sufficient condition for π to be an immersion.

Lemma 1.1. *Let $\pi : M^{2n+k} \to \mathbb{C}^{n+k}$ have components $(F_1, \ldots, F_{n+k})$. If either of the two equivalent conditions holds:*

1. *$dF_1 \wedge \cdots \wedge dF_{n+k} \neq 0$ and $\{dF\} \cup \{d\overline{F}\}$ spans $\mathbb{C}T^*M$*
2. *For $B = \{dF\}^\perp$, we have $\operatorname{rank} B = n$ and $B \cap \overline{B} = \{0\}$*

then $\mathbb{C}TM = A \oplus B$ *as above and* π *is a generic immersion.*

Proof. It is easy to see that these conditions are equivalent. Since B is the annihilator of a trivial bundle, there is a trivial bundle A of rank $n+k$ such that $\mathbb{C}TM = A \oplus B$. Further, for any map

$$(\pi_* \mathbb{C}TM) \cap T^{0,1} = \pi_*(\{dF\}^\perp),$$

so $\operatorname{rank}\{dF\} = n+k$ implies $\operatorname{rank} \pi_* \mathbb{C}TM \cap T^{0,1}(\mathbb{C}^{n+k}) = n$. Further, since $\{dF\} \cup \{d\bar{F}\}$ spans $\mathbb{C}T^*M$, π_* is injective. Thus π is a generic immersion.

We will use a result of Gromov to prove the converse of Theorem 1.1. Then we will have:

Theorem 1.2. *M^{2n+k} has a generic immersion into $\mathbb{C}^{n+k}$ if and only if $\mathbb{C}TM = A \oplus B$ with A trivial of rank $n+k$ and $B \cap \overline{B} = \{0\}$.*

Here are three well known corollaries about totally real immersions. For the first two, we may use the following lemma, which is of independent interest.

Lemma 1.2. *Let F be a complex vector bundle over a manifold M with* $\operatorname{rank} F \geq (\dim\ M)/2$. *If F is stably trivial, then F is trivial.*

Recall that a complex bundle is *stably trivial* if it becomes trivial when a product bundle $M \times \mathbb{C}^k$ is added to it. For a real bundle, one adds $M \times \mathbb{R}^k$.

Proof of Lemma. It will suffice to show that if $\operatorname{rank} F = k \geq (\dim\ M)/2$ and $F \oplus (M \times \mathbb{C})$ is trivial, then also F is trivial. We use the standard fibration $S^{2k+1} \to \mathrm{BU}(k) \to \mathrm{BU}(k+1)$ with projection p and inclusion of the fiber i, noting that $S^{2k+1} = \mathrm{U}(k+1)/\mathrm{U}(k)$. Let $f : M \to \mathrm{BU}(k)$ be a classifying map for F. Then $p \circ f$ is nullhomotopic, and so by the homotopy lifting property there is a map $g : M \to S^{2k+1}$ so that $f \simeq i \circ g$. Since $\dim M \leq 2k$, the cellular approximation theorem (taking M to be a CW-complex, or replacing it by a homotopy equivalent CW-complex of dimension at most $2k$) implies that g is nullhomotopic. Hence also f is nullhomotopic, and F is trivial.

Remark 3. *As a complement to the lemma which will be useful in section 5, we note that when* $\operatorname{rank} F > (\dim M)/2$, *the sphere bundle $S(F)$ has a cross-section, which implies that F is the Whitney sum of a sub-bundle and a trivial complex line bundle. Taking M to be a CW-complex, one easily constructs a cross-section of the sphere bundle by induction over the skeleta of M.*

If TM is stably trivial, then so is $\mathbb{C}TM$. So TM stably trivial implies the existence of totally real immersions. This gives the following two corollaries. The third is even simpler, because TM^3 itself is trivial.

Corollary 1. *Any compact orientable M^2 has a totally real immersion into $\mathbb{C}^2$.*

Corollary 2. *Every sphere S^n has a totally real immersion into $\mathbb{C}^n$.*

Corollary 3. *Any compact orientable M^3 has a totally real immersion into $\mathbb{C}^3$.*

Remark 4. *As we explain in sections* 4 *and* 5, *among compact orientable surfaces only the torus has a totally real* embedding. *On the other hand, every compact orientable M^3 has a totally real embedding into $\mathbb{C}^3$ [9]. Again see section* 4.

Note that the triviality of TM also implies, via Smale-Hirsch theory, that M^3 immerses into $\mathbb{R}^4$. Any real hypersurface in $\mathbb{C}^2$ is generic. Thus any compact orientable M^3 has a generic immersion into $\mathbb{C}^2$.

