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ARMAND BOREL

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The Swiss mathematician Armand Borel died August 11, 2003, in Princeton from a rapidly evolving cancer. Few foreign mathematicians had as many connections with France. He was a student of Leray, he took part in the Cartan seminar, and he published more than twenty papers in collaboration with our colleagues Lichnerowicz and Tits, as well as with me. He was a member of Bourbaki for more than twenty years, and he became a foreign member of the Académie des Sciences in 1981. French mathematicians feel that it is one of their own who has died.

He was born in La Chaux-de-Fonds in 1923 and was an undergraduate at Eidgenössische Technische Hochschule of Zürich (the "Poly"). There he met H. Hopf, who gave him a taste for topology, and E. Stiefel, who introduced him to Lie groups and their root systems. He spent the year 1949–50 in Paris, with a grant from the CNRS[1]. A good choice (for us, as well as for him), Paris being the very spot where what Americans have called "French Topology" was being created, with the courses from Leray at the Collège de France and the Cartan seminar at the École Normale Supérieure. Borel was an active participant in the Cartan seminar while closely following Leray's courses. He managed to understand the famous "spectral sequence", not an easy task, and he explained it to me so well that I have not stopped using it since. He began to apply it to Lie groups, and to the determination of their cohomology with integer coefficients. That work would make a thesis, defended at the Sorbonne (with Leray as president) in 1952, and published immediately in the *Annals of Mathematics*. Meanwhile Borel returned to Switzerland. He did not stay long. He went for two years (1952–54) to the Institute for Advanced Study in Princeton and spent the year 1954–55 in Chicago, where he benefited from the presence of André Weil by learning algebraic geometry and number theory. He returned to Switzerland, this time to Zürich, and in 1957 the Institute for Advanced Study offered him a position as permanent professor, a post he occupied until his death (he became a professor emeritus in 1993).

He was a member of the academies of sciences of the USA and of Finland. He received the Brouwer Medal in 1978, the AMS Steele Prize in 1991, and the Balzan Prize in 1992. His *Œuvres* have been collected in four volumes, published by Springer-Verlag in 1983 (volumes I, II, III) and in 2001 (volume IV).

Borel's works have a profound unity: they relate in almost every case to *group theory*, and more particularly to Lie groups and algebraic groups. The central nature of group theory has been known for a long time. In [Œ IV, p. 381] Borel quoted a sentence from Poincaré, dating from 1912, saying this: "Group theory is, so to speak, all of mathematics, stripped of its content and reduced to a pure form."[2]

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[1] Centre National de la Recherche Scientifique.

[2] "La théorie des groupes est, pour ainsi dire, la mathématique entière, dépouillée de sa matière et réduite à une forme pure."

We today would no longer employ terms so extreme: “all of mathematics” seems too much, and “stripped of its content” seems unfair. Nevertheless the importance of group theory is much more evident now than in Poincaré’s day, and that is true in areas as different as geometry, number theory, and theoretical physics. It seems that, in his youth, Borel consciously made the decision to explore and go further into everything related to Lie theory; and this is what he did, during nearly sixty years.

I am not going to try to make an exhaustive list of the results he obtained. I shall confine myself to what I know best.

Topology of Lie Groups and of Their Classifying Spaces. As said above, this is the subject of his thesis ([Œ 23]). The objective is the determination of the cohomology with coefficients in the integers $\mathbb{Z}$ (torsion included) of compact Lie groups and of their classifying spaces. Borel uses Leray’s theory, and sharpens it by proving a difficult result of the type:

$$\text{“exterior algebra (fiber)} \Longrightarrow \text{polynomial algebra (base).”}$$

(The proof is so intricate that, according to Borel, “I do not know whether it has had any serious reader other than J. Leray and E. B. Dynkin.”)

This led to the introduction of the “torsion primes” of a compact Lie group G (for example, 2, 3, and 5 for G of type E_8). He showed that these primes play a special role in the structure of the finite commutative subgroups of G ([Œ 24, 51, 53]). It has since been found that they also occur in the Galois cohomology of G , and in particular in the theory of the “essential dimension”.

Linear Algebraic Groups. His article on this subject ([Œ 39]) played a fundamental role (it in particular served as the point of departure for the classification by Chevalley [Che] of semisimple groups in terms of root systems). In it Borel established the main properties of maximal connected solvable groups (now called “Borel subgroups”) and of maximal tori. The proofs are astonishingly simple; they rest in great part on a lemma saying that every linear connected solvable group that acts algebraically on a nonempty projective variety has a fixed point.

The point of view of [Œ 39] is “geometric”: the given group G is defined over a ground field k that is assumed algebraically closed. The same assumption occurs in [Che]. The case of a field that is not algebraically closed is however of great interest, as much for geometers (É. Cartan, for $k = \mathbb{R}$) as for number theorists (k = number field, or p-adic field). Borel (and, independently, Tits) constructed a “relative” theory, based on maximal split k-tori and the corresponding root systems. Borel and Tits published their results together ([Œ 66, 94]); the theory obtained in this way carries their name today; it is invaluable as long as the group, assumed simple, is isotropic, that is, contains nontrivial unipotent elements. (The anisotropic case is in the domain of “Galois cohomology”, and is still not completely understood.) Borel and Tits completed their results by describing the homomorphisms that are not necessarily algebraic (called, curiously, “abstract”) between groups of the form $G(k)$, cf. [Œ 82, 97].

