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Attractors for a Cahn-Hilliard-Navier-Stokes Model with Delays

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Abstract

In this article, we study a coupled Cahn-Hilliard-Navier-Stokes model with delays in a two-dimensional domain. The model consists of the Navier-Stokes equations for the velocity, coupled with a Cahn-Hilliard model for the order (phase) parameter. We prove the existence of an attractor using the theory of pullback attractors.

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1 Introduction

It is well accepted that the incompressible Navier-Stokes (NS) equation governs the motions of single-phase fluids such as air or water. On the other hand, we are faced with the difficult problem of understanding the motion of binary fluid mixtures, that is fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids, [9]. For instance, this approach is used in [1] to describe cavitation phenomena in a flowing liquid. The model consists of the NS equation coupled with the phase-field system, [2, 9, 8, 10]. In the isothermal compressible case, the existence of a global weak solution is proved in [7]. In the incompressible isothermal case, neglecting chemical reactions and other forces, the model reduces to an evolution system which governs the fluid velocity v and the order parameter ϕ. This system can be written as a NS equation coupled with a convective Allen-Cahn equation, [9]. The associated initial and boundary value problem was studied in [9] in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space which possesses a global attractor. They also established the existence of an exponential attractor. This entails that the global attractor has a finite fractal dimension, which is estimated in [9] in terms of some model parameters. The dynamic of simple single-phase fluids has been widely investigated although some important issues remain unresolved, [17]. In the case of binary fluids, the analysis is even more complicate and the mathematical studied is still at it infancy as noted in [9].

As noted in [8], the mathematical analysis of binary fluid flows is far from being well understood. For instance, the spinodal decomposition under shear consists of a two-stage evolution of a homogeneous initial mixture: a phase separation stage in which some macroscopic patterns appear, then

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a shear stage in which these patters organize themselves into parallel layers (see, e.g. [16] for experimental snapshots). This model has to take into account the chemical interactions between the two phases at the interface, achieved using a Cahn-Hilliard approach, as well as the hydrodynamic properties of the mixture (e.g., in the shear case), for which a Navier-Stokes equations with surface tension terms acting at the interface are needed. When the two fluids have the same constant density, the temperature differences are negligible and the diffuse interface between the two phases has a small but non-zero thickness, a well-known model is the so-called "Model H" (cf. [11]). This is a system of equations where an incompressible Navier-Stokes equation for the (mean) velocity v is coupled with a convective Cahn-Hilliard equation for the order parameter ϕ, which represents the relative concentration of one of the fluids.

In [4, 5, 6], the authors studied the NS equations in which the forcing term contains some hereditary features. The model can be used for instance to control a system by applying a force which takes into account not only the present state of the system, but also the history of the solutions. The existence and uniqueness of solutions to the 2D NS equations with delays was investigated in [4] and the asymptotic behavior of the solutions is studied in [5]. The existence of attractors for the 2D NS equations with delays is proved in [6]. In [3], the authors studied the existence of an attractor for the 3D Lagrangian averaged Navier-Stokes $\alpha-$ (3D LAN-α) model with delays. Instead of working directly with the 3D LAN-α model, they proved the existence of attractors for an abstract delay model and then applied the result to the 3D LAN-α model.

In this article, we study an CH-NS model with delays. We prove the existence of an attractor when the external force contains some delays following some ideas of [6, 3]. Let us note that the coupling between the Navier-Stokes and the Cahn-Hilliard systems makes the analysis more involved.

The article is divided as follows. In the next section, we introduce the AC-NS model with delays and its mathematical setting. The main result appear in the third section.

