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MICROLOCALIZATION AND STATIONARY PHASE*

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Respectfully dedicated to the memory of Armand Borel

0. Introduction. In [6], G. Laumon defined a set of functors, called by him local Fourier transformations, which allow to analyze the local structure of the ℓ-adic Fourier transform of a constructible ℓ-adic sheaf $\mathcal{G}$ on the affine line in terms of the local behaviour of $\mathcal{G}$ at infinity and at the points where it is not lisse. These local Fourier transforms play a major rôle in his cohomological interpretation of the local constants and in his proof of the product formula (see loc. cit. and [5]).

In this article, we are concerned with differential systems defined over a field K of characteristic zero. We define a set of functors which allow to prove a stationary phase formula, expressing the formal germ at infinity of the $\mathcal{D}$-module theoretic Fourier transform of a holonomic $K[t]\langle\partial_t\rangle$-module $\mathbb{M}$ in terms of the formal germs defined by $\mathbb{M}$ at its singular points and at infinity. These functors might therefore be regarded as formal analogues of Laumon's local Fourier transformations.

When the module $\mathbb{M}$ is of exponential type (in Malgrange's sense, see [8]), such a stationary phase formula is implicit in the work B. Malgrange (loc. cit. and [7]). In this case, the only transformation needed is the one given by the functor of formal microlocalization which, as it is probably known, should correspond to the transformation labelled $(0, \infty')$ by Laumon. The main point of section 1 below is to treat the case of a $K[t]\langle\partial_t\rangle$-module with arbitrary slopes at infinity. In this case, one is forced to introduce another functor (corresponding to Laumon's (∞, ∞') transformation) to keep track of the contribution coming from the germ at infinity defined by $\mathbb{M}$. A transcendental construction of this functor was explained by B. Malgrange to the author, however, the construction we give here is algebraic. The main advantage of having an algebraic definition is that, in spite of the fact that we cannot avoid some transcendental arguments in the course of the proof of the stationary phase formula, the final statement is valid for any field of characteristic zero. We point up that, independently, S. Bloch and H. Esnault have defined in [2] formal analogues of Laumon's local Fourier transformations for germs of meromorphic connections. In this case, their constructions give the same objects as ours, but both their methods and their applications significatively differ from those in the present article.

When the base field K is the field of complex numbers, from the formal statements we can obtain a 1-Gevrey variant of the stationary phase formula, and we use it to give a decomposition theorem for germs of meromorphic connections (the decompositions obtained are much rougher than the decomposition according to formal slopes, but they hold at the s-Gevrey level, $s > 0$). In section 2 we define an analogue of Laumon's $(\infty, 0')$ local Fourier transform, and we use it for the study of the singularity at zero of the Fourier transform of $\mathbb{M}$ and to establish a long exact sequence of vanishing cycles. In section 3 we make a modest attempt to transpose part of the above constructions

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into the p-adic setting. We define a ring of p-adic microdifferential operators of finite order, we prove a division theorem for them and we show that, in some cases, the corresponding microlocalization functor has the relation one would expect with the p-adic Fourier transform (in a sense which is made precise in the introduction to section 3).

We will use some results from $\mathcal{D}$-module theory for which we refer e.g. to [14], [15]. Our proof of the formal stationary phase formula follows the leitfaden of the one given by C. Sabbah in [15] for modules with regular singularities. I thank C. Sabbah, B. Malgrange, G. Christol and W. Messing for their useful remarks. I thank also G. Lyubeznik and S. Sperber for their invitation to the University of Minnesota, during which part of this work was done.

1. Formal stationary phase. We will use the following notations:

i) Unless otherwise stated, all modules over a non-commutative ring will be left modules. We denote by $\mathbb{M}$ a holonomic module over the Weyl algebra $\mathbb{W}_t = K[t]\langle\partial_t\rangle$ (that is, we assume that $\mathbb{M}$ is finitely generated over $\mathbb{W}_t$ and for each $m \in \mathbb{M}$ there is an operator $P \in \mathbb{W}_t - \{0\}$ such that $P \cdot m = 0$). The rank of $\mathbb{M}$ is defined as $\mathrm{rank}(\mathbb{M}) := \dim_{K(t)} K(t) \otimes_{K[t]} \mathbb{M}$. Let $\overline{K}$ be an algebraic closure of K. There is a maximal Zariski open subset $U \subset \mathbb{A}^1_{\overline{K}}$ such that the restriction of $\mathbb{M}$ to U is of finite type over the ring of regular functions on U, by definition the set of singular points of $\mathbb{M}$ is $Sing(\mathbb{M}) = \mathbb{A}^1_{\overline{K}} - U$. We will assume the points of $Sing(\mathbb{M})$ are in K.

ii) The Fourier antiinvolution is the morphism of K-algebras $\mathbb{W}_t \to \mathbb{W}_\eta$ given by $t \mapsto -\partial_\eta$, $\partial_t \mapsto \eta$. The Fourier transform of $\mathbb{M}$ is defined as the $\mathbb{W}_\eta$-module $\widehat{\mathbb{M}} := \mathbb{W}_\eta \otimes_{\mathbb{W}_t} \mathbb{M}$, where $\mathbb{W}_\eta$ is regarded as a right $\mathbb{W}_t$-module via the Fourier morphism. If $m \in \mathbb{M}$, we put $\widehat{m} = 1 \otimes m \in \widehat{\mathbb{M}}$.

iii) We will set $\mathcal{K}_{\eta^{-1}} := K[[\eta^{-1}]][\eta]$ and we consider on this field the derivation $\partial_{\eta^{-1}} = -\eta^2\partial_\eta$. If $\mathcal{V}$ is a $\mathcal{K}_{\eta^{-1}}$-vector space, a connection on $\mathcal{V}$ is a K-linear map $\nabla : \mathcal{V} \to \mathcal{V}$ satisfying the Leibniz rule

$$\nabla(\alpha \cdot v) = \partial_{\eta^{-1}}(\alpha) \cdot v + \alpha \cdot \nabla(v) \quad \text{for all} \quad \alpha \in \mathcal{K}_{\eta^{-1}}, \ v \in \mathcal{V}.$$

iv) We set $\mathbb{M}_\infty = K[[t^{-1}]]\langle\partial_t\rangle \otimes_{K[t^{-1}]\langle\partial_t\rangle} \mathbb{M}[t^{-1}]$, and for $c \in K$ we put $t_c = t - c$, $\mathbb{M}_c = K[[t_c]]\langle\partial_{t_c}\rangle \otimes_{\mathbb{W}_t} \mathbb{M}$.

We will consider the following rings:

i) *The ring $\mathcal{F}^{(c,\infty)}$ of formal microdifferential operators:* Let $c \in K$. For $r \in \mathbb{Z}$, we denote by $\mathcal{F}^{(c,\infty)}[r]$ the set of formal sums

$$\sum_{i \le r} a_i(t_c)\, \eta^i \quad \text{where} \quad a_i(t_c) \in K[[t_c]], r \in \mathbb{Z}.$$

We put $\mathcal{F}^{(c,\infty)} = \cup_r \mathcal{F}^{(c,\infty)}[r]$. For $P, Q \in \mathcal{F}^{(c,\infty)}$, their product is defined by the formula

$$P \cdot Q = \sum_{\alpha \geqslant 0} \frac{1}{\alpha!} \ \partial_\eta^\alpha P \cdot \partial_{t_c}^\alpha Q \ \in \mathcal{F}^{(c,\infty)} ,$$

where the product on the right hand side is the usual, commutative product. With this multiplication, $\mathcal{F}^{(c,\infty)}$ becomes a filtered ring and $\mathcal{F}^{(c,\infty)}[0]$ is a subring. One has a morphism of K-algebras given by

$$\begin{array}{rcl} \varphi^{(c,\infty)}: \mathbb{W}_t & \longrightarrow & \mathcal{F}^{(c,\infty)} \\ t & \mapsto & t_c + c \\ \partial_t & \mapsto & \eta \end{array}$$

which endows $\mathcal{F}^{(c,\infty)}$ with a structure of $(\mathbb{W}_t, \mathbb{W}_t)$-bimodule.

ii) *The ring $\mathcal{F}^{(\infty,\infty)}$:*
For $r \in \mathbb{Z}$, we denote by $\mathcal{F}^{(\infty,\infty)}[r]$ the set of formal sums

$$\sum_{i \leq r} a_i(t^{-1})\, \eta^i \quad \text{where} \quad a_i(t^{-1}) \in K[[t^{-1}]]\,, r \in \mathbb{Z}.$$

We put $\mathcal{F}^{(\infty,\infty)} = \cup_r \mathcal{F}^{(\infty,\infty)}[r]$. If $P, Q \in \mathcal{F}^{(\infty,\infty)}$, their product is given as above by

$$P \cdot Q = \sum_{\alpha \geq 0} \frac{1}{\alpha\,!}\ \partial_\eta^\alpha P \cdot \partial_t^\alpha Q\ .$$

Again, $\mathcal{F}^{(\infty,\infty)}$ is a filtered ring and $\mathcal{F}^{(\infty,\infty)}[0]$ is a subring. One has a morphism of K-algebras

$$\begin{array}{rcl} \varphi^{(\infty,\infty)}: K[t^{-1}]\langle\partial_t\rangle & \longrightarrow & \mathcal{F}^{(\infty,\infty)} \\ t^{-1} & \mapsto & t^{-1} \\ \partial_t & \mapsto & \eta \end{array}$$

(notice that on the ring $K[t^{-1}]\langle\partial_t\rangle$, one has the relation $[\partial_t, t^{-1}] = -t^{-2}$). The morphism $\varphi^{(\infty,\infty)}$ endows $\mathcal{F}^{(\infty,\infty)}$ with a structure of $(\,K[t^{-1}]\langle\partial_t\rangle\,,\, K[t^{-1}]\langle\partial_t\rangle\,)$ - bimodule.

If $P = \sum_{i\in\mathbb{Z}} a_i(t_c)\,\eta^i \in \mathcal{F}^{(c,\infty)}$, the order of P is the largest integer r such that $a_r(t_c) \neq 0$, and we define the principal symbol of P as $\sigma(P) = a_{ord(P)}(t_c)$. We define similarly the order and the principal symbol of an operator $P \in \mathcal{F}^{(\infty,\infty)}$. Principal symbols are multiplicative, in the sense that $\sigma(P \cdot Q) = \sigma(P) \cdot \sigma(Q)$.

We recall next some results which are well-known for the rings $\mathcal{F}^{(c,\infty)}$. The proofs for $\mathcal{F}^{(\infty,\infty)}$ follow a similar pattern (using the fact that the graded ring associated to the filtration on $\mathcal{F}^{(\infty,\infty)}$ is isomorphic to $K[[t^{-1}, x]][x^{-1}]$), and therefore they are omitted.

(1.1) Division theorem (see e.g. [1, Ch.4, 2.6.]). *Let $F \in \mathcal{F}^{(c,\infty)}$ and assume that $\sigma(F) = t_c^m\, b(t_c)$ where $b(0) \neq 0$. Then, for all $G \in \mathcal{F}^{(c,\infty)}$, there exist unique $Q \in \mathcal{F}^{(c,\infty)}$ and $R_0, \ldots, R_{m-1} \in \mathcal{K}_{\eta^{-1}}$ such that*

$$G = Q \cdot F + R_{m-1}\, t_c^{m-1} + \ldots + R_0.$$

The same statement holds for the ring $\mathcal{F}^{(\infty,\infty)}$, replacing t_c by t^{-1}.

Proposition ([1, Ch.4, 2.1 and 2.9]). *The rings $\mathcal{F}^{(c,\infty)}$ and $\mathcal{F}^{(\infty,\infty)}$ are left and right noetherian.*

Proposition ([1, Ch.5, §5]). *$\mathcal{F}^{(c,\infty)}$ is a flat left and right $\mathbb{W}_t$-module. $\mathcal{F}^{(\infty,\infty)}$ is a flat left and right $K[t^{-1}]\langle\partial_t\rangle$-module.*

We will consider the following modules:

i) *The (ordinary) microlocalization of* $\mathbb{M}$ *at* $c \in K$: It is defined as

$$\mathcal{F}^{\,(\,c,\infty)}(\mathbb{M}) := \mathcal{F}^{\,(\,c,\infty)} \otimes_{\mathbb{W}_t} \mathbb{M}.$$

where $\mathcal{F}^{\,(\,c,\infty)}$ is viewed as a right $\mathbb{W}_t$-module via $\varphi^{(c,\infty)}$. It has a structure of $\mathcal{K}_{\eta^{-1}}$-vector space with a connection given by left multiplication by
$\eta^2 \cdot (t_c + c) = \eta^2 \cdot t$. Notice that

$$\mathcal{F}^{\,(\,c,\infty)}(\mathbb{M}) \cong \mathcal{F}^{\,(\,c,\infty)} \otimes_{K[[t_c]]\langle\partial_{t_c}\rangle} (K[[t_c]]\langle\partial_{t_c}\rangle \otimes_{\mathbb{W}_t} \mathbb{M}) = \mathcal{F}^{\,(\,c,\infty)} \otimes_{K[[t_c]]\langle\partial_{t_c}\rangle} \mathbb{M}_c\,,$$

thus $\mathcal{F}^{\,(\,c,\infty)}(\mathbb{M})$ depends only on the formal germ $\mathbb{M}_c$.

ii) *The* $(\,\infty\,,\infty\,)$*-microlocalization of* $\mathbb{M}$: It is defined as

$$\mathcal{F}^{\,(\,\infty,\infty)}(\mathbb{M}) := \mathcal{F}^{\,(\,\infty,\infty)} \otimes_{K[t^{-1}]\langle\partial_t\rangle} \mathbb{M}\,[t^{-1}]\,,$$

where $\mathcal{F}^{\,(\,\infty,\infty)}$ is viewed as a $K[t^{-1}]\langle\partial_t\rangle$-module via the morphism $\varphi^{(\,\infty,\infty\,)}$. Again, it has a structure of $\mathcal{K}_{\eta^{-1}}$-vector space with a connection defined by

$$\nabla(\alpha\otimes m) := \partial_{\eta^{-1}}(\alpha)\otimes m \,+\, \eta^2\alpha\otimes t\cdot m\,,$$

and it depends only on $\mathbb{M}_\infty$, since one has

$$\mathcal{F}^{\,(\,\infty,\infty)}(\mathbb{M}) := \mathcal{F}^{\,(\,\infty,\infty)} \otimes_{K[[t^{-1}]]\langle\partial_t\rangle} \mathbb{M}_\infty\,.$$

By flatness of $\varphi^{(\,0,\infty\,)}$ and $\varphi^{(\,\infty,\infty\,)}$, both microlocalizations define exact functors.