Another proof of Corollary 2. This is a direct verification of the fact that $\mathbb{C}TS^n$ is trivial. It was shown to one of the authors several years ago by Gerardo Mendoza. Using parallel translation in $\mathbb{R}^{n+1}$ and the usual correspondence of the vector spaces $T_0(\mathbb{R}^{n+1})$ and $\mathbb{R}^{n+1}$, we identify each $v \in T_pS^n$ with its translation to the origin. Let $\mathbf{n} = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$, $\mathbf{s} = -\mathbf{n}$, $\mathcal{U}^+ = S^n - \mathbf{s}$ and $\mathcal{U}^- = S^n - \mathbf{n}$. For $v \in T_{\mathbf{n}}(S^n)$, we have $v = (\tilde{v}, 0), \tilde{v} \in \mathbf{R}^n$, but, allowing the abuse of notation, we write this as $v = (v, 0)$. The same abuse holds for $v \in T_{\mathbf{s}}(S^n)$.

Each $v \in T_{\mathbf{n}}(S^n)$ determines the parametrized great circle

$$\gamma_v^+(t) = ((\sin t)\, v, \cos t),$$

and each $v \in T_{\mathbf{s}}(S^n)$ determines the parametrized great circle

$$\gamma_v^-(t) = ((\sin t)\, v, -\cos t).$$

Consider some vector $w \in T_{\mathbf{n}}(S^n)$. If w is normal to v, then parallel translation along γ_v^+ within S^n coincides with parallel translation within $\mathbb{R}^{n+1}$ and so produces the vector w at the point $\gamma_v^+(t)$. If $w = av$ then parallel transport of w along γ_v^+ produces $a\dot{\gamma}_v^+(t)$ at $\gamma_v^+(t)$. Thus for a general w, parallel transport produces the vector

$$\phi_+(t, v, w) = (w - (w \cdot v)v + (w \cdot v)(\cos t)\, v, -(w \cdot v) \sin t)$$

at the point $\gamma_v^+(t)$. Likewise, starting from $\mathbf{s}$, parallel transport of w' along $\gamma_v^-(t)$ produces

$$\phi_-(t, v, w') = (w' - (w' \cdot v)v + (w' \cdot v)(\cos t)\, v, (w' \cdot v) \sin t).$$

Using $\phi_+(t, v, w)$ we identify $TS^n|\{U^+ - \mathbf{n}\}$ with

$$\{(t, v, w) : 0 < t < \pi, |v| = 1, w \in T_{\mathbf{n}}(S^n)\}.$$

and similarly for $T_{\mathbf{s}}(S^n)^n|\{\mathcal{U}^- - \mathbf{s}\}$. It is now easy to determine the gluing map

$$T(S^n)|\{\mathcal{U}^+ - \mathbf{n}\} \to T((S^n)|\{\mathcal{U}^- - \mathbf{s}\}.$$

A point (t, v, w) of $T(S^n)|\{\mathcal{U}^+ - \mathbf{n}\}$ is glued to a point (t, v, w') of $T(S^n)|\{\mathcal{U}^- - \mathbf{s}\}$ when

$$\phi_+(t, v, w) = \phi_-(t, v, w').$$

It is sufficient to solve this equation when $t = \pi/2$. So the equations become

$$(w - (w \cdot v)v, -(w \cdot v)v) = (w' - (w' \cdot v)v, (w \cdot v)v).$$

Hence

$$w' = w - 2(w \cdot v)v.$$

Thus at the point $v \in S^{n-1}$, the gluing map takes w to its reflection across the plane normal to v. Write this map as

$$\begin{aligned} w' &= w - (w \cdot v)v - (w \cdot v)v \\ &= (\pi_{v^\perp} - \pi_v)w \end{aligned}$$

and use the homotopy

$$w' = \pi_{v^\perp} - e^{i\tau\pi}\pi_v$$

for $0 \leq \tau \leq 1$.

An *almost CR structure* on M is a complex sub-bundle $B \subset \mathbb{C}TM$ for which $B \cap \overline{B} = \{0\}$. This is a *CR structure* when, in addition, B is involutive. That is, if u and v are sections of B over some open set, then the vector field bracket $[u, v]$ is also a section of B over that set. See, for instance [4], §2.1. In particular, a generically immersed manifold has an induced CR structure. The CR bundle H defined in (1) is the underlying real bundle of B.

COROLLARY 4. *If M^{2n+k} admits an almost CR structure B of complex rank n for which $\mathbb{C}TM/B$ is trivial, then B may be deformed, through almost CR structures, into a CR structure.*

Proof of Corollary 4. This is actually a corollary to the proof of Theorem 1.2 presented in the next section. The resulting CR structure is induced by a generic immersion into $\mathbb{C}^{n+k}$.