2 The CH-NS model and its mathematical setting

2.1 Governing equations

In this article, we study a 2D Cahn-Hilliard-Navier-Stokes system with delays. More precisely, we assume that the domain $\mathcal{M}$ of the fluid is a bounded domain in $\Re^2$. Then, we consider the system

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v + \nabla p - \mathcal{K}\mu\nabla\phi = g(t) + \mathcal{G}(t, u_t), \\ \text{div } v = 0, \\ \dfrac{\partial \phi}{\partial t} + v\cdot\nabla\phi - \Delta\mu = 0, \\ \mu = -\epsilon\Delta\phi + \alpha f(\phi), \end{cases} \tag{2.1}$$

in $\mathcal{M}\times(0,+\infty)$.

In (2.1), the unknown functions are the velocity $v = (v_1, v_2)$ of the fluid, its pressure p and the order (phase) parameter ϕ. The quantity μ is the variational derivative of the following free energy functional

$$\mathcal{F}(\phi) = \int_{\mathcal{M}}\Big(\frac{\epsilon}{2}|\nabla\phi|^2 + \alpha F(\phi)\Big)ds, \tag{2.2}$$

where, e.g., $F(r) = \displaystyle\int_0^r f(\zeta)d\zeta$. Here, the constants $\nu > 0$, $\epsilon > 0$ and $\mathcal{K} > 0$ correspond to the kine-

matic viscosity of the fluid, the mobility constant and the capillarity (stress) coefficient respectively. Here $\epsilon,\ \alpha > 0$ are two physical parameters describing the interaction between the two phases. In particular, ϵ is related with the thickness of the interface separating the two fluids, [8].
A typical example of potential F is that of logarithmic type (see [8]). However, this potential is often replaced by a polynomial approximation of the type $F(r) = \gamma_1 r^4 - \gamma_2 r^2$, γ_1, γ_2 being positive constants. As noted in [8], $(2.1)_1$ can be replaced by

$$\frac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v + \nabla\tilde{p} = -\mathcal{K}\mathrm{div}\ (\nabla\phi\oplus\nabla\phi) + Q(t-\tau(t),(v,\phi)(t-\tau(t))), \tag{2.3}$$

where $\tilde{p} = p - \mathcal{K}(\frac{\epsilon}{2}|\nabla\phi|^2 + \alpha F(\phi))$, since $\mathcal{K}\mu\nabla\phi = \mathcal{K}(\frac{\epsilon}{2}|\nabla\phi|^2 + \alpha F(\phi)) - \mathcal{K}\mathrm{div}\ (\nabla\phi\oplus\nabla\phi)$. The stress tensor $\nabla\phi\oplus\nabla\phi$ is considered the main contribution modeling capillary forces due to surface tension at the interface between the two phases of the fluid.
Regarding the boundary conditions for these models, as in [8] we assume that the boundary conditions for ϕ are the natural no-flux condition

$$\partial_\eta\phi = \partial_\eta\mu = 0, \ \text{ on } \partial\mathcal{M}\times(0,\infty), \tag{2.4}$$

where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$ and η is the outward normal to $\partial\mathcal{M}$. These conditions ensure the mass conservation. In fact, from $\partial_\eta\mu = 0$ on $\partial\mathcal{M}\times(0,\infty)$, we have the conservation of the following quantity

$$\langle\phi(t)\rangle = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}}\phi(x,t)dx, \tag{2.5}$$

where $|\mathcal{M}|$ stands for the Lebesgue measure of $\mathcal{M}$. More precisely, we have

$$\langle\phi(t)\rangle = \langle\phi(0)\rangle,\ \forall t \geq 0. \tag{2.6}$$

Concerning the boundary condition for u, we assume the Dirichlet (no-slip) boundary condition

$$v = 0, \ \text{ on } \partial\mathcal{M}\times(0,\infty). \tag{2.7}$$

Therefore we assume that there is no relative motion at the fluid-solid interface.
The initial condition is given by

$$\begin{aligned} &u(\tau) = (v,\phi)(\tau) = u_0 = (v_0,\phi_0),\\ &u(t) = (v,\phi)(t) = \vartheta(t) = (\vartheta_1,\vartheta_2)(t)\ t\in(\tau-h,\tau), \end{aligned} \tag{2.8}$$

where $u_0 = (v_0,\phi_0)$ and ϑ are given initial data at $t = \tau$ and in the interval $(-h,0)$ respectively, and $h > 0$ is a fixed time.
The terms $g(t)$ and $\mathcal{G}(t,u_t)$ are the external forcing depending eventually on the the history of the solution $u = (v,\phi)$.