For the proof of the formal stationary phase formula we will need the following result:

Proposition 1. *Let* $Q(t^{-1},\partial_t) = \sum_{v=1}^n b_v(t^{-1})\,\partial_t^v \in K[t^{-1}]\langle\partial_t\rangle$ *be such that there is at least one index* $v \in \{1,\dots,n\}$ *with* $b_v(0)\neq 0$. *Then, there is an isomorphism of* $K[t^{-1}]\langle\partial_t\rangle$*-modules*

$$\mathcal{F}^{\,(\,\infty,\infty)} \otimes_{K[t^{-1}]\langle\partial_t\rangle} \frac{K[t^{-1}]\langle\partial_t\rangle}{K[t^{-1}]\langle\partial_t\rangle\cdot Q} \cong \mathcal{F}^{\,(\,\infty,\infty)} \otimes_{K[t^{-1}]\langle\partial_t\rangle} \frac{K[t,t^{-1}]\langle\partial_t\rangle}{K[t,t^{-1}]\langle\partial_t\rangle\cdot Q}\,.$$

Proof. We show first that the natural map

$$\beth: \frac{K[t^{-1}]\langle\partial_t\rangle}{K[t^{-1}]\langle\partial_t\rangle\cdot Q} \longrightarrow \frac{K[t,t^{-1}]\langle\partial_t\rangle}{K[t,t^{-1}]\langle\partial_t\rangle\cdot Q}$$

is injective. Assume we have $A(t,t^{-1},\partial_t) \in K[t,t^{-1}]\langle\partial_t\rangle$ such that $A(t,t^{-1},\partial_t)\cdot Q(t^{-1},\partial_t) \in K[t^{-1}]\langle\partial_t\rangle$, we have to show that in fact $A \in K[t^{-1}]\langle\partial_t\rangle$. Write $A = \sum_u a_u(t,t^{-1})\,\partial_t^u$ with $a_u(t,t^{-1}) \in K[t,t^{-1}]$. Let v_0 be the largest index with $b_{v_0}(0)\neq 0$ and let $k_0\in\mathbb{N}$ be the largest exponent of t appearing in the Laurent polynomials $\{a_u\}_u$ (if $a_u \in K[t^{-1}]$ for all u, we are done). Let u_0 be the largest index such that a_{u_0} contains a monomial $\beta t^{k_0}\neq 0$, $\beta\in K$. Set $j_0 = u_0+v_0$. The coefficient of $\partial_t^{j_0}$ in $A\cdot Q$ is

$$\sum_{\substack{u,v,\alpha\\ j_0=u+v-\alpha}} \frac{1}{\alpha\,!}\,u(u-1)\dots(u-\alpha+1)\,a_u\,\frac{d^\alpha b_v}{d\,t^\alpha}\ .$$

The monomial $\beta b_{v_0}(0)t^{k_0}$ appearing in the summand corresponding to $u = u_0, v = v_0, \alpha = 0$ cannot be cancelled, because in the other summands either $u > u_0$, and then in $a_u\, d^\alpha b_v/d\,t^\alpha$ all powers of t appear with exponent strictly smaller than k_0, or else $u \leqslant u_0$, and then $d^\alpha b_v/d\,t^\alpha \in t^{-1}K[t^{-1}]$, thus the exponents of t in these summands are also strictly smaller than k_0. Since $A \cdot Q \in K[t^{-1}]\langle\partial_t\rangle$, we conclude that $k_0 = 0$, which proves the injectivity of $\daleth$.

By flatness of $\mathcal{F}^{(\infty,\infty)}$ over $K[t^{-1}]\langle\partial_t\rangle$, the map $\mathrm{Id}_{\mathcal{F}^{(\infty,\infty)}} \otimes \daleth$ is injective as well. In order to show that it is a surjection, we prove first the following statement:

Claim: For all $P \in K[t,t^{-1}]\langle\partial_t\rangle$, there exists a polynomial $p(x) \in K[x] - \{0\}$ such that $p(\partial_t) \cdot P \in K[t,t^{-1}]\langle\partial_t\rangle \cdot Q + K[t^{-1}]\langle\partial_t\rangle$.

Proof of the claim. Let us denote by $\Omega \subseteq K[t,t^{-1}]\langle\partial_t\rangle$ the set of differential operators $P \in K[t,t^{-1}]\langle\partial_t\rangle$ satisfying the condition of the claim, notice that Ω is closed under addition. It suffices to show that $t^i \in \Omega$ for all $i \geqslant 1$, because if there is a $p(x)$ with $p(\partial_t) \cdot t^i = \alpha(t,t^{-1},\partial_t) \cdot Q + \beta(t^{-1},\partial_t)$ then, multiplying on the right by ∂_t^j, we obtain that $\partial_t^j \cdot t^i \in \Omega$ for all $i,j \geqslant 0$, and then we are done. We prove $t^i \in \Omega$ by induction on $i \geqslant 1$: By our hypothesis on Q there exists a non-zero polynomial $p(x) \in K[x]$ such that $Q = p(\partial_t) + \beta(t^{-1},\partial_t)$ with $\beta(t^{-1},\partial_t) \in t^{-1}K[t^{-1}]\langle\partial_t\rangle$ (we will denote $p' = \frac{dp}{dx}$). Now, multiplying this equality on the left by t and using $t\,p(\partial_t) = p(\partial_t)\,t - p'(\partial_t)$, case $i = 1$ follows. For the induction step, assume we have

$$p(\partial_t) \cdot t^i = \alpha(t,t^{-1},\partial_t) \cdot Q + \beta(t^{-1},\partial_t).$$

Since we have also $p(\partial_t)\,t^{i+1} = t\,p(\partial_t)t^i + p'(\partial_t)\,t$, substituting we get

$$p(\partial_t)\,t^{i+1} = t(\alpha\,Q + \beta) + p'(\partial_t)\,t$$

Now, from the case $i = 1$ it follows that we have $t\,\beta(t^{-1},\partial_t) \in \Omega$ and $p'(\partial_t)\,t \in \Omega$, so the claim is proved. Given

$$F(t^{-1},\eta) \otimes P(t,t^{-1},\partial_t) \in \mathcal{F}^{(\infty,\infty)} \otimes \frac{K[t,t^{-1}]\langle\partial_t\rangle}{K[t,t^{-1}]\langle\partial_t\rangle \cdot Q}$$

choose $p(x) \neq 0$ such that $p(\partial_t) \cdot P \in K[t,t^{-1}]\langle\partial_t\rangle \cdot Q + K[t^{-1}]\langle\partial_t\rangle$. By the division theorem $p(\eta)$ is invertible in $\mathcal{F}^{(\infty,\infty)}$, thus we have $F \otimes P = F \cdot p(\eta)^{-1} \otimes p(\partial_t)\,P \in \mathrm{Im}\,[\mathrm{Id}_{\mathcal{F}^{(\infty,\infty)}} \otimes \daleth]$, and then the proposition is proved. ◻

Definition. If $\mathbb{N}$ is a $\mathbb{W}_\eta$-module, its formal germ at infinity is the $\mathcal{K}_{\eta^{-1}}$-vector space $\mathcal{N}_\infty = \mathcal{K}_{\eta^{-1}} \otimes_{K[\eta]} \mathbb{N}$, endowed with the connection defined by

$$\nabla(\alpha \otimes n) = \partial_{\eta^{-1}}(\alpha) \otimes n - \alpha \otimes \eta^2 \partial_\eta n\,.$$

The main result of this section is:

Theorem (formal stationary phase). *Let K be a field of characteristic zero, let $\mathbb{M}$ be a holonomic $K[t]\langle\partial_t\rangle$-module. Then, after a finite extension of the base field K, the map*

$$\Upsilon : \widehat{\mathcal{M}}_\infty \longrightarrow \bigoplus_{c \in Sing\mathbb{M} \cup \{\infty\}} \mathcal{F}^{(c,\infty)}(\mathbb{M})$$

given by $\Upsilon(\alpha\otimes\widehat{m}) = \oplus_c\ \alpha\otimes m$ *is an isomorphism of* $\mathcal{K}_{\eta^{-1}}$*-vector spaces with connection.*

Proof. The connections on the right hand side have been chosen so that the map is a morphism of $\mathcal{K}_{\eta^{-1}}$-vector spaces with connection, we have to show it is an isomorphism. We will assume all $\mathbb{W}_t$-modules appearing in what follows have K-rational singularities (which can be achieved after a finite extension of K). Consider first the Dirac modules $\delta_c = \mathbb{W}_t/\mathbb{W}_t(t-c)$. It is easy to check that we have

$$\mathcal{F}^{(\infty,\infty)}(\delta_c) = 0\,,\ \mathcal{F}^{(d,\infty)}(\delta_c) = 0 \text{ if } d\neq c\,,$$
$$\mathcal{K}_{\eta^{-1}}\otimes\widehat{\delta_c} = \mathcal{K}_{\eta^{-1}} = \mathcal{F}^{(c,\infty)}(\delta_c)\,,$$

so the theorem follows in this case.

For an arbitrary holonomic module $\mathbb{M}$, there is a differential operator $P\in\mathbb{W}_t$ and a $\mathbb{W}_t$-module with punctual support $\mathbb{K}$ so that one has an exact sequence

$$0\longrightarrow\mathbb{K}\longrightarrow\mathbb{W}_t/\mathbb{W}_t\cdot P\longrightarrow\mathbb{M}\longrightarrow 0\,.$$

A holonomic module with punctual support is a finite direct sum of Dirac δ-modules. Since both the global and the local Fourier transforms are exact functors, it will suffice to consider the case of a quotient of $\mathbb{W}_t$ by an operator. Moreover, given $\mathbb{M} = \mathbb{W}_t/\mathbb{W}_t\cdot P_1$, there is an operator $P_2\in\mathbb{W}_t$ such that one has

$$0\longrightarrow\mathbb{K}_1\longrightarrow\mathbb{M}\longrightarrow\mathbb{M}\,[t^{-1}] = \mathbb{W}_t/\mathbb{W}_t\cdot P_2\longrightarrow\mathbb{K}_2\longrightarrow 0$$

where $\mathbb{K}_i$ are supported at zero for $i = 1,2$ ([14, 4.2]). Thus, we will assume in what follows that $\mathbb{M} = \mathbb{M}\,[t^{-1}] = \mathbb{W}_t/\mathbb{W}_tP$, we write $P(t,\,\partial_t) = \sum_{i=0}^{d} a_i(t)\partial_t^i$ with $a_i(t)\in K[t]$.

Step 1: *The map* $\Upsilon_c : \widehat{\mathcal{M}}_\infty\to\mathcal{F}^{(c,\infty)}(\mathbb{M})$*, given by the composition of* Υ *with the projection onto* $\mathcal{F}^{(c,\infty)}(\mathbb{M})$*, is exhaustive for all* $c\in Sing(\mathbb{M})\cup\{\infty\}$*.*

Assume first that $c = 0$. Then $P(t,\eta)\in\mathcal{F}^{(0,\infty)}$ has a principal symbol of the form $\sigma(P) = t^{m_0}\,b(t)$ with $b(0)\neq 0$, $m_0\geqslant 0$, and we have

$$\mathcal{F}^{(0,\infty)}(\mathbb{M}) = \mathcal{F}^{(0,\infty)}/\mathcal{F}^{(0,\infty)}\cdot P.$$

Given $G\in\mathcal{F}^{(0,\infty)}$, by the division theorem

$$G\equiv\sum_{i=0}^{m_0-1} R_i\cdot t^i\qquad(\mathrm{mod}\ \mathcal{F}^{(0,\infty)}\,P)$$

where $R_i\in\mathcal{K}_{\eta^{-1}}$ $(0\leq i\leq m_0-1)$. Then, a preimage of the class of G in $\mathcal{F}^{(0,\infty)}(\mathbb{M})$ under Υ_c is given by

$$\sum_{i=0}^{m_0-1} R_i\otimes\widehat{t^i}\ \in\ \mathcal{K}_{\eta^{-1}}\otimes\widehat{\mathbb{M}}\,,$$

where $\widehat{t^i}$ denotes the element $1\otimes t^i\in\widehat{\mathbb{M}} = \mathbb{W}_\eta\otimes\mathbb{M}$. The proof for arbitrary $c\in K$ is done in the same way, using the division theorem in $\mathcal{F}^{(c,\infty)}$ (notice that the assumption $\mathbb{M} = \mathbb{M}[t^{-1}]$ has not been used up to now).