We can relate the data in Theorem 1.2 to the Smale-Hirsch Immersion Theorem. Recall that this latter states, among other things, that a manifold M has an immersion into a higher dimensional manifold N provided there exists a bundle map, injective on the fibers, of TM into TN. The decomposition $\mathbb{C}TM = A \oplus B$ defines a projection $\mathbb{C}TM \to A$ and its restriction $TM \to A$. We claim that this restriction is injective. For, assume there is a real vector which projects to zero. Then this real vector is in B. But $B \cap \overline{B} = \{0\}$ and so this vector must be zero. Thus our map is injective. Since A is trivial, we obtain an injective map of TM into $T\mathbb{R}^{2(n+k)}$. And the Smale-Hirsch Theorem guarantees that there is an immersion of M^{2n+k} into $\mathbb{C}^{n+k}$. In particular, our proof of Theorem 1.2 contains a proof of the Smale-Hirsch Theorem in this context. One can hope to go a step further in this direction and relate the generic immersions of M^{2n+k} into X to the set of bundle maps $\Phi : \mathbb{C}TM \to T^{1,0}(X)$, where X is a complex manifold of dimension $n + k$.

2. Proof of Theorem 1.2. Here is the essence of the proof.

CLAIM. *Let M be a manifold of dimension N and let $\theta_1, \ldots, \theta_r$ be global one-forms with $\theta_1 \wedge \cdots \wedge \theta_r$ different from zero at each point of M. Then there exist complex-valued functions $F_1, \ldots, F_r$ and a deformation of one-forms $\theta_1^t, \ldots, \theta_r^t$ such that*

$$\begin{aligned} \theta_j^0 &= \theta_j \\ \theta_1^t \wedge \cdots \wedge \theta_r^t &\neq 0, \ \text{for } 0 \leq t \leq 1 \\ \{\theta_1^t, \ldots, \theta_r^t, \overline{\theta_1^t}, \ldots, \overline{\theta_r^t}\} &\ \text{spans } \mathbb{C}T^*M, \\ \theta_j^1 &= dF_j. \end{aligned}$$

Further, $\max_M |F|$ *can be made less than any preassigned positive number.*

Note that the real analogue of this result is clearly false. The existence of a real global one-form which never is zero certainly does not imply the existence of a real function without critical points. See also the discussion following the statement of Lemma 2.2 below.

Theorem 1.2 follows immediately from this claim. Take $\{a_1, \ldots, a_{n+k}\}$ to be a global basis for A. Define corresponding one-forms by

$$\theta_i(a_j) = \delta_{ij}$$
$$\theta_i|_B = 0.$$

Note that $\theta_1 \wedge \cdots \wedge \theta_{n+k} \neq 0$ and $B = \{\theta_1, \ldots, \theta_{n+k}\}^\perp$.

Define $\pi : M \to \mathbb{C}^{n+k}$ by setting $\pi = (F_1, \ldots, F_{n+k})$. Since

$$\pi^*(dz_1 \wedge \cdots \wedge dz_{n+k}) = dF_1 \wedge \cdots \wedge dF_{n+k} \neq 0,$$

π would be a generic immersion provided it is an immersion at all. That it is an immersion follows from the fact that $\{dF_1, \ldots, d\overline{F_{n+k}}\}$ spans $\mathbb{C}T^*M$.

We now outline a proof of the claim. We will use the notation of [7] and in particular Theorem 18.4.1 on page 171 which states that if $R \subset X^{(1)}$ is an open ample relation then all forms of the h-principle hold. Here is the basic set-up. Note that $r \leq N \leq 2r$.

$$\dim\ M = N$$
$$X = M \times \mathbb{C}^r = M \times (\mathbb{R} \oplus \mathbb{R})^r$$
$$Z = \Lambda^1_{\mathbb{C}}(M)^r = \mathbb{C}T^*(M)^r = (\Lambda^1(M) \oplus \Lambda^1(M))^r$$
$$d : \operatorname{Sec} X \to \operatorname{Sec} Z \text{ is given by } d(f_1, \ldots, f_r) = (df_1, \ldots, df_r).$$

For the jet space $X^{(1)} = J^1(M, X)$ we use local coordinates

$$\{(p, c, a_i^j),\ i = 1, \ldots, r,\ j = 1, \ldots, N\}$$

where

$$p \in M, \quad c \in \mathbb{C}^r, \text{ and for each } i \text{ and } j,\ a_i^j \in \mathbb{C}.$$

The point (p, c, a_i^j) may be thought of as the 1-jet at $p \in M$ of $f(x) = (f^1(x), \ldots, f^r(x))$, with $f^i = c_i + \sum_{j=1}^N a_i^j x_j$. The differential relation R is given by

$$R = \{(p, c, a_i^j)\}$$

where

$$a_1^j dx_j \wedge \cdots \wedge a_r^k dx_k \neq 0$$

and

$$\{a_1^j dx_j, \ldots, a_r^k dx_k, \overline{a_1^j} dx_j, \ldots, \overline{a_r^k} dx_k\} \text{ spans } \mathbb{C}T_p^* M.$$