2.2 Mathematical setting

We first recall from [8] a weak formulation of (2.1)-(2.8). Hereafter, we assume that the domain $\mathcal{M}$ is bounded with a smooth boundary $\partial\mathcal{M}$ (e.g., of class C^2). We also assume that $f\in C^3(\mathfrak{R})$ satisfies

$$\begin{cases} \lim\limits_{|r|\to+\infty} f^{'}(r) > 0, \\ |f^i(r)| \leq c_f(1+|r|^{k-i+1}),\ \forall r\in\mathfrak{R},\ i = 0,1,2,3, \end{cases} \tag{2.9}$$

where c_f is some positive constant and $k \in [2, +\infty)$ is fixed.
If X is a real Hilbert space with inner product $(\cdot,\cdot)_X$, we will denote the induced norm by $|\cdot|_X$, while X^* will indicate its dual. We set

$$\mathcal{V}_1 = \{u \in C_c^\infty(\mathcal{M}) : \ \text{div } u = 0 \text{ in } \mathcal{M}\}.$$

We denote by H_1 and V_1 the closure of $\mathcal{V}_1$ in $(L^2(\mathcal{M}))^2$ and $(H_0^1(\mathcal{M}))^2$ respectively. The scalar product in H_1 is denoted by $(\cdot,\cdot)_{L^2}$ and the associated norm by $|\cdot|_{L^2}$. Moreover, the space V_1 is endowed with the scalar product

$$((u,v)) = \sum_{i=1}^{2} (\partial_{x_i} u, \partial_{x_i} v)_{L^2}, \ \ \|u\| = ((u,u))^{1/2}.$$

We now define the operator A_0 by

$$A_0 v = \mathcal{P}\Delta v, \ \forall v \in D(A_0) = H^2(\mathcal{M}) \cap V_1,$$

where $\mathcal{P}$ is the Leray-Helmotz projector in $L^2(\mathcal{M})$ onto H_1. Then, A_0 is a self-adjoint positive unbounded operator in H_1 which is associated with the scalar product defined above. Furthermore, A_0^{-1} is a compact linear operator on H_1 and $|A_0 \cdot|_{L^2}$ is a norm on $D(A_0)$ that is equivalent to the H^2-norm.
Hereafter, we set

$$H_2 = L^2(\mathcal{M}), \ V_2 = H^1(\mathcal{M}), \ H = H_1 \times H_2, \ V = V_1 \times V_2. \tag{2.10}$$

We will denote by $\lambda_1 > 0$ a positive constant such that

$$\lambda_1 |w|_{L^2}^2 \le \|w\|^2 \ \forall w \in V_1, \ \lambda_1 \|\psi\|^2 \le \ |A_N \psi|_{L^2}^2 \ \forall \psi \in H^2(\mathcal{M}). \tag{2.11}$$

Then we introduce the linear nonnegative unbounded operator on $L^2(\mathcal{M})$

$$A_N \phi = -\Delta\phi, \ \forall \phi \in D(A_N) = \{\phi \in H^2(\mathcal{M}), \ \partial_\eta \phi = 0, \ \text{ on } \partial\mathcal{M}\}, \tag{2.12}$$

and we endow $D(A_N)$ with the norm $|A_N \cdot|_{L^2} + |\langle \cdot \rangle|_{L^2}$, which is equivalent to the H^2-norm. Also we define the linear positive unbounded operator on the Hilbert space $L_0^2(\mathcal{M})$ of the L^2- functions with null mean

$$B_N \phi = -\Delta\phi, \ \forall \phi \in D(B_N) = D(A_N) \cap L_0^2(\mathcal{M}). \tag{2.13}$$