We consider now the case $c = \infty$. Write $\widehat{d} = \max_{j=1}^{d}\{\deg a_j(t)\}$, set $Q(t^{-1}, \partial_t) = t^{-\widehat{d}} P(t, \partial_t)$. Since Q satisfies the hypothesis of proposition 1 above, we have

$$\frac{K[t^{-1}]\langle\partial_t\rangle}{K[t^{-1}]\langle\partial_t\rangle \cdot Q} \cong \frac{K[t,t^{-1}]\langle\partial_t\rangle}{K[t,t^{-1}]\langle\partial_t\rangle \cdot Q} = \mathbb{M}$$

as $K[t^{-1}]\langle\partial_t\rangle$-modules. Then we have

$$\mathcal{F}^{(\infty,\infty)}(\mathbb{M}) \cong \frac{\mathcal{F}^{(\infty,\infty)}}{\mathcal{F}^{(\infty,\infty)} \cdot Q},$$

and notice that the principal symbol of Q is of the form $\sigma(Q) = t^{\deg(a_d(t)) - \widehat{d}} \cdot b(t^{-1})$ with $b(0) \neq 0$. Given $G \in \mathcal{F}^{(\infty,\infty)}$, by the division theorem

$$G \equiv \sum_i R_i \cdot t^{-i} \pmod{\mathcal{F}^{(\infty,\infty)} Q}$$

where $i \in \{0, \ldots, \widehat{d} - \deg(a_d(t))\}$. Since $\mathbb{M} = \mathbb{M}[t^{-1}]$, we have that $\sum_i R_i \otimes \widehat{t^{-i}} \in \widehat{\mathcal{M}}_\infty$ is a preimage of G under Υ_∞.

Step 2: $\dim_{\mathcal{K}_{\eta^{-1}}} \widehat{\mathcal{M}}_\infty = \dim_{\mathcal{K}_{\eta^{-1}}} \left[\left(\bigoplus_{c \in Sing\,\mathbb{M}} \mathcal{F}^{(c,\infty)}(\mathbb{M})\right) \oplus \mathcal{F}^{(\infty,\infty)}(\mathbb{M}_\infty)\right]$.

Put $a_d(t) = \prod_{c \in Sing\,\mathbb{M}} (t-c)^{m_c}$. It follows from the existence and uniqueness of division for $\mathcal{F}^{(c,\infty)}$ and $\mathcal{F}^{(\infty,\infty)}$ that we have $\dim_{\mathcal{K}_{\eta^{-1}}} \mathcal{F}^{(c,\infty)}(\mathbb{M}) = m_c$, $\dim_{\mathcal{K}_{\eta^{-1}}} \mathcal{F}^{(\infty,\infty)}(\mathbb{M}) = \widehat{d} - \deg(a_d(t))$. Since $\widehat{d} = \operatorname{rank}(\widehat{\mathbb{M}}) = \dim_{\mathcal{K}_{\eta^{-1}}} \widehat{\mathcal{M}}_\infty$, the claimed equality follows.

Step 3:

a) All slopes of $\mathcal{F}^{(0,\infty)}(\mathbb{M})$ *are strictly smaller than* $+1$.

b) For $c \in K - \{0\}$*, all slopes of* $\mathcal{F}^{(c,\infty)}(\mathbb{M})$ *are equal to* $+1$.

c) All slopes of $\mathcal{F}^{(\infty,\infty)}(\mathbb{M})$ *are strictly greater than* $+1$.

We assume first that the base field K is the field $\mathbb{C}$ of complex numbers. Then, there is a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module $\mathbb{N}$ which is only singular at 0 and ∞, the singularity at infinity is regular and for the singularity at zero we have $\mathbb{C}[[t]]\langle\partial_t\rangle \otimes_{\mathbb{W}_t} \mathbb{N} \cong \mathbb{C}[[t]]\langle\partial_t\rangle \otimes_{\mathbb{W}_t} \mathbb{M}$ (this is the transcendental step in the proof, see [7], it follows that $\mathcal{F}^{(0,\infty)}(\mathbb{M}) \cong \mathcal{F}^{(0,\infty)}(\mathbb{N})$). Let $\mathbb{L} \twoheadrightarrow \mathbb{N}$ be a surjection where $\mathbb{L}$ is the quotient of $\mathbb{C}[t]\langle\partial_t\rangle$ by the left ideal generated by a single differential operator. Then we have

$$\begin{tikzcd}
\mathcal{K}_{\eta^{-1}} \otimes_{\mathbb{C}[\eta^{-1}]} \widehat{\mathbb{L}} \arrow[r, two heads] \arrow[d, "\Upsilon_0^{\mathbb{L}}"] & \mathcal{K}_{\eta^{-1}} \otimes_{\mathbb{C}[\eta^{-1}]} \widehat{\mathbb{N}} \arrow[d, "\Upsilon_0^{\mathbb{N}}"] \\
\mathcal{F}^{(0,\infty)}(\mathbb{L}) \arrow[r, two heads] & \mathcal{F}^{(0,\infty)}(\mathbb{N})
\end{tikzcd}$$

where we denote $\Upsilon_0^{\mathbb{L}}$ (respectively, $\Upsilon_0^{\mathbb{N}}$) the map defined in step 1 for the module $\mathbb{L}$ (resp., for $\mathbb{N}$) and the horizontal arrows are surjective. By step 1, the map $\Upsilon_0^{\mathbb{L}}$ is onto, and then so is $\Upsilon_0^{\mathbb{N}}$. But the behavior of formal slopes under Fourier transform is well-known (see e.g. [8, V.1]), in particular the slopes of $\mathcal{K}_{\eta^{-1}} \otimes_{\mathbb{C}[\eta]} \widehat{\mathbb{N}}$ are strictly smaller than $+1$, and a) follows.

For b), take now a $\mathbb{C}[t]\langle\partial_t\rangle$-module $\mathbb{N}^c$ with singularities only at $c \in \mathbb{C} - \{0\}$ and at infinity, and such that one has an isomorphism of formal germs $\mathbb{C}[[t_c]]\langle\partial_{t_c}\rangle \otimes \mathbb{N}^c \cong \mathbb{C}[[t_c]]\langle\partial_{t_c}\rangle \otimes \mathbb{M}$ and the singularity at infinity of $\mathbb{N}^c$ is regular. Then (as in a) above) the corresponding map $\Upsilon_c^{\mathbb{N}^c}$ is onto, and since $\mathcal{K}_{\eta^{-1}} \otimes_{\mathbb{C}[\eta]} \widehat{\mathbb{N}^c}$ has only slope $+1$ at infinity, b) follows.

For the proof of c) consider first a $\mathbb{C}[t]\langle\partial_t\rangle$-module $\mathbb{N}^0$ with a regular singularity at zero, no other singularity in $\mathbb{C}$, and $\mathbb{N}^0_\infty \cong \mathbb{M}_\infty$ (then $\mathcal{F}^{(\infty,\infty)}(\mathbb{M}) \cong \mathcal{F}^{(\infty,\infty)}(\mathbb{N}^0)$). Let $\mathbb{L}^0 \twoheadrightarrow \mathbb{N}^0$ be a surjection where $\mathbb{L}^0$ is the quotient of $\mathbb{C}[t]\langle\partial_t\rangle$ by a single differential operator. Then $\mathbb{L}^0[t^{-1}]$ is given by a single differential operator as well and, as in case a) above, the map

$$\Upsilon_\infty^{\mathbb{N}^0[t^{-1}]} : \mathcal{K}_{\eta^{-1}} \otimes_{\mathbb{C}[\eta]} \widehat{\mathbb{N}^0[t^{-1}]} \longrightarrow \mathcal{F}^{(\infty,\infty)}(\mathbb{N}^0[t^{-1}]) \cong \mathcal{F}^{(\infty,\infty)}(\mathbb{M})$$

is onto (the last isomorphism holds because $\mathbb{M}_\infty \cong \mathbb{N}^0_\infty = \mathbb{N}^0[t^{-1}]_\infty$).

Since the slopes of $K_{\eta^{-1}} \otimes_{\mathbb{C}[\eta]} \widehat{\mathbb{N}^0[t^{-1}]}$ are either zero or strictly greater than $+1$, the same holds for $\mathcal{F}^{(\infty,\infty)}(\mathbb{M})$.

Let now $\mathbb{N}^1$ denote the pull back of $\mathbb{N}^0$ by the translation $t \mapsto t+1$. In the same way we get a surjection

$$\Upsilon_\infty^{\mathbb{N}^1[t^{-1}]} : \mathcal{K}_{\eta^{-1}} \otimes_{\mathbb{C}[\eta]} \widehat{\mathbb{N}^1[t^{-1}]} \longrightarrow \mathcal{F}^{(\infty,\infty)}(\mathbb{N}^1[t^{-1}]) \cong \mathcal{F}^{(\infty,\infty)}(\mathbb{M})$$

and the slopes of $K_{\eta^{-1}} \otimes_{\mathbb{C}[\eta]} \widehat{\mathbb{N}^1}$ are greater or equal than $+1$. Thus all slopes of $\mathcal{F}^{(\infty,\infty)}(\mathbb{M})$ must be strictly greater than $+1$, and then we are done when the base field is $\mathbb{C}$.

In the general case, there is a subfield $K_1 \subset K$ of finite transcendence degree over $\mathbb{Q}$ such that the module $\mathbb{M}$ is defined over K_1 and all its singular points are K_1-rational. Choosing an embedding of fields $K_1 \hookrightarrow \mathbb{C}$, the statement follows from the complex case treated above.

If $\mathcal{N}$ is a $\mathcal{K}_{\eta^{-1}}$-vector space with connection and $q \in \mathbb{Q}$, we denote by $\mathcal{N}_q$ its subspace of formal slope q (see e.g. [14, 5.3.1]), and we denote $\mathcal{N}_{<q}$ its subspace of slopes strictly smaller than q. If $\varphi : \mathcal{L} \to \mathcal{N}$ is a morphism, we denote by $\varphi_{<q} : \mathcal{L}_{<q} \to \mathcal{N}_{<q}$ the induced morphism (and similarly, we let $\varphi_{>q}$ denote the restriction of φ to the subspaces of slopes strictly bigger than q). For $c \in \mathbb{C}$, let $\mathcal{E}^c$ denote the one dimensional $\mathcal{K}_{\eta^{-1}}$- vector space with connection given by $\nabla(1) = c \cdot \eta^2$. It is easy to see that, if $c \neq 0$, then $\mathcal{E}^c$ has slope $+1$. Let $\tau_c : K[t] \to K[t]$ the translation given by $t \mapsto t+c$. One has an isomorphism of $\mathcal{K}_{\eta^{-1}}$- vector spaces with connection $\mathcal{E}^c \otimes \mathcal{F}^{(c,\infty)}(\mathbb{M}) \cong \mathcal{F}^{(0,\infty)}(\tau_c^*\mathbb{M})$ (from this isomorphism one can get an alternative proof of b) above). Notice also that $\mathcal{E}^c \otimes \mathcal{E}^{-c} \cong (\mathcal{K}_{\eta^{-1}}, \partial_{\eta^{-1}})$.

Step 4: Υ *is an isomorphism.* We remark first that if $\mathcal{M}$ is $\mathcal{K}_{\eta^{-1}}$- vector space with connection such that all its slopes are strictly smaller than $+1$, then for $c \neq 0$ the twisted vector space with connection $\mathcal{E}^c \otimes_{\mathcal{K}_{\eta^{-1}}} \mathcal{M}$ has only slope $+1$ (this can be easily seen using the structure theorem of formal meromorphic connections, [14, Theorem 5.4.7]).

Set $\widehat{\mathcal{M}}^c_\infty := \mathcal{E}^{-c}\otimes(\mathcal{E}^c\otimes\widehat{\mathcal{M}}_\infty)_{<1}\subset\widehat{\mathcal{M}}_\infty$. If $c,d\in Sing(\mathbb{M})$ are distinct, then the map

$$\text{id }\otimes\Upsilon_{c\,|\widehat{\mathcal{M}}^d_\infty}:\mathcal{E}^c\otimes\widehat{\mathcal{M}}^d_\infty\longrightarrow\mathcal{E}^c\otimes\mathcal{F}^{(c,\infty)}(\mathbb{M})\cong\mathcal{F}^{(0,\infty)}(\tau_c^*\mathbb{M})$$

is the zero map, because its source is purely of slope $+1$ and its target has slopes strictly smaller than $+1$. Tensoring with $\mathcal{E}^{-c}$, it follows that $\Upsilon_{c\,|\widehat{\mathcal{M}}^d_\infty}:\widehat{\mathcal{M}}^d_\infty\to\mathcal{F}^{(c,\infty)}(\mathbb{M}_c)$ is the zero map as well.