All repeated indices are summed from 1 to N. Let P_a be any coordinate principal subspace, say,

$$\begin{aligned} P_a &= \{(p, c, a_i^j) : a_i^j \text{ has a fixed value for } j = 2, \ldots, N \text{ and } i = 1, \ldots, r\} \\ &= \{(p, c, \zeta, a_i^2, \ldots, a_i^N) : \zeta \in \mathbb{C}^r, a_i^j \text{ fixed }\}. \end{aligned}$$

So P_a is a copy of $\mathbb{C}^r$ and

$$R \cap P_a = \{(p, c, \zeta, a)\}$$

where

$$(\zeta_1 dx_1 + a_1^j dx_j) \wedge \cdots \wedge (\zeta_r dx_1 + a_r^k dx_k) \neq 0 \tag{2}$$

and

$$\{\zeta_1 dx_1 + a_1^j dx_j, \ldots, \overline{\zeta_r} dx_1 + \overline{a_r^j} dx_j\} \text{ spans } \mathbb{C}T^*M. \tag{3}$$

Let $A(\zeta)$ denote the $r \times N$ matrix (ζ, a). So the first column of A is the vector ζ and the j^{th} column, for $j = 2, \ldots, N$ is the vector a_i^j. Let

$$B(\zeta) = \begin{pmatrix} \zeta & a \\ \bar{\zeta} & \bar{a} \end{pmatrix}$$

be the associated $2r \times N$ matrix.

We think of p and c as fixed and identify $R \cap P_a$ with

$$\{\zeta : \operatorname{rank} A(\zeta) = r \text{ and } \operatorname{rank} B(\zeta) = N\}. \tag{4}$$

R is ample in the direction P_a provided either $R \cap P_a$ is empty or the convex hull of each component of $R \cap P_a$ is all of P_a.

Equation (2) is equivalent to $\operatorname{rank} A = r$ and Equation (3) is equivalent to $\operatorname{rank} B = N$.

Lemma 2.1. *Assume that $R \cap P_a$ is non-empty. Then* $\operatorname{rank} a = r$ *or* $r-1$ *and* $\operatorname{rank} \begin{pmatrix} a \\ \bar{a} \end{pmatrix} = N$ *or* $N-1$.

Proof. We are assuming that for some ζ the rank of $A(\zeta)$ is r and the rank of $B(\zeta)$ is N. Since the column space of the matrix $A(\zeta) = (\zeta, a)$ has dimension r, the column space of the matrix a has dimension r or $r-1$. Similarly for $B(\zeta)$.

We proceed to prove that $R \cap P_a$ is ample by considering each of these four possibilities on the ranks. We use the following simple result.

Lemma 2.2. *Let Q be a real plane in $\mathbb{R}^q$ with* $\operatorname{codim} Q \geq 2$. *Then $\mathbb{R}^q - Q$ is connected and its convex hull is all of $\mathbb{R}^q$.*

In particular, the lemma applies to the span of $r-1$ complex vectors in $\mathbb{C}^r$. Thus the complement of a plane of (complex) dimension $r-1$ in $\mathbb{C}^r$ is ample but the complement of a plane of dimension $r-1$ in $\mathbb{R}^r$ is not. This is the reason that the (false) analogue of results such as Theorem 1.2 cannot be proved in this way.

1. $\operatorname{rank} a = r$ and $\operatorname{rank} \begin{pmatrix} a \\ \bar{a} \end{pmatrix} = N$

   Here $R \cap P_a = P_a$ and so R is ample in the direction P_a.
2. $\operatorname{rank} a = r-1$ and $\operatorname{rank} \begin{pmatrix} a \\ \bar{a} \end{pmatrix} = N$

   Here $R \cap P_a = \{\zeta : \zeta \notin \operatorname{Colspace}(a)\}$ and this space is connected. Its convex hull is P_a. Let $A(\zeta : a)$ and $A(z : a)$ both be in $R \cap P_a$ (using the identification (4)). Thus ζ and z are each independent of the column vectors of B. Now B defines, via its columns, a complex hyperplane in $\mathbb{C}^r$ and ζ and z are points of $\mathbb{C}^r$ which do not lie on this plane. Clearly, there is a path which connects ζ and z and avoids the plane.

3. $\operatorname{rank} a = r$ and $\operatorname{rank}\begin{pmatrix} a \\ \overline{a} \end{pmatrix} = N-1$

   Now we have

$$R \cap P_a = \left\{ \zeta : \begin{pmatrix} \zeta \\ \overline{\zeta} \end{pmatrix} \notin \text{Colspace}\begin{pmatrix} a \\ \overline{a} \end{pmatrix} \right\} \tag{5}$$

$$= \mathbb{C}^r - \operatorname{span}_{\mathbb{R}} \text{Colspace}(a). \tag{6}$$

   Again, this is an ample set.

4. $\operatorname{rank} a = r-1$ and $\operatorname{rank}\begin{pmatrix} a \\ \overline{a} \end{pmatrix} = N-1$

   It is easy to see that in this case, $\zeta \in R \cap P_a$ if and only if $\zeta \notin \text{Col}(a)$. Thus $R \cap P_a$ is ample.