Note that B_N^{-1} is a compact linear operator on $L_0^2(\mathcal{M})$. More generally, we can define B_N^s for any $s \in \mathfrak{R}$, noting that $|B_N^{s/2} \cdot|_{L^2}, \ s > 0$, is an equivalent norm to the canonical H^s- norm on $D(B_N^{s/2}) \subset H^s(\mathcal{M}) \cap L_0^2(\mathcal{M})$. Also note that $A_N = B_N$ on $D(B_N)$. If ϕ is such that $\phi - \langle\phi\rangle \in D(B_N^{s/2})$, we have that $|B_N^{s/2}(\phi - \langle\phi\rangle)|_{L^2} + |\langle\phi\rangle|_{L^2}$ is equivalent to the H^s-norm. Moreover, we set $H^{-s}(\mathcal{M}) = (H^s(\mathcal{M}))^*$, whenever $s < 0$.
We introduce the bilinear operators B_0, B_1 (and their associated trilinear forms b_0, b_1) as well as the coupling mapping R_0, which are defined from $D(A_0) \times D(A_0)$ into H, $D(A_0) \times D(A_N)$ into $L^2(\mathcal{M})$, and $L^2(\mathcal{M}) \times (D(A_N) \cap H^3(\mathcal{M}))$ into H_1, respectively. More precisely, we set

$$\begin{aligned}
&(B_0(u,v),w) = \int_{\mathcal{M}} [(u\cdot\nabla)v]\cdot w dx = b_0(u,v,w), \ \forall u,v,w \in D(A_0),\\
&(B_1(u,\phi),\rho) = \int_{\mathcal{M}} [(u\cdot\nabla)\phi]\rho dx = b_1(u,\phi,\rho), \ \forall u \in D(A_0), \ \phi,\rho \in D(A_N),\\
&(R_0(\mu,\phi),w) = \int_{\mathcal{M}} \mu[\nabla\phi\cdot w]dx = b_1(w,\phi,\mu), \ \forall w \in D(A_0), \ \phi \in D(A_N) \cap H^3(\mathcal{M}), \ \mu \in L^2(\mathcal{M}).
\end{aligned} \tag{2.14}$$

Note that

$$R_0(\mu,\phi) = \mathcal{P}\mu\nabla\phi.$$

We recall that B_0, B_1 and R_0 satisfy the following estimates

$$\begin{aligned}
&|B_0(u,v)|_{V_1^*} \le c|u|_{L^2}^{1/2}\|u\|^{1/2}\|v\|,\ \forall u,v \in V_1,\\
&|B_0(u,v)|_{L^2} \le c|u|_{L^2}^{1/2}\|u\|^{1/2}\|v\|^{1/2}|A_0v|_{L^2}^{1/2},\ \forall u \in V_1,\ v \in D(A_0),
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
&|B_1(u,\phi)|_{V_2^*} \le c|u|_{L^2}^{1/2}\|u\|^{1/2}\|\phi\|,\ \forall u \in V_1, \phi \in V_2,\\
&|B_1(u,\phi)|_{L^2} \le c|u|_{L^2}^{1/2}\|u\|^{1/2}\|\phi\|^{1/2}|A_N\phi|_{L^2}^{1/2},\ \forall u \in V_1,\ \phi \in D(A_N),
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
&|R_0(A_N\phi,\rho)|_{V_1^*} \le c|A_N\phi|_{L^2}^{1/2}|\phi|_{H^3}^{1/2}\|\rho\|,\ \forall \phi \in D(A_N),\ \rho \in V_2,\\
&|R_0(A_N\phi,\rho)|_{L^2} \le c\|\rho\|^{1/2}|A_N\rho|_{L^2}^{1/2}|A_N\phi|_{L^2}^{1/2}|\phi|_{H^3}^{1/2},\ \forall \phi \in D(A_N), \rho \in D(A_N^{3/2}).
\end{aligned} \tag{2.17}$$