Since by step one Υ_c is onto, also is the map id $\otimes\Upsilon_c:\mathcal{E}^c\otimes\widehat{\mathcal{M}}_\infty\to\mathcal{E}^c\otimes\mathcal{F}^{(c,\infty)}(\mathbb{M})$. Since $\mathcal{E}^c\otimes\mathcal{F}^{(c,\infty)}(\mathbb{M})\cong\mathcal{F}^{(0,\infty)}(\tau_c^*\mathbb{M})$ has slopes strictly smaller than $+1$, the restriction

$$(\text{ id }\otimes\Upsilon_c)_{<1}:(\mathcal{E}^c\otimes\widehat{\mathcal{M}}_\infty)_{<1}\longrightarrow\mathcal{E}^c\otimes\mathcal{F}^{(c,\infty)}(\mathbb{M})$$

is onto as well, thus tensoring with $\mathcal{E}^{-c}$ follows that $\Upsilon_{c\,|\widehat{\mathcal{M}}^c_\infty}$ is onto.

Let Υ_∞ denote the composition of Υ with the projection onto $\mathcal{F}^{(\infty,\infty)}(\mathbb{M})$. The restriction $\Upsilon_{\infty,>1}:\widehat{\mathcal{M}}_{\infty,>1}\to\mathcal{F}^{(\infty,\infty)}(\mathbb{M})$ is onto while for $c\in K$, the maps $\Upsilon_{c,>1}$ are zero. Notice also that if $c\in Sing(\mathbb{M})$, then

$$\widehat{\mathcal{M}}^c_\infty\cap(\oplus_{d\neq c}\widehat{\mathcal{M}}^d_\infty\oplus\widehat{\mathcal{M}}_{\infty,>1})=\{0\},$$

because tensoring both $\widehat{\mathcal{M}}^c_\infty$ and $\oplus_{d\neq c}\widehat{\mathcal{M}}^d_\infty$ with $\mathcal{E}^c$, one obtains two subspaces of $\mathcal{E}^c\otimes\widehat{\mathcal{M}}_\infty$ with different slopes. Also, $\widehat{\mathcal{M}}_{\infty,>1}\cap(\oplus_c\widehat{\mathcal{M}}^c_\infty)=\{0\}$. Thus the map

$$(\oplus_c\Upsilon_c)\,\oplus\,\Upsilon_{\infty,>1}:(\oplus_c\widehat{\mathcal{M}}^c_\infty)\,\oplus\,\widehat{\mathcal{M}}_{\infty,>1}\to\Big(\bigoplus_{c\in Sing\mathbb{M}}\mathcal{F}^{(c,\infty)}(\mathbb{M}_c)\Big)\oplus\,\mathcal{F}^{(\infty,\infty)}(\mathbb{M}_\infty)$$

is an epimorphism, and then

$$\begin{gathered}\dim_{\mathcal{K}_{\eta^{-1}}}(\oplus_c\widehat{\mathcal{M}}^c_\infty)\,\oplus\,\widehat{\mathcal{M}}_{\infty,>1}\geqslant\\ \dim_{\mathcal{K}_{\eta^{-1}}}\Big(\bigoplus_{c\in Sing\mathbb{M}}\mathcal{F}^{(c,\infty)}(\mathbb{M}_c)\Big)\oplus\,\mathcal{F}^{(\infty,\infty)}(\mathbb{M}_\infty)=\dim_{\mathcal{K}_{\eta^{-1}}}\widehat{\mathcal{M}}_\infty,\end{gathered}$$

the last equality by step 2. It follows that $\widehat{\mathcal{M}}_\infty=(\oplus_c\widehat{\mathcal{M}}^c_\infty)\oplus\widehat{\mathcal{M}}_{\infty,>1}$, $\Upsilon=(\oplus_c\Upsilon_c)\oplus\Upsilon_{>1}$ and Υ is an isomorphism, as was to be proved. ◻

For the rest of this section we will assume that the base field K is the field $\mathbb{C}$ of complex numbers. In this case, one can consider the subrings of $\mathcal{F}^{(c,\infty)}$ $(c\in\mathbb{C}\cup\{\infty\})$ consisting of convergent microdifferential operators. These rings are well known for $c\in\mathbb{C}$, see for example [11] or [1], we will show that an analogous definition can be given for $c=\infty$.

Definition. We denote by $\mathcal{E}$ the set of formal series

$$\sum_{i\leq r}a_i(z)\,\eta^i\ \ ,r\in\mathbb{Z}\ \ a_i(z)\in\mathbb{C}[[z]]$$

such that

a) There exists a $\rho_0>0$ such that all series $a_i(z)$ are convergent in the disk $|\,z\,|<\rho_0$.

b) There exists a $0 < \rho < \rho_0$ and a $\theta > 0$ such that the series

$$\sum_{k \geqslant 0} \|a_{-k}(z)\|_{\rho} \frac{\theta^k}{k\,!}$$

is convergent, where $\|a_{-k}(z)\|_{\rho} = \sup_{|z| \leqslant \rho} \mid a_{-k}(z) \mid$.

Given $c \in \mathbb{C}$, we denote $\mathcal{E}^{\,(c,\infty)}$ the image of $\mathcal{E}$ by the map

$$\begin{aligned} \mathcal{E} &\longrightarrow \mathcal{F}^{\,(\,c,\infty)} \\ \sum_{i \leq r} a_i(z)\,\eta^i &\longrightarrow \sum_{i \leq r} a_i(t_c)\,\eta^i \end{aligned}$$

and similarly, we denote $\mathcal{E}^{\,(\infty,\infty)}$ the image of $\mathcal{E}$ by the map

$$\begin{aligned} \mathcal{E} &\longrightarrow \mathcal{F}^{\,(\,\infty,\infty)} \\ \sum_{i \leq r} a_i(z)\,\eta^i &\longrightarrow \sum_{i \leq r} a_i(t^{-1})\,\eta^i\ . \end{aligned}$$

One can prove that $\mathcal{E}^{\,(c,\infty)}$ is in fact a subring of $\mathcal{F}^{\,(\,c,\infty)}$ and that the division theorem (1.1) holds also for the rings $\mathcal{E}^{\,(c,\infty)}$ (see loci cit.). For $c = \infty$ these facts can be proved in a similar way, only some small modifications are needed. To illustrate them, we prove in some detail that $\mathcal{E}^{\,(\infty,\infty)}$ is a ring using a slight variation of the seminorms of Boutet de Monvel-Kree ([1, Chap. 4, §3]):

If (t, η) are coordinates in $\mathbb{C}^2$ and $\delta > 0$, we denote $\Delta_{\,\delta} \subset \mathbb{C}^2$ the open subset defined by the inequalities $\mid \eta \mid < \delta, \ \mid t \mid > \delta^{-1}$. Given series $a_{m-k}(z) \in \mathbb{C}[[z]]$ $(\,k \geqslant 0\,)$, convergent in some common disk $\mid z \mid < \delta_0$, put $F = \sum_{k \geqslant 0} a_{m-k}(t^{-1})\eta^{m-k}$ and for $0 < \delta < \delta_0$ consider the series

$$N_m(F;\delta;x) := \sum_{k,\alpha,\beta \geqslant 0} \frac{2^{-k+1}\ k\,!}{(k+\alpha)!\ (k+\beta)!}\ \|\partial_t^{\alpha}\,\partial_{\eta}^{\beta}\, a_{m-k}(t^{-1})\eta^{m-k}\|_{\Delta_{\delta}}\ x^{2k+\alpha+\beta}$$

where $\|\partial_t^{\alpha}\,\partial_{\eta}^{\beta}\, a_{m-k}(t^{-1})\eta^{m-k}\|_{\Delta_{\delta}} := \sup_{(t,\eta)\in\Delta_{\delta}}\{\mid \partial_t^{\alpha}\,\partial_{\eta}^{\beta}\, a_{m-k}(t^{-1})\eta^{m-k} \mid\}$.

Proposition. *i) If $F \in \mathcal{E}^{\,(\infty,\infty)}$, then there exist δ, $x > 0$ such that $N_m(F;\delta;x)$ is convergent.*

ii) If there exist δ, $x > 0$ such that $N_m(F;\delta;x)$ is convergent, then $F \in \mathcal{E}^{\,(\infty,\infty)}$.

iii) If $F, G \in \mathcal{E}^{\,(\infty,\infty)}$, then there is a $\delta > 0$ such that $N_{ord(FG)}(F \cdot G;\delta;x) \leqslant N_{ord(F)}(F;\delta;x) \cdot N_{ord(G)}(G;\delta;x)$, and thus $F \cdot G \in \mathcal{E}^{\,(\infty,\infty)}$.

Proof (cf. [1, Ch.4,§3]): i). If $F \in \mathcal{E}^{\,(\infty,\infty)}$, then there exist constants $A, C_1, \delta > 0$ such that

$$\|a_{m-k}(t^{-1})\eta^{m-k}\|_{\Delta_{2\delta}} \leqslant A \cdot k\,! \cdot C_1^k \ \text{ for all } k \geqslant 0.$$

From Cauchy's inequalities we get

$$(1) \qquad \|\partial_t^{\alpha}\,\partial_{\eta}^{\beta}\, a_{m-k}(t^{-1})\eta^{m-k}\|_{\Delta_{\delta}} \leqslant \alpha\,! \cdot \beta\,! \cdot \delta^{-\alpha-\beta} \cdot \|a_{m-k}(t^{-1})\eta^{m-k}\|_{\Delta_{2\delta}}$$

for δ small enough. Since $\alpha!\,k! \leqslant (\alpha+k)!$, $\beta!\,k! \leqslant (\beta+k)!$, putting $C_2 = \max\{\sqrt{C_1}, \delta^{-1}\}$, we get from (1) that $N_m(F;\delta;x) \leqslant A \sum 2^{-k+1}(C_2\,x)^{2k+\alpha+\beta}$, and this series is convergent for $0 < x < C_2^{-1}$.

Items ii) and iii) are proved as for the usual microdifferential operators, see loc. cit. □

The proof of the division theorem for the ring $\mathcal{E}^{(0,\infty)}$, as given in [1], relies on a series of combinatorial identities and on Cauchy's inequalities for analytic functions defined in polydisks. For the ring $\mathcal{E}^{(\infty,\infty)}$, these arguments can be modified as done above, and one obtains that the division theorem (1.1) holds also for the ring $\mathcal{E}^{(\infty,\infty)}$.

Definition ([12]). For $s \in \mathbb{R}^+$, we denote by $\mathcal{K}^s_x$ the field $\mathbb{C}\{x\}_s[x^{-1}]$, where $\mathbb{C}\{x\}_s$ is the ring of s-Gevrey series on the variable x, this is the ring of series $\sum_{i\geqslant 0} a_i\, x^i$, $a_i \in \mathbb{C}$, such that $\sum_{i\geqslant 0}(a_i/(i!)^s)\, x^i$ has non-zero convergence radius. In particular, $\mathcal{K}^0_x$ will denote the field $\mathbb{C}\{x\}[x^{-1}]$ of germs of meromorphic functions. For later use, we briefly recall the behavior under ramification of vector spaces with connection over these fields (cf. [7, (1.3)]): Let q be a positive integer, $s \in \mathbb{R}^+$ and set $\sigma = s/q$. The assignment $y \mapsto z^q$ defines a morphism of fields $\pi : \mathcal{K}^s_y \longrightarrow \mathcal{K}^\sigma_z$. Then:

i) If $\mathcal{V}$ is a $\mathcal{K}^s_y$-vector space with connection ∇_y, we put $\pi^*(\mathcal{V}) := \mathcal{K}^\sigma_z \otimes_{\mathcal{K}^s_y} \mathcal{V}$, and we endow this $\mathcal{K}^\sigma_z$-vector space with the connection ∇ defined by

$$z\nabla(\varphi \otimes v) = q\,(\varphi \otimes y\nabla_y(v)) + (z\frac{d\varphi}{dz} \otimes v).$$

If the slopes of $\mathcal{V}$ are $\lambda_1, \dots, \lambda_r$, those $\pi^*(\mathcal{V})$ are $q\,\lambda_1, \dots, q\,\lambda_r$.

ii) If $\mathcal{V}$ is a $\mathcal{K}^\sigma_z$-vector space with connection ∇_z, we denote $\pi_*(\mathcal{V})$ the set $\mathcal{V}$ regarded as a vector space over $\mathcal{K}^s_y$ by restriction of scalars and endowed with the connection $\nabla := \frac{1}{q\,z^{q-1}}\nabla_z$. If the slopes of $\mathcal{V}$ are $\lambda_1, \dots, \lambda_r$, those $\pi_*(\mathcal{V})$ are $\lambda_1/q, \dots, \lambda_r/q$ (each one repeated q times).

Definition. If $\mathbb{N}$ is a $\mathbb{W}_\eta$-module, the s-Gevrey germ at infinity defined by $\mathbb{N}$ is the $\mathcal{K}^s_{\eta^{-1}}$-vector space $\mathcal{K}^s_{\eta^{-1}} \otimes_{\mathbb{C}[\eta]} \mathbb{N}$, endowed with the same connection as in the formal case (that is, $\nabla(\alpha \otimes n) = \partial_{\eta^{-1}}(\alpha) \otimes n - \alpha \otimes \eta^2 \partial_\eta n$).