This covers all cases.

R is an ample relation. Since it is clearly open, we have that all forms of the h-principle hold. In particular, the Claim is established. More explicitly, using local coordinates we write $\theta_i = a_i^j dx_j$. Let $G : M \to R \subset J^1(M, X)$ be given by

$$G(p) = (p, 0, a_i^j).$$

G is a formal solution (in the sense of Gromov, see [7], page 56) and so, by the h-principle, G is homotopic to a genuine solution. That is, there exists $F : M \to R \subset J^1(M, X)$ of the form

$$F(p) = (p, F_1(p), \ldots, F_r(p), dF_1(p), \ldots, dF_r(p))$$

and, since F maps into R, at each $p \in M$ we have

$$dF_1 \wedge \cdots \wedge dF_r \neq 0 \text{ and } \{dF_1, \ldots, \overline{dF_r}\} \text{ spans } \mathbb{C}T^*M.$$

Further, by the C^0-dense version of the h-principle, for any given $\epsilon > 0$, the homotopy may be chosen to satisfy

$$\max_{p \in M} |F(p)| < \epsilon.$$

As we show in the next section, the h-principle does not hold for complex structures. That is, the h-principle may not be used, as it is used above, to prove that if M^{2n} admits a complex bundle B with $B \oplus \overline{B} = \mathbb{C}TM$ and $\mathbb{C}TM/B$ trivial then M admits an involutive bundle B_1 with $B_1 \oplus \overline{B_1} = \mathbb{C}TM$. However, it may be used to prove that if M admits a real bundle B with TM/B trivial, then M admits a real involutive bundle B_1 with the same rank as that of B. Hence there is a foliation of M with tangent bundle B_1. This is an exercise on page 106 of [11]. (This is actually somewhat different than what we have done above, since the foliation cannot be produced by a map into some $\mathbb{R}^n$.)

3. Complex structures. We have seen that if

$$\mathbb{C}TM^{2n+k} = A \oplus B$$

with A trivial and of rank $n+k$, and $B \cap \overline{B} = \{0\}$, then M has a generic immersion. This immersion induces a CR structure. The one-forms which annihilate B are deformed into exact one-forms. These one-forms annihilate some other bundle B_1 and, because of exactness, B_1 is involutive. That is, if B is an almost CR structure and $\mathbb{C}TM/B$ is trivial, then B may be deformed to a CR structure.

Recall that we have always assumed $k \geq 1$. What if $k = 0$? Then

$$\mathbb{C}TM = A \oplus B.$$

where we may take $A = \overline{B}$. So the assumption that $\mathbb{C}TM/B$ is trivial implies that B is trivial which then implies that TM is trivial. We have this question: Let M be parallelizable and let B be a trivial bundle which defines an almost complex structure. Does it follow that B may be deformed to a complex structure?

It is easy to see where the proof of our claim breaks down in this context. In fact, it is not only the proof which fails: as was pointed out to the authors by L. Lempert and P. Wong, these conditions on M do not imply that M admits a complex structure. There are parallelizable four dimensional manifolds which do not admit any complex structures (Yau [19]). We want to indicate where the proof of the claim breaks down. It is enough to exhibit some P_a for which $R \cap P_a$ is not ample. To do this, let

$$a = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \end{pmatrix}.$$

Then

$$A = \begin{pmatrix} \zeta_1 & 1 & 0 & i \\ \zeta_2 & 0 & 1 & 0 \end{pmatrix}$$

and rank $A = 2$ for all ζ. We have

$$B = \begin{pmatrix} \zeta_1 & 1 & 0 & i \\ \zeta_2 & 0 & 1 & 0 \\ \overline{\zeta_1} & 1 & 0 & -i \\ \overline{\zeta_2} & 0 & 1 & 0 \end{pmatrix}.$$

So $R \cap P_a = \{\zeta : \text{rank } B = 4\}$. Let $\zeta_1 = a + ib$. Subtract a times the second column and b times the fourth column from the first column to get

$$\begin{pmatrix} 0 & 1 & 0 & i \\ \zeta_2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -i \\ \overline{\zeta_2} & 0 & 1 & 0 \end{pmatrix}.$$

This matrix, and hence also B, has rank four if and only if $Im\ \zeta_2 \neq 0$. Thus, in real coordinates

$$R \cap P_a = \{(x, y, u, v) : v \neq 0\}$$

and so consists of two connected components. The convex hull of either one of these two components is not all of P_a. So R is not ample.

4. Manifolds admitting totally real immersions and embeddings into $\mathbb{C}^N$. For the convenience of readers, we survey some results concerning the existence of totally real immersions of a smooth N-manifold M into $\mathbb{C}^N$, as well as the more delicate question of the existence of totally real embeddings into $\mathbb{C}^N$. These results were largely obtained during the 1980's. Our coverage is regretably selective. The most complete studies of these questions were made by F. Forstnerič [9] and M. Audin [3], and rely heavily on work by M. Gromov in the 1970's.