Arithmetic Groups, Stability, Representations, ... It is to Borel and Harish-Chandra that we owe the basic results on *arithmetic subgroups* of reductive groups over number fields: finite generation, cocompactness in the anisotropic case, finite covolume in the semisimple case, cf. [Œ 54, 58]. These results have great importance for number theory. Borel completed them in a series of papers ([Œ 59, 61, 70, 74, 88, 99]), as well as in [1]. Several themes are intertwined:

- Compactification of quotients: that of Baily-Borel ([Œ 63, 69]) in the complex analytic case; that of [Œ 90, 98] in the real case, using manifolds with corners. In the two cases, it is the Tits building of the group that dictates what has to be added "at infinity".
- Generalization to S-arithmetic groups and to adelic groups ([Œ 60, 91, 105]); here the use of the Tits building must be completed by that of Bruhat-Tits buildings at nonarchimedean places.
- Infinite-dimensional representations, and the Langlands program: [3], [6], and [Œ 103, 106, 112].
- Relations between the cohomology of arithmetic groups and that of symmetric spaces.

This last theme leads Borel to one of his most beautiful results: a stability theorem ([Œ 93, 100, 118]) that gives the determination of the ranks of the *K-theory groups of $\mathbb{Z}$* (and, more generally, of the ring of integers of any number field). This leads him to the definition of a *regulator*, which he shows is essentially a *value of a zeta* function at an integer point ([Œ 108]).

History of Mathematics. In the last ten years of his life, Borel published a series of articles of a kind at once historical and mathematical on the following topics:

- Topology: on D. Montgomery ([Œ 357]), J. Leray ([Œ 164]), and A. Weil ([Œ 168]).
- Group theory: on H. Weyl ([Œ 132]), C. Chevalley ([Œ 143]), and E. Kolchin ([Œ 171]).
- Special relativity ([Œ 173]).

Some of his papers have been reprinted, and completed, in the last book that he published, his *Essays in the History of Lie Groups and Algebraic Groups* ([7]). It is a fascinating book, which leads us through a century of group theory, from Sophus Lie to Chevalley, past É. Cartan and H. Weyl. A "guided tour", with such a guide, what a pleasure!

The work of Borel is not at all limited to the texts that I have just invoked. He was an enthusiastic organizer. We owe to him several particularly successful seminars and summer schools, notably on group actions ([1]), algebraic groups ([3], [2]), continuous cohomology ([5]), modular forms ([4]), and D-modules ([6]). I wish to mention also his contribution to Bourbaki (from 1950 until his retirement in 1973, cf. [Œ 165]), to which he brought both common sense and expertise. The chapters on Lie groups ([LIE]) owe a great deal to him. Borel received the Balzan Prize in 1992, with the following citation: *"For his fundamental contributions to the theory of Lie groups, algebraic groups and arithmetic groups, and for his indefatigable action in favour of high quality in mathematical research and of the propagation of new ideas."*

One would not know how to say it better.

REFERENCES

[Œ] *Œuvres: Collected Papers*, Springer-Verlag, Berlin; vol. I, II, III, 1983; vol. IV, 2001.

[1] *Introduction aux groupes arithmétiques*, Actualités Sci. Indust., vol. 1341, Hermann, Paris, 1969.

[2] (Edited with G. D. Mostow) *Algebraic Groups and Discontinuous Subgroups*, Proc. Sympos. Pure Math., vol. IX, American Mathematical Society, Providence, RI, 1966.

[3] (Written with R. Carter, C. W. Curtis, N. Iwahori, T. A. Springer, R. Steinberg) *Seminar on Algebraic Groups and Related Finite Groups*, Lecture Notes in Math., vol. 131, Springer-Verlag, Berlin, 1970.

[4] (Edited with W. Casselman) *Automorphic Forms, Representations and L–Functions*, Proc. Sympos. Pure Math., vol. XXXIII, Parts I and II, American Mathematical Society, Providence, RI, 1979.

[5] (Written with N. R. Wallach) *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*, Ann. of Math. Stud., vol. 94, Princeton University Press, Princeton, NJ, and University of Tokyo Press, Tokyo, 1980; 2nd edition, Math. Surveys Monogr., vol. 67, American Mathematical Society, Providence, RI, 2000.

[6] (Written with P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, F. Ehlers) *Algebraic D–Modules*, Perspect. Math., vol. 2, Academic Press, Inc., Boston, MA, 1987.

[7] *Essays in the History of Lie Groups and Algebraic Groups*, History Math., vol. 21, American Mathematical Society, Providence, RI, and London Mathematical Society, Cambridge, 2001.

[Che] C. Chevalley, *Classification des groupes algébriques*, Séminaire École Normale Supérieure 1956–1958, Secrétariat de Mathématiques, Institut Henri Poincaré, Paris, 1958; 2nd edition, Springer-Verlag, to appear.

[LIE] N. Bourbaki, *Éléments de Mathématique: Groupes et Algèbres de Lie*, Chaps. I to IX, Hermann, Paris, and Masson, Paris, 1960 to 1982.