As in the formal case, given a holonomic $\mathbb{W}_t$-module $\mathbb{M}$ and $c \in \mathbb{C}$, its microlocalization $\mathcal{E}^{(c,\infty)}(\mathbb{M}) := \mathcal{E}^{(c,\infty)} \otimes_{\mathbb{W}_t} \mathbb{M}$ is a $\mathcal{K}^1_{\eta^{-1}}$-vector space endowed with the connection given by left multiplication by $\eta^2\, t$ and one defines similarly $\mathcal{E}^{(\infty,\infty)}(\mathbb{M})$. We have

Theorem (1-Gevrey stationary phase). *Let $\mathbb{M}$ be a holonomic $\mathbb{W}_t$-module. Then the map*

$$\Upsilon^{Gev} : \mathcal{K}^1_{\eta^{-1}} \otimes_{\mathbb{C}[\eta]} \widehat{\mathbb{M}} \longrightarrow \bigoplus_{c \in\, Sing\mathbb{M} \,\cup\, \{\infty\}} \mathcal{E}^{(c,\infty)}(\mathbb{M})$$

given by $\Upsilon^{Gev}(\alpha \otimes \widehat{m}) = \oplus_c\ \alpha \otimes m$ is an isomorphism of $\mathcal{K}^1_{\eta^{-1}}$-vector spaces with connection.

Proof. Again as in the formal case, the map Υ^{Gev} is a morphism of $\mathcal{K}^1_{\eta^{-1}}$-vector spaces with connection, and we have to prove that it is an isomorphism. The case of a module with punctual support is easy and left to the reader, so we assume that $\mathbb{M} = \mathbb{W}_t/\mathbb{W}_t \cdot P$. Then, it follows from the division theorem for the rings $\mathcal{E}^{(c,\infty)}$ $(c \in Sing(\mathbb{M}) \cup \{\infty\})$, that the dimension of the source and the target of Υ^{Gev} are equal. So, it will be enough to prove that the map

$$id_{\mathbb{C}[[\eta^{-1}]][\eta]} \otimes \Upsilon^{Gev} : \widehat{\mathcal{M}}_\infty \longrightarrow \mathbb{C}[[\eta^{-1}]][\eta] \otimes_{\mathcal{K}^1_{\eta^{-1}}} (\oplus_c \mathcal{E}^{(c,\infty)}(\mathbb{M}))$$

is injective. But we have a commutative diagram of $\mathbb{C}[[\eta^{-1}]][\eta]$-vector spaces

$$\begin{tikzcd} \widehat{\mathcal{M}}_\infty \arrow[rr, "id \otimes \Upsilon^{Gev}"] \arrow[dr, "\Upsilon"'] & & \mathbb{C}[[\eta^{-1}]][\eta] \otimes_{\mathcal{K}^1_{\eta^{-1}}} (\oplus_c \mathcal{E}^{(c,\infty)}(\mathbb{M})) \arrow[dl, "\mathfrak{m}"] \\ & \oplus_c \mathcal{F}^{(c,\infty)}(\mathbb{M}) & \end{tikzcd}$$

where $\mathfrak{m}$ is given by $\mathfrak{m}(\varphi \otimes (\oplus_c \xi_c)) = \oplus_c \varphi \cdot \xi_c$. Since Υ is an isomorphism (by formal stationary phase), we are done. $\square$

Remarks. i) The theorem is proved in [8] when $Sing(\mathbb{M}) = \{0\}$ and the singularity at infinity of $\mathbb{M}$ has slopes strictly smaller than $+1$, by a different method.

ii) It is a consequence of the above proof that the map $\mathfrak{m}$ is also an isomorphism. Notice that this fails if $\mathbb{M}$ is not holonomic, for example the multiplication map $\mathbb{C}[[\eta^{-1}]][\eta] \otimes_{\mathcal{K}^1_{\eta^{-1}}} \mathcal{E}^{(c,\infty)} \longrightarrow \mathcal{F}^{(c,\infty)}$ is clearly not onto.

iii) Let $\mathbb{M} = \mathbb{W}_t/\mathbb{W}_t \cdot P(t,\partial_t)$ be a holonomic $\mathbb{W}_t$-module such that its formal slopes at infinity are smaller or equal than $+1$. Set $\mathcal{D}^i = \mathcal{K}^i_{\eta^{-1}}\langle\partial_{\eta^{-1}}\rangle$ $(i = 0, 1)$. For some $k \geqslant 0$ we will have $Q = \eta^{-k} P(-\partial_\eta, \eta) \in \mathbb{C}[\eta^{-1}]\langle\partial_{\eta^{-1}}\rangle$ (using $\partial_\eta = -\eta^{-2}\partial_{\eta^{-1}}$). Consider the two complexes of $\mathbb{C}$-vector spaces

$$C_i : \qquad 0 \longrightarrow \mathcal{K}^i_{\eta^{-1}} \xrightarrow{Q} \mathcal{K}^i_{\eta^{-1}} \longrightarrow 0 \qquad (i = 0, 1).$$

There is an obvious morphism of complexes $C_0 \to C_1$. Since the formal slopes at infinity of $\mathbb{M}$ are smaller or equal than $+1$, it follows from the results of J.P.Ramis in [12, 1.5.11, 1.5.14] that this morphism is a quasi-isomorphism. Since the complex C_i is quasi-isomorphic to $\mathbb{R}\mathrm{Hom}_{\mathcal{D}^i}(\mathcal{K}^i_{\eta^{-1}} \otimes \widehat{\mathbb{M}}, \mathcal{K}^i_{\eta^{-1}})$, by the theorem above we have a quasi-isomorphism of solution complexes

$$\mathbb{R}\mathrm{Hom}_{\mathcal{D}^0}(\mathcal{K}^0_{\eta^{-1}} \otimes \widehat{\mathbb{M}}, \mathcal{K}^0_{\eta^{-1}}) \cong \oplus_{c \in Sing\mathbb{M}} \mathbb{R}\mathrm{Hom}_{\mathcal{D}^1}(\mathcal{E}^{(c,\infty)}(\mathbb{M}), \mathcal{K}^1_{\eta^{-1}}) .$$

That is, we can compute microlocally the stalk at infinity of the complex of meromorphic solutions of $\widehat{\mathbb{M}}$.

iv) The theorem above can be applied to study to which extent the formal decomposition of a $\mathbb{C}\{x\}[x^{-1}]$-vector space with connection given by its slopes holds at the s-Gevrey level, namely one has:

THEOREM. *Let $s \in \mathbb{R}^+ - \{0\}$ and let $\mathcal{V}$ be a finitely dimensional $\mathcal{K}^s_x$-vector space with connection. Then there exist $\mathcal{K}^s_x$-vector spaces with connection $\mathcal{V}_{<1/s}, \mathcal{V}_{=1/s}, \mathcal{V}_{>1/s}$ of formal slopes strictly smaller than $\frac{1}{s}$ (respectively, equal to $\frac{1}{s}$, strictly greater than $\frac{1}{s}$) and an isomorphism of $\mathcal{K}^s_x$-vector spaces with connection*

$$\mathcal{V} \cong \mathcal{V}_{<1/s} \oplus \mathcal{V}_{=1/s} \oplus \mathcal{V}_{>1/s}.$$

Proof. We consider first the case $s = 1$, set $u = 1/x$. From a theorem of Malgrange–Ramis ([12, 3.2.13], [13, 7.1], cf. also [7]) on algebraization of s-Gevrey spaces with connection, it follows that there is a holonomic $\mathbb{C}[u]\langle\partial_u\rangle$-module $\mathbb{N}$ and an isomorphism of $\mathcal{K}^1_x$-vector spaces with connection $\mathcal{V} \cong \mathcal{K}^1_x \otimes \mathbb{N}$ (the right hand side denotes the 1-Gevrey germ at infinity defined by $\mathbb{N}$, although the coordinate at infinity is labelled now x instead of η^{-1}). Let $\mathbb{M}$ denote the inverse Fourier transform of $\mathbb{N}$. Then, putting $\mathcal{V}_{<1} = \mathcal{E}^{(0,\infty)}(\mathbb{M})$, $\mathcal{V}_{=1} = \oplus_{c\in Sing\mathbb{M}-\{0\}}\mathcal{E}^{(c,\infty)}(\mathbb{M})$ and $\mathcal{V}_{>1} = \mathcal{E}^{(\infty,\infty)}(\mathbb{M})$, the claimed decomposition is the one given by the 1-Gevrey stationary phase formula.

We consider next the case $s = \frac{1}{q}$, q a positive integer (compare [7, (2.2)]). Consider the map $\pi : \mathcal{K}^1_y \longrightarrow \mathcal{K}^s_x$ given by $y \mapsto x^q$. By the previous case, we will have

$$\pi_*(\mathcal{V}) \cong \pi_*(\mathcal{V})_{<1} \oplus \pi_*(\mathcal{V})_{=1} \oplus \pi_*(\mathcal{V})_{>1}.$$

We have a surjective morphism of $\mathcal{K}^s_x$-vector spaces with connection $\alpha : \pi^*(\pi_*(\mathcal{V})) = \mathcal{K}^s_x \otimes \pi_*(\mathcal{V}) \longrightarrow \mathcal{V}$ given by $\alpha(\varphi \otimes v) = \varphi \cdot v$. Notice that all formal slopes of $\pi^*(\pi_*(\mathcal{V})_{<1})$ are strictly smaller than q, those of $\pi^*(\pi_*(\mathcal{V})_{=1})$ are equal to q, and those of $\pi^*(\pi_*(\mathcal{V})_{>1})$ are strictly bigger than q. Denote $\alpha_{<q}$ the restriction of α to $\pi_*(\pi^*(\mathcal{V})_{<1})$ (similarly in the cases $= q$, $> q$). Since the filtration by slopes is strict, we have a decomposition

$$\mathcal{V} \cong \alpha_{<q}(\pi_*(\pi^*(\mathcal{V})_{<1})) \oplus \alpha_{=q}(\pi_*(\pi^*(\mathcal{V})_{=1})) \oplus \alpha_{>q}(\pi_*(\pi^*(\mathcal{V})_{>1}))$$

as desired.

Assume now $s = \frac{p}{q}$ where $p, q \geqslant 1$ are integers. We consider the morphism $\pi : \mathcal{K}^s_x \longrightarrow \mathcal{K}^{1/q}_z$ given by $x \mapsto z^p$. By the previous case we have a decomposition of $\mathcal{K}^{1/q}_z$- vector spaces with connection

$$\pi^*(\mathcal{V}) \cong \pi^*(\mathcal{V})_{<q} \oplus \pi^*(\mathcal{V})_{=q} \oplus \pi^*(\mathcal{V})_{>q} \ ,$$

and an injective morphism of $\mathcal{K}^s_x$-vector spaces with connection $\beta : \mathcal{V} \longrightarrow \pi_*(\pi^*(\mathcal{V}))$ given by $v \mapsto 1 \otimes v$. Denote $\beta_{<q/p}$ the composition of β with the projection $\pi_*(\pi^*(\mathcal{V})) \twoheadrightarrow \pi_*(\pi^*(\mathcal{V})_{<q})$ and similarly for $\beta_{=q/p}$, $\beta_{>q/p}$. Again by strictness of the filtration by formal slopes, we have a decomposition

$$\mathcal{V} \cong (\mathrm{Ker}(\beta_{=q/p}) \cap \mathrm{Ker}(\beta_{>q/p})) \oplus (\mathrm{Ker}(\beta_{<q/p}) \cap \mathrm{Ker}(\beta_{>q/p})) \oplus (\mathrm{Ker}(\beta_{<q/p}) \cap \mathrm{Ker}(\beta_{=q/p}))$$

which, because of the behavior of slopes under π_*, fulfills the required conditions.

Finally, if $s \in \mathbb{R}^+ - \mathbb{Q}$, choose a rational number $0 < p/q < s$ such that no formal slope of $\mathcal{V}$ is in the interval $[1/s, q/p]$. By the Malgrange-Ramis algebraization theorem we can assume $\mathcal{V} \cong \mathcal{K}^s_x \otimes \mathcal{W}$, where $\mathcal{W}$ is a $\mathcal{K}^{p/q}_x$-vector space with connection. Then, the previous case applies to $\mathcal{W}$ and the proof is complete. ∎

2. The singularity at zero of $\widehat{\mathbb{M}}$ and an exact sequence of vanishing cycles. In this section we introduce one more variant of the microlocalization functors (which should correspond to Laumon's $(\infty, 0')$ local Fourier transform). Our aim is to establish the existence of a sequence of vanishing cycles analogous to [5, Theorem 10], [6, proof of 3.4.2]. In this section we will work over the complex numbers and we will consider only the convergent version of the $(\infty, 0)$-microlocalization, the corresponding formal version can be obtained just by dropping all convergence conditions.