If M^N admits a totally real *immersion* into $\mathbb{C}^N$, then the complexified tangent bundle $\mathbb{C}TM$ is a trivial complex N-plane bundle. Applying Theorem 1.2 with $B = 0$ (a case covered by Gromov's work in the 1970's), the triviality of $\mathbb{C}TM$ implies the existence of such an immersion. This applies whenever the tangent bundle TM is stably trivial.

Indeed, T. Duchamp [6] has given a K-theoretic classification of totally real immersions into $\mathbb{C}^N$. Assuming that one such immersion exists, he establishes a bijection between regular homotopy classes of totally real immersions of M^N into $\mathbb{C}^N$ and the K-theory group $K^1(M)$ (which can be defined as homotopy classes of continuous maps from M to the infinite unitary group U).

Knowing that all spheres S^N admit totally real immersions into $\mathbb{C}^N$, it is next interesting to examine the real projective spaces $\mathbb{R}\mathrm{P}^N$. Using the known K-theory of real projective spaces, one shows easily that the complexified tangent bundle of $\mathbb{R}\mathrm{P}^N$ is nontrivial for $N \neq 1, 3, 7$. Since $\mathbb{R}\mathrm{P}^1 \approx S^1, \mathbb{R}\mathrm{P}^3 \approx \mathrm{SO}(3)$, and $\mathbb{R}\mathrm{P}^7$ are parallelizable (use Cayley number for the latter), each of them admits a totally real immersion.

If M^N admits a totally real immersion into $\mathbb{C}^N$, the triviality of $\mathbb{C}TM$ implies that $TM \oplus TM$ is also a trivial real bundle, and so the squares of all Stiefel-Whitney class of M vanish (Audin [2]). L. Smith and R. Stong [16] have examined the "exotic" cobordism theory based on compact manifolds M for which $\mathbb{C}TM$ is trivial, and concluded that the resulting cobordism classes form a polynomial ring

$$\mathbb{Z}/2\,[z_5, z_9, z_{11}, \ldots]$$

on odd dimensional generators $z_k = [M^k]$ for all odd dimensions so that $k+1$ is not a power of 2.

As is conventional, $[M^k]$ denotes the cobordism class of a k-dimensional manifold M^k. The product of cobordism classes is induced by the Cartesian product of manifolds, and the sum is induced by disjoint union. There is a single generator for the cobordism ring in each of the odd dimensions indicated above. To establish that a manifold represents a nonzero cobordism class, it is sufficient to show that $\mathbb{C}TM$ is trivial and that some Stiefel-Whitney number is nonzero. The cobordism class of a manifold M^k is a generator if the Stiefel-Whitney number $s_k[M^k]$ is nonzero, where s_k is the polynomial in the Stiefel-Whitney classes w_i that equals the sum of kth powers of variables $x_1, \ldots, x_k$ when each w_i is written as the ith elementary symmetric polynomial of these variables.

Since the 5-manifold SU(3)/SO(3), commonly known as the Wu manifold, admits a totally real embedding into $\mathbb{C}^5$ (Audin [3], Prop. 0.8), and $w_2 w_3 \neq 0$ ([13], p. 393), one can take this homogeneous space as a representative of z_5. Indeed, an exploration of the problem of deciding which homogeneous spaces $\mathrm{SU}(n)/\mathrm{SO}(n)$ admit totally real embeddings for $n \geq 3$ has been carried out, based on the results of Audin [3]. One needs to work out some elementary properties of the mod 2 cohomology ring and the second Stiefel-Whitney class of the tangent bundle for $\mathrm{SU}(n)/\mathrm{SO}(n)$. The conclusion is that for $n \geq 3$, a totally real embedding exists in the majority of cases; namely, the only cases in which this is not assured (but might still exist) are when

$$n = 8, \quad n \equiv 0 \pmod{16}, \quad \text{and} \quad n \equiv -1 \pmod{8}.$$

In the case $n \equiv 8 \pmod{16}$ with $n > 8$, so that $n = 16l + 8$, $\mathrm{SU}(n)/\mathrm{SO}(n)$ has dimension $8[4(2l+1)^2 + l] + 3$, and one needs to know that $4(2l+1)^2 + l$ is *not* a power of 2 in order to apply Audin [3], Thm. 0.5(c); we thank R. E. Stong for proving that this is true for $l > 0$, by an argument based on Pell's equation.