Definition. We denote $\mathcal{E}^{(\infty,0)}$ the set of formal sums

$$P = \sum_{i \leqslant r} a_i(\eta)\, t^i \quad , \quad r \in \mathbb{Z}$$

such that there exists a $\rho_0 > 0$ so that all series $a_i(\eta)$ are convergent in the disk of radius ρ_0 centered at 0, and there exists a $0 < \rho < \rho_0$ and a $\theta > 0$ such that the series

$$\sum_{k \geqslant 0} \|a_{-k}(\eta)\|_\rho \frac{\theta^k}{k\,!}$$

is convergent, where $\|a_{-k}(\eta)\|_\rho = \sup_{|z| \leqslant \rho} \mid a_{-k}(z) \mid$. We consider in $\mathcal{E}^{(\infty,0)}$ the multiplication rule given by

$$P \cdot Q = \sum_{\alpha \geqslant 0} \frac{1}{\alpha\,!} \; \partial_t^\alpha P \cdot \partial_\eta^\alpha Q$$

and the morphism of $\mathbb{C}$-algebras

$$\begin{aligned} \varphi^{(\infty,0)} : \mathbb{W}_t &\longrightarrow \mathcal{E}^{(\infty,0)} \\ t &\mapsto -t \\ \partial_t &\mapsto \eta \end{aligned}$$

which endows $\mathcal{E}^{(\infty,0)}$ with a structure of $(\mathbb{W}_t, \mathbb{W}_t)$-bimodule. It is not difficult to see that $\varphi^{(\infty,0)}$ is flat.

Remark. While the ring $\mathcal{E}^{(\infty,0)}$ is nothing but $\mathcal{E}^{(0,\infty)}$, with the rôles of the variables t and η interchanged, the morphism $\varphi^{(\infty,0)}$ is *not* obtained in the same way from $\varphi^{(0,\infty)}$ (the morphism we considered in section 1). The morphism we get from $\varphi^{(0,\infty)}$ interchanging t and η will be denoted

$$\begin{aligned} \mu : \mathbb{W}_\eta &\longrightarrow \mathcal{E}^{(\infty,0)} \\ \eta &\mapsto \eta \\ \partial_\eta &\mapsto t\ . \end{aligned}$$

Definitions. i) Given a $\mathbb{W}_t$-module $\mathbb{M}$, we put

$$\mathcal{E}^{(\infty,0)}(\mathbb{M}) := \mathcal{E}^{(\infty,0)} \otimes_{\mathbb{W}_t} \mathbb{M}$$

where $\mathcal{E}^{(\infty,0)}$ is viewed as a left $\mathbb{W}_t$-module *via* $\varphi^{(\infty,0)}$.
ii) Given a $\mathbb{W}_\eta$-module $\mathbb{N}$, we put

$$\mu(\mathbb{N}) := \mathcal{E}^{(\infty,0)} \otimes_{\mathbb{W}_\eta} \mathbb{N}$$

where now $\mathcal{E}^{(\infty,0)}$ is viewed as a left $\mathbb{W}_\eta$-module *via* μ.

Both $\mathcal{E}^{(\infty,0)}(\mathbb{M})$ and $\mu(\mathbb{N})$ have a structure of $\mathbb{C}\{t^{-1}\}[t]$-vector spaces and of $\mathbb{C}\{\eta\}\langle\partial_\eta\rangle$-modules, where the action of ∂_η is, by definition, given by left multiplication by t. Notice that in fact $\mu(\mathbb{N})$ is nothing but $\mathcal{E}^{(0,\infty)}(\mathbb{N})$ with the variables t and η interchanged. In section 1 we considered in the $(0,\infty)$-microlocalization only the structure of $\mathbb{C}\{\eta^{-1}\}[\eta]$-vector space (i.e., of $\mathbb{C}\{t^{-1}\}[t]$-vector space after our interchange of variables), while now the structure of $\mathbb{C}\{\eta\}\langle\partial_\eta\rangle$-module will be considered as well, and in fact it will play the major rôle. Notice also that $\mathcal{E}^{(\infty,0)}(\mathbb{M})$ depends only on the 1-Gevrey germ at infinity defined by $\mathbb{M}$ and $\mu(\mathbb{N})$ depends only on $\mathbb{N}_0=\mathbb{C}\{\eta\}\langle\partial_\eta\rangle\otimes_{\mathbb{W}_\eta}\mathbb{N}$.

(2.1) Proposition. *Let $\mathbb{M}$ be a holonomic $\mathbb{W}_t$-module. Then the map*

$$\begin{aligned}\Upsilon^0:\mu(\widehat{\mathbb{M}})&\longrightarrow\mathcal{E}^{(\infty,0)}(\mathbb{M})\\ \alpha\otimes\widehat{m}&\longrightarrow\alpha\otimes m\end{aligned}$$

is an isomorphism of $\mathbb{C}\{\eta\}\langle\partial_\eta\rangle$-modules and of $\mathbb{C}\{t^{-1}\}[t]$-vector spaces.

Proof. It is easy to check that the map is a morphism both of $\mathbb{C}\{\eta\}\langle\partial_\eta\rangle$-modules and of $\mathbb{C}\{t^{-1}\}[t]$-vector spaces, we have to prove that it is an isomorphism. As for the stationary phase formulas, the theorem reduces to the case of a Dirac δ-module (then one has $\mu(\widehat{\mathbb{M}})=\mathcal{E}^{(\infty,0)}(\mathbb{M})=0$), and the case $\mathbb{M}=\mathbb{W}_t/\mathbb{W}_t\cdot P(t,\partial_t)$.

In this last case, we have $\widehat{\mathbb{M}}=\mathbb{W}_\eta/\mathbb{W}_\eta\cdot P(-\partial_\eta,\eta)$, and both the source and the target of the map Υ^0 are isomorphic to $\mathcal{E}^{(\infty,0)}/\mathcal{E}^{(\infty,0)}\cdot P(-t,\eta)$. The map Υ^0 composed with these isomorphisms is the identity map, and the proposition is thus proved. $\square$

Let τ be a coordinate in the affine line and let $\mathbb{N}$ be a holonomic $\mathbb{C}\{\tau\}\langle\partial_\tau\rangle$-module. We recall next the formalism of solutions and microsolutions of $\mathbb{N}$, following [8]: For $r>0$, denote by D_r the disk in the complex plane centered at $\tau=0$ and of radius r, by $\widetilde{D_r^*}$ the universal covering space of $D_r-\{0\}$, and by $\mathcal{O}(D_r)$ (respectively, $\mathcal{O}(\widetilde{D_r^*})$) the ring of holomorphic functions on D_r (respectively, on $\widetilde{D_r^*}$). Put $\widetilde{\mathcal{C}}(D_r)=\mathcal{O}(\widetilde{D_r^*})/\mathcal{O}(D_r)$. Set $\mathcal{O}:=\operatorname{indlim}_{r\to0}\mathcal{O}(D_r)(=\mathbb{C}\{\tau\})$, $\widetilde{\mathcal{O}}:=\operatorname{indlim}_{r\to0}\widetilde{\mathcal{O}}(D_r)$, $\widetilde{\mathcal{C}}:=\operatorname{indlim}_{r\to0}\widetilde{\mathcal{C}}(D_r)$, $\mathcal{D}:=\mathcal{O}\langle\partial_\tau\rangle$. We denote by $\mathbb{N}\mapsto D\mathbb{N}$ the duality functor in the category of holonomic left $\mathcal{D}$-modules. Recall that if $\mathbb{N}=\mathcal{D}/\mathcal{D}P$, then $D\mathbb{N}=\mathcal{D}/\mathcal{D}\,{}^tP$ where ${}^tP$ denotes the transposed differential operator (see e.g. [15, V.1]).

Following Malgrange, we put

i) $\Psi(\mathbb{N}):=\operatorname{Hom}_{\mathcal{D}}(D\mathbb{N},\widetilde{\mathcal{O}})$ (the $\mathbb{C}$-vector space of "nearby cycles" of $\mathbb{N}$).

ii) $\Phi(\mathbb{N}):=\operatorname{Hom}_{\mathcal{D}}(D\mathbb{N},\widetilde{\mathcal{C}})$ (the $\mathbb{C}$-vector space of microsolutions of $\mathbb{N}$ or "vanishing cycles").

Between these vector spaces there are morphisms $can:\Psi(\mathbb{N})\mapsto\Phi(\mathbb{N})$ (induced by the quotient map $can:\widetilde{\mathcal{O}}\to\widetilde{\mathcal{C}}$) and $var:\Phi(\mathbb{N})\mapsto\Psi(\mathbb{N})$ (induced by the only map $var:\widetilde{\mathcal{C}}\to\widetilde{\mathcal{O}}$ such that $var\circ can=T-Id$, where T is the monodromy on $\widetilde{\mathcal{O}}$). The map can is an isomorphism if $\mathbb{N}\cong\mu(\mathbb{N})$, the map var is an isomorphism if $\mathbb{N}\cong\mathbb{N}[\tau^{-1}]$. The assignment $\mathbb{N}\mapsto(\Psi(\mathbb{N}),\Phi(\mathbb{N}),can,var)$ is functorial. The behavior of this spaces under localization and microlocalization is the following:

a) For the localization we have $\Phi(\mathbb{N}[\tau^{-1}])\cong\Psi(\mathbb{N}[\tau^{-1}])\cong\Psi(\mathbb{N})$.

b) For the microlocalization we have $\Psi(\mu(\mathbb{N}))\cong\Phi(\mu(\mathbb{N}))\cong\Phi(\mathbb{N})$.

For a), recall that both the kernel and cokernel of $\mathbb{N}\mapsto N[\tau^{-1}]$ are a direct sum of Dirac δ_0's and $\Psi(\delta_0)=0$ ($\delta_0=\mathcal{D}/\mathcal{D}\cdot\partial_\tau$). Similarly, the kernel and cokernel of

$\mathbb{N} \mapsto \mu(\mathbb{N})$ is a direct sum of copies of the $\mathcal{D}$-module $\mathcal{O}$ (see e.g. [7, 4.11.b]) and $\Phi(\mathcal{O}) = 0$, from which b) follows.

Given a holonomic $\mathbb{W}_\tau$-module $\mathbb{M}$ we denote by $DR(\mathbb{M})$ its De Rham complex (see e.g. [8, I.2]) and by $\mathrm{Sol}(\mathbb{M})_0 = \mathbb{R}\mathrm{Hom}_{\mathcal{D}}(\mathcal{O} \otimes_{\mathbb{W}_\tau} \mathbb{M}, \, , \mathcal{O})[1]$ the stalk at zero of its solution complex. We denote by $\mathbb{H}^*_c(\mathbb{A}^1_{\mathbb{C}}, DR(\mathbb{M}))$ the hypercohomology with compact supports of the De Rham complex of $\mathbb{M}$.

The following proposition follows essentially from results of B. Malgrange. It shows that the De Rham cohomology with compact supports of a holonomic $\mathbb{W}_t$-module which has slopes at infinity strictly smaller than $+1$ can be computed locally in terms of the germs defined by $\mathbb{M}$ at its singular points and at infinity.

Proposition (exact sequence of vanishing cycles). *Let $\mathbb{M}$ be a holonomic $\mathbb{W}_t$-module such that all its formal slopes at infinity are strictly smaller than $+1$. Then there is an exact sequence of $\mathbb{C}$-vector spaces*

$$0 \to \mathbb{H}^1_c(\mathbb{A}^1_{\mathbb{C}}, DR(\mathbb{M})) \to \oplus_{c \in Sing(\mathbb{M})} \Phi(\mathbb{M}_c) \to \Phi(\mathcal{E}^{(\infty,0)}(\mathbb{M})) \to \mathbb{H}^2_c(\mathbb{A}^1_{\mathbb{C}}, DR(\mathbb{M})) \to 0$$

where $\mathbb{M}_c := \mathbb{C}\{t_c\}\langle\partial_t\rangle \otimes_{\mathbb{W}_t} \mathbb{M}$.

Proof. From the exact sequence $0 \longrightarrow \mathcal{O} \longrightarrow \widetilde{\mathcal{O}} \longrightarrow \widetilde{\mathcal{C}} \longrightarrow 0$, we get

$$0 \to \mathrm{Hom}_{\mathcal{D}}(D\widehat{\mathbb{M}}_0, \mathcal{O}) \to \Psi(\widehat{\mathbb{M}}_0) \to \Phi(\widehat{\mathbb{M}}_0) \to \mathrm{Ext}^1_{\mathcal{D}}(D\widehat{\mathbb{M}}_0, \mathcal{O}) \to 0$$

(since $\mathrm{Ext}^1_{\mathcal{D}}(D\widehat{\mathbb{M}}_0, \mathcal{O}) = 0$, see e.g. [8, II.3]). We have also quasiisomorphisms

$$\mathbb{R}\Gamma_c(\mathbb{A}^1_{\mathbb{C}}, DR(\mathbb{M})) \cong \mathrm{Sol}\,(\widehat{D\mathbb{M}})_0\,[-1] \cong \mathrm{Sol}(\,D\widehat{\mathbb{M}})_0\,[-1],$$

the first one follows from [8, VI, 2.9 and VII, 1.1] (since we assume that the slopes at infinity of $\mathbb{M}$ are strictly smaller than $+1$), and the second one holds because Fourier transform and duality commute up to the transformation given by $t \mapsto -t, \partial_t \mapsto -\partial_t$ (see e.g. [15, V.2.b]).

From an element of $\oplus_{c \in Sing\mathbb{M}} \Phi(\mathbb{M}_c)$ we get, by the Laplace transform considered in [8, chap. XII], a multivaluated solution of $\widehat{\mathbb{M}}$ defined on a half-plane in $\mathbb{C}$ [loc.cit., XII, 1.2]. Under our hypothesis, the module $\widehat{\mathbb{M}}$ is singular only at zero and at infinity, so this solution can be analytically prolonged and determines univocally an element of $\Psi(\widehat{\mathbb{M}}_0)$. This assignment establishes an isomorphism of complex vector spaces $\oplus_{c \in Sing\mathbb{M}} \Phi(\mathbb{M}_c) \simeq \Psi(\widehat{\mathbb{M}}_0)$. On the other hand, by proposition (2.1) we have also an isomorphism

$$\Phi(\widehat{\mathbb{M}}_0) \cong \Phi(\mu(\widehat{\mathbb{M}})) \cong \Phi(\mathcal{E}^{(\infty,0)}(\mathbb{M})),$$

and the proposition follows. ◻

Remark. Using b) above, the long exact sequence in the proposition can be rewritten in terms of spaces of nearby cycles instead of spaces of microsolutions, namely one has an exact sequence

$$\begin{aligned} 0 \to \mathbb{H}^1_c(\mathbb{A}^1_{\mathbb{C}}, \mathrm{DR}(\mathbb{M})) &\to \oplus_{c \in Sing(\mathbb{M})} \Psi(\mu(\mathbb{M}_c)) \\ &\to \Psi(\mathcal{E}^{(\infty,0)}(\mathbb{M})) \to \mathbb{H}^2_c(\mathbb{A}^1_{\mathbb{C}}, \mathrm{DR}(\mathbb{M})) \to 0. \end{aligned}$$

3. p-adic microdifferential operators of finite order. B. Malgrange proved in [7] (and it follows also from the 1-Gevrey stationary phase theorem proved in section 1) that if $\mathbb{M}$ is a holonomic $\mathbb{W}_t$-module, singular only at zero and at infinity, and such that the formal slopes of the singularity at infinity are smaller than $+1$, then one has an isomorphism of $\mathcal{K}^1_{\eta^{-1}}$-vector spaces with connection

$$\mathcal{K}^1_{\eta^{-1}} \otimes_{\mathbb{C}[\eta]} \widehat{\mathbb{M}} \cong \mathcal{E}^{(0,\infty)}(\mathbb{M}) \ .$$

Notice that, in case the singularity at infinity of $\mathbb{M}$ is regular, these $\mathbb{W}_t$-modules are analogous to the ℓ-adic canonical prolongations of Gabber and Katz, which play a major rôle in G. Laumon work ([6], [5]). So, it might be of interest to find a p-adic analogue of Malgrange's result.

For the rest of this section we assume that K is a spherically complete p-adic field (e.g., a finite extension of $\mathbb{Q}_p$), and we denote by $|\cdot|$ its absolute value, normalized by the condition $|\, p \,| = p^{-1}$. It seems that a reasonable p-adic version of the ring $\mathbb{C}\{\eta^{-1}\}_1$ would be the ring of power series $\sum_{i\geqslant 0} a_i\eta^{-i}$, $a_j \in K$, such that $\sum_{i\geqslant 0} i!\, a_i\, \eta^{-i}$ is convergent for $|\, \eta^{-1} \,| < 1$. If we set $\omega = \sqrt[p-1]{1/p} \in \mathbb{R}$, it follows easily from the classical bounds $\omega^{k-1} < |\, k! \,| < (k+1)\omega^k$ that this ring is nothing but the ring $\mathcal{A}_{\eta^{-1}}(\omega)$ of power series in η^{-1} convergent in the disk $|\, \eta^{-1} \,| < \omega$. Pursuing this analogy, to the field $\mathcal{K}^1_{\eta^{-1}}$ would correspond the ring $\mathcal{A}_{\eta^{-1}}(\omega)[\eta]$, which for simplicity will be denoted $\mathcal{A}(\omega)[\eta]$ in the sequel. Its elements are the Laurent series $\sum_{j\leqslant r} a_j\, \eta^j$ with $a_j \in K$, $r \in \mathbb{Z}$, such that for all $0 < \rho < \omega$, $\limsup_{j\to-\infty} |\, a_j \,|\, \rho^{-j} = 0$.

In this section we define a ring $\Phi^{(0,\infty)}$ of p-adic microdifferential operators and a corresponding microlocalization functor $\mathbb{M} \mapsto \Phi^{(0,\infty)}(\mathbb{M})$. Let $\mathbb{M}_P = K[t]\langle\partial_t\rangle/K[t]\langle\partial_t\rangle \cdot P$ be a holonomic $K[t]\langle\partial_t\rangle$-module given by a single differential operator. Assume $\mathbb{M}_P$ is singular only at zero and infinity, the singularity at infinity has formal slopes smaller or equal that $+1$, and the singularity at zero is solvable at radius 1 ([3, 8.7]). Then we will prove that there is an isomorphism of $\mathcal{A}(\omega)[\eta]$-modules with connection

$$\mathcal{A}(\omega)[\eta] \otimes_{K[\eta]} \widehat{\mathbb{M}}_P \cong \Phi^{(0,\infty)}(\mathbb{M}_P) \ ,$$

where $\widehat{\mathbb{M}}_P$ denotes now the p-adic Fourier transform of $\mathbb{M}_P$ (defined below). This isomorphism might be regarded as a p-adic analogue of the theorem of Malgrange quoted above.

We denote by $\mathcal{A}_t(1)$ the ring of power series in the variable t with coefficients in K which are convergent for $|\, t \,| < 1$. For all $0 < \lambda < 1$, the ring $\mathcal{A}_t(1)$ is endowed with the norm

$$|\sum_{i\geqslant 0} a_i\, t^i\, |_\lambda = \sup_i \{\, |\, a_i \,|\, \lambda^i \,\} \in \mathbb{R}^+.$$

Let $r \in \mathbb{Z}$ be an integer, set $\omega = \sqrt[p-1]{1/p} \in \mathbb{R}$. We denote by $\Phi^{(0,\infty)}[r]$ the set of all Laurent series $\sum_{j\leqslant r} a_j(t)\, \eta^j$ with $a_j(t) \in \mathcal{A}_t(1)$, such that for all $\lambda, \rho \in \mathbb{R}$ with $0 < \rho < \omega \cdot \lambda < \omega$, one has

$$\limsup_{j\to-\infty} |\, a_j(t) \,|_\lambda\ \rho^{-j} = 0.$$

Equivalently, for all λ, ρ in the range above there is a $C > 0$ such that for all $j \leqslant r$ one has $\mid a_j(t) \mid_\lambda \leqslant C \cdot \rho^j$ (in fact, it is clear that given any $0 < \lambda_0 < 1$, it is enough to check this condition holds for $\lambda > \lambda_0$). We put $\Phi^{(0,\infty)} = \bigcup_{r\in\mathbb{Z}} \Phi^{(0,\infty)}[r]$.

Proposition. *If* $F = \sum f_u \eta^u \in \Phi^{(0,\infty)}$ *and* $G = \sum g_v \eta^v \in \Phi^{(0,\infty)}$, *then*

$$F \cdot G = \sum_{\alpha \geqslant 0} \frac{1}{\alpha!} \ \partial_\eta^\alpha F \cdot \partial_t^\alpha G \ \in \Phi^{(0,\infty)}.$$

Proof. Write $F \cdot G = \sum r_j(t) \eta^j$. We have

$$\begin{aligned} \mid r_j(t) \mid_\lambda \rho^{-j} &\leqslant \max_{j=u+v-\alpha} \left\{ \mid \frac{1}{\alpha!} \ u(u-1)\dots(u-\alpha+1) \ f_u \ \frac{d^\alpha g_v}{dt^\alpha} \mid_\lambda \rho^{-j} \right\} \\ &\leqslant \max_{j=u+v-\alpha} \left\{ \mid u(u-1)\dots(u-\alpha+1) \mid \mid f_u \mid_\lambda \mid g_v \mid_\lambda \ \frac{1}{\lambda^\alpha} \ \rho^{-j} \right\} \\ &\leqslant \sup_u\{\mid f_u \mid_\lambda \rho^{-u}\} \ \cdot \ \sup_v\{\mid g_v \mid_\lambda \rho^{-v}\} \cdot (\frac{\rho}{\lambda})^\alpha \end{aligned}$$

where the second inequality follows from the Cauchy inequalities. If $j \mapsto -\infty$ then either $u \mapsto -\infty$ or $v \mapsto -\infty$ or $\alpha \mapsto \infty$, and then we are done. ☐

Definition. The filtered ring $\Phi^{(0,\infty)}$ will be called the ring of *p-adic microdifferential operators of finite order*. The order and the principal symbol of a microdifferential operator are defined as in the formal case. Notice that $\Phi^{(0,\infty)}[0] \subset \Phi^{(0,\infty)}$ is a filtered subring and that one has $\mathcal{A}(\omega)[\eta] = K[[\eta^{-1}]][\eta] \cap \Phi^{(0,\infty)}$.

Definitions. If $F = \sum_{u \leqslant m} f_u(t) \eta^u \in \mathcal{A}_t(1)[[\eta^{-1}]][\eta]$ and $0 < \rho < \omega \cdot \lambda < \omega$, we put

$$\|F\|_{\lambda,\rho} = \sup_u\{\mid f_u(t) \mid_\lambda \rho^{-u}\} \ .$$

Notice that we have $F \in \Phi^{(0,\infty)}$ if and only if $\|F\|_{\lambda,\rho} < \infty$ for all λ, ρ in the range above. From the proof of the preceding proposition follows that if $F, G \in \Phi^{(0,\infty)}$, then we have $\|F \cdot G\|_{\lambda,\rho} \leqslant \|F\|_{\lambda,\rho} \cdot \|G\|_{\lambda,\rho}$. The subscript λ, ρ will be omitted if no confusion may arise. We will say that $F = \sum_u f_u \eta^u \in \Phi^{(0,\infty)}$ is *dominant* if there is a $\lambda_0 < 1$ such that for all $\lambda_0 < \lambda < 1$ and $0 < \rho < \omega \cdot \lambda$, one has $\mid f_{ord(F)} \mid_\lambda \rho^{-ord(F)} = \|F\|_{\lambda,\rho}$.

We want to prove a division theorem for p-adic microdifferential operators of finite order. We will make implicit use of the following lemma, its proof is elementary and left to the reader:

Lemma. *Let* $f(t) = t^m b(t) \in \mathcal{A}_t(1)$, *where* $b(t)$ *is invertible in* $\mathcal{A}_t(1)$ *and* $m \geqslant 0$. *Then, for each* $\varphi \in \mathcal{A}_t(1)$, *there are unique* $q \in \mathcal{A}_t(1)$ *and* $r \in K[t]$ *of degree smaller or equal than* $m-1$ *such that* $\varphi = f \cdot q + r$, *and for all* $0 < \lambda < 1$, $\mid r \mid_\lambda \leqslant \mid \varphi \mid_\lambda$, *and* $\mid f \mid_\lambda \cdot \mid q \mid_\lambda \leqslant \mid \varphi \mid_\lambda$.

Theorem. *Let* $F \in \Phi^{(0,\infty)}$ *be dominant and assume that* $\sigma(F) = t^m b(t)$ *where* $b(t) \in \mathcal{A}_t(1)$ *is invertible. Then, for all* $G \in \Phi^{(0,\infty)}$ *there exist unique* $Q \in \Phi^{(0,\infty)}$ *and* $R_0, \dots, R_{m-1} \in \mathcal{A}(\omega)[\eta]$ *such that*

$$G = Q \cdot F + t^{m-1} R_{m-1} + \dots + R_0.$$

The remainder $t^{m-1} R_{m-1} + \ldots + R_0$ *can also be written in a unique way in the form* $S_{m-1}\, t^{m-1} + \ldots + S_0$ *with* $S_i \in \mathcal{A}(\omega)[\eta]$.

Proof. It is easy to see that if F is dominant, then the product $\eta^{-ord(F)}F$ is also dominant, so we can assume F is of order zero. Multiplying G by a suitable power of η we may also assume that $\mathrm{ord}(G) = 0$. The existence of a unique formal solution $Q = \sum_{j\leqslant 0} q_j(t)\,\eta^j$, $R_i = \sum_{j\leqslant 0} r_{i,j}\,\eta^j$ $(0 \leqslant i \leqslant m-1)$ to the division problem formulated above is well-known, the solution can be obtained as follows (cf. [1, Ch.4, Theorem 2.6]): One constructs inductively power series $q_0, q_{-1}, \ldots \in \mathcal{A}_t(1)$ such that

$$G - (q_0 + \ldots + q_j\,\eta^{-j})F = H_{j-1} + K_{j-1} \quad \text{with} \quad H_{j-1} \in \Phi^{(0,\infty)}[j-1]$$
$$\text{and} \quad K_{j-1} \in t^{m-1}\,\mathcal{A}(\omega)[\eta] + \ldots + \mathcal{A}(\omega)[\eta].$$

Assume $q_0, \ldots, q_{j+1}$ have already been found and put

$$\varphi_j = g_j - \sum_{(j)} \frac{1}{\alpha!}\, v\,(v-1)\ldots(v-\alpha+1)\; q_v\; \frac{d^\alpha f_u}{dt^\alpha}$$

where the sum runs over those v, u, α with $j = v+u-\alpha$, $\alpha \geqslant 0$ and $j+1 \leqslant v \leqslant 0$ (it is understood that the product $v\,(v-1)\ldots(v-\alpha+1)$ is replaced by 1 if $\alpha = 0$). Then, the next series q_j is the quotient of the division of φ_j by $\sigma(F)$, that is, it is defined by the equality $\varphi_j = t^m\, b(t)\, q_j + r_j$, where $q_j \in \mathcal{A}_t(1)$ and $r_j(t) = \sum_{i=0}^{m-1} r_{i,j} t^i$ is a polynomial of degree $m-1$ at most. The formal solution to the division problem is given by the quotient $Q = \sum_{j\leqslant 0} q_j(t)\,\eta^j$ and the series $R_i = \sum_{j\leqslant 0} r_{i,j}\,\eta^j \in \mathcal{A}(\omega)[\eta]$ $(1 \leqslant i \leqslant m-1)$.