Let M^N admit a totally real embedding $\pi : M^N \to \mathbb{C}^N$, and denote by J the standard complex structure on $T\mathbb{C}^N$. Then $TM \oplus JTM = \pi^* T\mathbb{C}^N$, so we can identify the normal bundle of M in $\mathbb{C}^N$ with JTM. Since J maps TM isomorphically onto JTM, the normal bundle to M in $\mathbb{C}^N$ is isomorphic to the tangent bundle TM. Now assume that M is compact and oriented; it then follows that the Euler class of the normal bundle to M in $\mathbb{C}^N$ is zero (Milnor and Stasheff [15], p. 120), and so the Euler class $e(TM)$ must vanish. Hence, if M^N is compact and oriented and admits a totally real embedding into $\mathbb{C}^N$, then the Euler characteristic $\chi(M)$ must vanish. This argument is due to Wells [18].

Which spheres S^N admit totally real embeddings? Since $\chi(S^N) = 2$ for even N, we restrict attention to odd dimensions. Of course, S^1 is a totally real submanifold of $\mathbb{C}$. The first challenging case is S^3, for which an explicit totally real embedding into $\mathbb{C}^3$ was found by Ahern and Rudin [1].

Turning to odd spheres with $N > 3$, Gromov ([11], p. 193) outlined a proof that no totally real embedding $S^N \to \mathbb{C}^N$ exists. A detailed argument has been given by Stout and Zame [17] in the case of S^7; they rule out other odd spheres by citing Kervaire's result [12] that any embedding of S^N in $\mathbb{C}^N$ has trivial normal bundle, so also trivial tangent bundle, which forces N to be $1, 3$, or 7.

As was the case for totally real immersions, it is also interesting to examine totally real embeddings of real projective spaces $\mathbb{RP}^N$. Only $\mathbb{RP}^1 \approx S^1, \mathbb{RP}^3 \approx \mathrm{SO}(3)$, and $\mathbb{RP}^7$ admit totally real immersions, so the challenges remaining are posed by $\mathbb{RP}^3$ and $\mathbb{RP}^7$. For the case of $\mathbb{RP}^3$, Forstnerič [8] produced an explicit totally real embedding into $\mathbb{C}^3$, and then went on in [9] (based on Gromov [10]) to show that every compact orientable 3-manifold M^3 admits a totally real embedding into $\mathbb{C}^3$.

The case of $\mathbb{RP}^7$ was treated by Audin [3], as part of a thorough and wide-ranging study of Lagrangian immersions and totally real embeddings (Lagrangian immersions are special cases of totally real immersions; if one exists so does the other. But a manifold may admit a totally real embedding without admitting a Lagrangian embedding.) Audin shows that a totally real embedding $\mathbb{RP}^7 \to \mathbb{C}^7$ does indeed exist, as follows. It has been noted that $\mathbb{RP}^7$ is parallelizable. Hence there is a totally real immersion $\pi : \mathbb{RP}^7 \to \mathbb{C}^7$, which wecan assume to be a Lagrangian immersion. Let $d(\pi) \in \mathbb{Z}/2$ be the number (taken mod 2) of double points of an approximation to π with normal crossings; the vanishing of $d(\pi)$ is a necessary and sufficient condition for π to be regularly homotopic to an embedding. By Theorem 1.2 of Forstnerič [9], it follows from the vanishing of $d(\pi)$ that there is a totally real embedding of $\mathbb{RP}^7$ into $\mathbb{C}^7$. To establish that $d(\pi) = 0$, Audin [3] Theorem 0.5 (a) proves that $d(\pi) = \hat{\chi}_{\mathbb{Z}/2}(M)$, where for an odd dimensional compact manifold M with $\dim M = 2k+1$, $\hat{\chi}_{\mathbb{Z}/2}(M)$ denotes the Kervaire semi-characteristic ([14])

$$\hat{\chi}_{\mathbb{Z}/2}(M) = \sum_{i=0}^{k} \dim H^i(M; \mathbb{Z}/2) \mod 2.$$

It therefore suffices to verify (as one does immediately) that $\hat{\chi}_{\mathbb{Z}/2}(\mathbb{RP}^7) = 0$.

Finally we state several of Audin's results from the introduction of [3] in full generality:

1. If M^N is a compact connected orientable manifold of even dimension and M admits a totally real immersion into $\mathbb{C}^N$, then M admits a totally real embedding into $\mathbb{C}^N$ if and only if $\chi(M)$ vanishes.
2. If M^N is a compact connected *nonorientable* manifold of even dimension and M admits a totally real immersion into $\mathbb{C}^N$, then M admits a totally real embedding into $\mathbb{C}^N$ if and only if $\chi(M) \equiv 0 \mod 4$.
3. If M^N is a compact connected orientable manifold of dimension $N = 4k + 1 (k \geq 1)$ and M admits a totally real immersion into $\mathbb{C}^N$, then M admits a totally real embedding into $\mathbb{C}^N$ if and only if $\hat{\chi}_{\mathbb{Z}/2}(M) \in \mathbb{Z}/2$ vanishes.

One will find further results (and a wide selection of arguments) in Audin's paper, and in several other papers, including results stated in terms of regular homotopy of totally real immersions. It is our hope that some of the results collected here will point the way to more general results for generic immersions and embeddings.