We have to prove that this formal solution is convergent in our sense. Fix $0 < \rho < \omega \cdot \lambda < \omega$ and put $C_{\lambda,\rho} = \|G\|_{\lambda,\rho}/\|F\|_{\lambda,\rho}$. Notice that because of our hypothesis on F we have $\|F\|_{\lambda,\rho} = \mid f_0 \mid_\lambda$. We show next, by descending induction on $j \leqslant 0$, that $\mid q_j \mid \rho^{-j} \leqslant C_{\lambda,\rho}$. We have:

$$\begin{aligned}
\mid q_j \mid_\lambda \rho^{-j} &\leqslant \max\Big\{ \frac{1}{\mid f_0 \mid_\lambda} \mid g_j \mid_\lambda \rho^{-j}, \\
&\qquad \max_{u,v,\alpha}\Big\{ \frac{\mid v(v-1)\ldots(v-\alpha+1) \mid}{\mid f_0 \mid_\lambda \cdot \mid \alpha! \mid} \;\mid q_v \mid_\lambda \mid \frac{d^\alpha f_u}{dt^\alpha} \mid_\lambda \rho^{-j} \Big\}\Big\} \\
&\leqslant \max_{u,v,\alpha}\Big\{ \frac{\|G\|}{\|F\|}, \frac{1}{\lambda^\alpha\,\|F\|} \;\mid q_v \mid_\lambda \mid f_u \mid_\lambda \rho^{-j} \Big\} \\
&= \max_{u,v,\alpha}\Big\{ C_{\lambda,\rho}, \frac{1}{\lambda^\alpha\,\|F\|}\; (\mid q_v \mid_\lambda \rho^{-v})\; (\mid f_u \mid_\lambda \rho^{-u})\; \rho^\alpha \Big\} \\
&\leqslant \max\Big\{ C_{\lambda,\rho},\; C_{\lambda,\rho} \cdot (\frac{\rho}{\lambda})^\alpha \Big\} \;=\; C_{\lambda,\rho}\,,
\end{aligned}$$

where the second inequality follows from the Cauchy inequalities. This proves the convergence of the quotient as well as the first inequality of norms. The convergence of the series $R_0, \ldots, R_{m-1}$ is proved similarly and it is left to the reader. For the last statement, notice that there are unique $S_i = \sum_{j\leqslant 0} s_{i,j}\,\eta^j \in K[[\eta^{-1}]][\eta]$ $(0 \leqslant i \leqslant m-1)$, such that the remainder $t^{m-1} R_{m-1} + \ldots + R_0$ can be written in the form $S_{m-1}\,t^{m-1} + \ldots + S_0$, in fact one has

$$s_{i,j} = \sum_{k=0}^{m-j-1} (-1)^k\; r_{i+k,j+k} \frac{(i+k)! \cdot (j+k)!}{k! \cdot i! \cdot j!}.$$

From this formula follows easily that $S_i \in \mathcal{A}(\omega)[\eta]$ for $i = 0, \ldots, m-1$. □

Remark. It is unreasonable to expect a division theorem without some restriction on the divisor, for example if $\alpha \in K$ and we take $F = 1 - \alpha\,\eta^{-1} \in \Phi^{(0,\infty)}[0]$, its formal inverse is $\sum_{i \geqslant 0} \alpha^i \eta^{-i}$, which is not convergent in our sense for $|\,\alpha\,| > \omega^{-1}$.

We will assume that there is a $\pi \in K$ such that $\pi^{p-1} + p = 0$, which we fix from now on. Then, the morphism of K-algebras defined by

$$\begin{aligned} \varphi^{(0,\infty)} : \mathbb{W}_t &\longrightarrow \Phi^{(0,\infty)} \\ t &\mapsto t/\pi \\ \partial_t &\mapsto \pi \cdot \eta \end{aligned}$$

endows $\Phi^{(0,\infty)}$ with a structure of $(\mathbb{W}_t, \mathbb{W}_t)$-bimodule.

Definition. Let $\mathbb{M}$ be a $\mathbb{W}_t$-module. We define its p-adic $(0,\infty)$-microlocalization as the $\mathcal{A}(\omega)[\eta]$-module

$$\Phi^{(0,\infty)}(\mathbb{M}) := \Phi^{(0,\infty)} \otimes_{\mathbb{W}_t} \mathbb{M}\,,$$

endowed with the connection given by left multiplication by $\eta^2 \cdot t$.

Definition ([4] and [9]). If $\mathbb{M}$ is a $\mathbb{W}_t$-module, its p-adic Fourier transform is defined as $\widehat{\mathbb{M}} = \mathbb{W}_\eta \otimes_{\mathbb{W}_t} \mathbb{M}$, where $\mathbb{W}_\eta$ is regarded as a right $\mathbb{W}_t$-module via the K-algebra isomorphism given by $t \mapsto -\partial_\eta/\pi$, $\partial_t \mapsto \pi \cdot \eta$. If $m \in \mathbb{M}$, we put $\widehat{m} = 1 \otimes m \in \widehat{\mathbb{M}}$.

If τ is a coordinate, we denote by $\mathcal{R}_\tau(\theta)$ the Robba ring of power series $\sum_{i \in \mathbb{Z}} a_i\,\tau^i$, $a_i \in K$, convergent in some annulus $\theta - \epsilon < |\,\tau\,| < \theta$, $\epsilon > 0$, endowed with the derivation ∂_τ. Let $P(t, \partial_t) = \sum_{k=0}^{d} a_k(t)\partial_t^k \in K[t]\langle\partial_t\rangle$, set $\mathbb{M}_P := \mathbb{W}_t/\mathbb{W}_t\,P$. We make the following assumptions on the differential operator P:

i) P is singular only at zero and at infinity, and $\deg(a_d(t)) \geqslant \deg(a_i(t))$ for all $1 \leqslant i \leqslant d$ (that is, the formal slopes of the singularity at infinity of P are smaller or equal than $+1$).

ii) The $\mathcal{R}_t(1)-$module with connection $\mathcal{R}_t(1) \otimes_{K[t,t^{-1}]} \mathbb{M}_P$ is soluble at 1 (see [3, 8.7]).

As in the formal case, if $\mathbb{N}$ is a $\mathbb{W}_\eta$-module, on the $\mathcal{A}(\omega)[\eta]$-module $\mathcal{A}(\omega)[\eta] \otimes_{K[\eta]} \mathbb{N}$ we will consider the connection given by

$$\nabla(\alpha \otimes n) := \partial_{\eta^{-1}}(\alpha) \otimes n \,-\, \alpha \otimes \eta^2 \partial_\eta n.$$

Theorem. *Let $P \in K[t]\langle\partial_t\rangle$ satisfy the conditions i) and ii) above. Then the map*

$$\Upsilon : \mathcal{A}(\omega)[\eta] \otimes_{K[\eta]} \widehat{\mathbb{M}_P} \longrightarrow \Phi^{(0,\infty)}(\mathbb{M}_P)$$

given by $\Upsilon(\alpha \otimes \widehat{m}) = \alpha \otimes m$ is an isomorphism of $\mathcal{A}(\omega)[\eta]$-modules with connection.

Proof. It is easy to check that Υ is a morphism of $\mathcal{A}(\omega)[\eta]$-modules with connection, we have to prove that it is an isomorphism. We have

$$\Phi^{(0,\infty)}(\mathbb{M}_P) \cong \frac{\Phi^{(0,\infty)}}{\Phi^{(0,\infty)} \cdot P(t/\pi, \pi\,\eta)}.$$

By ii) we have (with the notations of [3]) $Ray(\mathcal{R}_t(1)\otimes_{K[t,t^{-1}]}\mathbb{M}_P, 1^-) = 1$, so for λ close to $+1$ we have $\|a_{d-i}(t)\|_\lambda \leqslant \lambda^{-i}\ \|a_d(t)\|_\lambda$ ([3, Corollaire 6.4]). Since $P(t,\partial_t)$ is singular only at zero and infinity, we have $a_d(t) = \alpha_d\, t^\delta$, with $\alpha_d \in K$, so $\|a_d(t/\pi)\|_\lambda = \omega^{-\delta}\ \|a_d(t)\|_\lambda$ and since $\delta \geqslant \deg\ a_{d-i}(t)$ for all $i = 0,\ldots,d$, we get $\|a_{d-i}(t/\pi)\|_\lambda \leqslant \omega^{-\delta}\ \|a_{d-i}(t)\|_\lambda$, so it follows that $\|a_{d-i}(t/\pi)\|_\lambda \leqslant \lambda^{-i}\ \|a_d(t/\pi)\|_\lambda$. If $0<\rho<\omega\lambda$ and λ is close to one,

$$\|a_d(t/\pi)\|_\lambda\, \omega^d\rho^{-d} \geqslant \|a_{d-i}(t/\pi)\|_\lambda\, \omega^{d-i}\cdot(\omega\,\lambda)^i\rho^{-d} \geqslant \|a_{d-i}(t/\pi)\|_\lambda\, \omega^{d-i}\cdot\rho^{-d+i}\ ,$$

which is just the condition of dominance for the microdifferential operator $P(t/\pi,\pi\,\eta)$. As in the formal case, it follows now from the division theorem that $\Phi^{(0,\infty)}(\mathbb{M}_P)$ is a free $\mathcal{A}(\omega)[\eta]$-module with basis $1\otimes t^i$, $i = 0,\ldots,\delta-1$, and since $\Upsilon(1\otimes(-1)^i\partial_\eta^i/\pi^i) = 1\otimes t^i$, the morphism Υ is surjective.

We compute next the rank over $\mathcal{A}(\omega)[\eta]$ of $\mathcal{A}(\omega)[\eta]\otimes_{\mathbb{C}[\eta]}\widehat{\mathbb{M}_P}$. Denote by $\alpha_i\in K$ the (possibly zero) coefficient of t^δ in $a_i(t)\in K[t]$. The coefficient of ∂_η^δ in the differential operator $\widehat{P} = P(-\partial_\eta/\pi,\pi\,\eta)$ will be the polynomial $q(\eta) = (-1)^\delta\sum_i \alpha_{d-i}\,\pi^{d-\delta-i}\,\eta^{d-i}\in K[\eta]$. By condition ii), $\mid\alpha_d\mid\geqslant\mid\alpha_i\mid$, which implies that the roots of $q(\eta)$ are either zero or of absolute value smaller than ω^{-1}. It follows that $q(\eta)$ is a unit of $\mathcal{A}(\omega)[\eta]$, and then the rank of $\mathcal{A}(\omega)[\eta]\otimes_{\mathbb{C}[\eta]}\widehat{\mathbb{M}_P}$ over $\mathcal{A}(\omega)[\eta]$ equals δ.

Thus Υ is an epimorphism between two $\mathcal{A}(\omega)[\eta]$-modules with connection of the same rank. Since the ring $\mathcal{A}(\omega)[\eta]$ is a localization of $\mathcal{A}_{\eta^{-1}}(\omega)$, it follows from [3, 8.1], that its kernel is zero and then the theorem is proved. □

Since $\mathcal{A}(\omega)[\eta]$ is a subring of $\mathcal{R}_{\eta^{-1}}(\omega)$, we have

Corollary. *Under the same hypothesis of the theorem above, there is an isomorphism of $\mathcal{R}_{\eta^{-1}}(\omega)$-modules with connection*

$$\mathcal{R}_{\eta^{-1}}(\omega)\otimes_{K[\eta]}\widehat{\mathbb{M}_P}\longrightarrow\mathcal{R}_{\eta^{-1}}(\omega)\otimes_{\mathcal{A}(\omega)[\eta]}\Phi^{(0,\infty)}(\mathbb{M}_P).$$

Remark. Given $P\in K[t]\langle\partial_t\rangle$ verifying the conditions of the previous theorem, one can consider as well the formal Fourier transform of $\widehat{\mathbb{M}_P}^{for}$ of $\mathbb{M}_P$ (that is, the one given by $t\mapsto-\partial_\eta$, $\partial_t\mapsto\eta$). One can also define a variant $\Phi_1^{(0,\infty)}$ of the ring $\Phi^{(0,\infty)}$ (taking $\omega=1$), it is easy to check that one obtains also a ring for which the division theorem holds. Then, similarly as in the theorem above, one can show that there is an isomorphism of $\mathcal{R}_{\eta^{-1}}(1)$-modules with connection

$$\mathcal{R}_{\eta^{-1}}(1)\otimes_{K[\eta]}\widehat{\mathbb{M}_P}^{for}\longrightarrow\mathcal{R}_{\eta^{-1}}(1)\otimes\Phi_1^{(0,\infty)}(\mathbb{M}_P).$$

However, unlike the p-adic Fourier transform, the formal Fourier transform does not extend to the weak completion of the Weyl algebra considered in [10] and [4], and I ignore whether it can be related to the sheaf-theoretic Fourier transform considered in [4].

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