5. Open Questions.

1. Here are two natural classes of problems for compact orientable manifolds. One is to distinguish those manifolds that admit totally real *embeddings* from those that admit only totally real *immersions*. Many cases of this problem have been settled (see the previous section), but others remain. The second is to extend to generic embeddings (or immersions) results that hold for totally real embeddings (or immersions). As an example of the first class of problems, consider the fact that Wells [18] showed that among compact orientable manifolds only those with zero Euler characteristic can have generic *embeddings*. (Thus the torus is the only 2-dimensional compact orientable manifold with a totally real embedding into $\mathbb{C}^2$. But, as we have seen in Corollary 1, all 2-dimensional compact orientable manifolds have totally real immersions into $\mathbb{C}^2$.) Wells' result of course does not give any information about odd dimensional manifolds. For dimension $n = 4k + 1$, Audin [3] showed that the Kervaire semi-characteristic must be zero to have a totally real embedding. This leaves open the more delicate case of $n = 4k+3$; Audin [3] gives partial result in this case, assuming that M is a Spin manifold of dimension $8k + 3$. Sometimes if M has a totally real immersion it also has a totally real embedding. So, for instance, if M^7 has a totally real immersion into $\mathbb{C}^7$, must it also have a totally real embedding? In general the answer is no. This follows from the well understood distinction between embeddings and immersions for spheres: S^n has a totally real immersion into $\mathbb{C}^n$ for each n (as above) but only S^1 and S^3 have totally real embeddings (Gromov [11], p. 193, Ahern-Rudin [1], Stout-Zame [17]). As an example of the second class of questions one could ask if every M^5 which admits an embedding into $\mathbb{C}^4$ and satisfies the necessary condition for generic immersions given by Theorem 1.2 has a generic embedding into $\mathbb{C}^4$ or if the Kervaire semi-characteristic is still restricted.
2. Another type of question starts with the observation that generic means each fiber of TM has only a complex subspace of the smallest possible rank. What if we ask for larger subspaces? Let m, N, l be integers with $0 \leq m - N \leq l \leq m$. When does there exist an immersion

$$\Phi : M^m \to \mathbb{C}^N$$

for which $\operatorname{rank} \Phi_* CTM \cap T^{0,1} = l$? For $l = m - N$, we are seeking a generic immersion. But for $l = m$ we would be seeking a complex submanifold and so we would fail unless M admits a Stein complex structure. So an obvious necessary condition would be that M is orientable and noncompact. Or one could drop the requirement the Φ is an immersion while keeping that $\Phi_* \mathbb{C}TM \cap T^{0,1}$ has constant rank.

3. Questions about CR structures might be easier than the corresponding ones about complex structures, in analogy with the observation that the proof of our Claim does not hold for complex structures.
   (a) What manifolds have almost CR structures?
   $\mathbb{C}TM^N$ has sub-bundles of all ranks (and trivial subbundles of all ranks less than or equal to N), in view of the remark following the proof of Lemma 1.2. Let B have rank $r \leq N$. What is the obstruction to deforming B to obtain $B \cap \overline{B} = \{0\}$? (For $N = 2n$ and $\operatorname{rank} B = n$, this is the question of which manifolds admit an almost complex structure.) This can be reformulated as an abstract bundle question. Let A be a complex bundle over M and let J be a fiberwise involution $J : A \to A$. Assume that the fixed point set of J is a subbundle C. Now let B be any subbundle of A. When can B be deformed through sub-bundles so that $B \cap C$ becomes only the zero section? The relation to CR structures is when $A = \mathbb{C}TM$, J is conjugation, and $C = TM$.
   (b) Assume M has an almost CR structure. What is the obstruction to deforming it to a CR structure?
   We have seen in Corollary 4 that $\mathbb{C}TM/B$ trivial is a sufficient condition. But maybe there is a much more general condition, perhaps just that M is open. Here is a related question: Drop the restriction that $B \cap \overline{B} = \{0\}$ and ask if B can be deformed to an involutive sub-bundle. Again, this is always possible when $\mathbb{C}TM/B$ is trivial and we seek more general conditions. Bott's topological obstruction for foliations will play a role here ([5]).
   (c) On the subject of deformations of CR structures: Can any C^∞ CR structure be deformed to a real analytic CR structure? The CR bundle B, thought of as an n-plane distribution can be deformed to a C^ω distribution, but how can we keep the Frobenius condition?
   (d) Forstnerič proved in [9] that if a totally real immersion is regularly homotopic to an embedding, then it is regularly homotopic to a totally real embedding. The proof uses Whitney's study of n-dimensional manifolds in $\mathbb{R}^{2n}$. Is there an analogous result for generic immersions? The interesting point would be to find additional conditions on the immersion to allow something like Whitney's methods to apply when $\dim M > \operatorname{codim} M$.

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