We recall that (due to the mass conservation) we have

$$\langle \phi(t) \rangle = \langle \phi(0) \rangle = M_0,\ \forall t > 0. \tag{2.18}$$

Thus, up to a shift of the order parameter field, we can always assume that the mean of ϕ is zero a the initial time and, therefore it will remain zero for all positive times. Hereafter, we assume that

$$\langle \phi(t) \rangle = \langle \phi(0) \rangle = 0,\ \forall t > 0. \tag{2.19}$$

We set

$$\mathbb{Y} = H_1 \times D(B_N^{1/2}). \tag{2.20}$$

The space $\mathbb{Y}$ is a complete metric space with respect to the norm

$$|(v,\phi)|_{\mathbb{Y}}^2 = \mathcal{K}^{-1}|v|_{L^2}^2 + \epsilon|\nabla\phi|_{L^2}^2. \tag{2.21}$$

We define the Hilbert space $\mathbb{V}$ by

$$\mathbb{V} = V_1 \times D(B_N), \tag{2.22}$$

endowed with the scalar products whose associated norm is

$$\|(v,\phi)\|_{\mathbb{V}}^2 = \|v\|^2 + |B_N\phi|_{L^2}^2. \tag{2.23}$$

Now we make the following assumptions on the external force $\mathcal{G}$. Hereafter, for a given Banach space X, we denote by C_X, L_X^2 and M_X respectively the spaces

$$C_X = C^0(-h,0;X),\ L_X^2 = L^2(-h,0;X),\ M_X = X \times L_X^2. \tag{2.24}$$

The space M_X is endowed with the norm

$$\|(u,\vartheta)\|_{M_X}^2 = \|u\|_X^2 + \int_{-h}^0 \|\vartheta(s)\|_X^2 ds. \tag{2.25}$$

First for $T > \tau$, $u : (\tau - h, T) \longrightarrow \mathbb{Y}$ and $t \in (\tau, T)$, we denote by u_t the function defined on $(-h,0)$ by

$$u_t(s) = u(t+s) \text{ for } s \in (-h,0). \tag{2.26}$$

We suppose that

$$\mathcal{G} : \Re \times C_{H_1} \longrightarrow H_1 \tag{2.27}$$

satisfies

$$\text{for any } v \in C_{H_1},\ t \in \Re \longmapsto \mathcal{G}(t,v) \text{ is measurable,} \tag{2.28}$$

$$\text{for any } t \in \Re,\ \mathcal{G}(t,0) = 0, \tag{2.29}$$

$$\begin{aligned} &\exists L_g > 0, \text{ s.t. } \forall t \in \Re,\ \forall u,v \in C_{H_1}, \\ &|\mathcal{G}(t,u) - \mathcal{G}(t,v)|_{L^2} \le L_g \|u - v\|_{C_{H_1}}, \end{aligned} \tag{2.30}$$

$$\begin{aligned} &\exists m_0 > 0, C_g > 0, \text{ s.t. } \forall m \in [0,m_0], \tau \le t \le T,\ u,v \in C^0(\tau - h, T; H_1) \\ &\int_\tau^t e^{ms} |\mathcal{G}(s,u_s) - \mathcal{G}(s,v_s)|_{L^2}^2 \le C_g \int_{\tau-h}^t e^{ms} |u(s) - v(s)|_{L^2}^2 ds, \end{aligned} \tag{2.31}$$

For $u \in C^0(\tau - h, T; H_1)$ we define the mapping $\mathcal{G}_u$ by

$$\begin{aligned} \mathcal{G}_u :\quad & [\tau, T] \longrightarrow H_1 \\ & t \longmapsto \mathcal{G}_u(t) = \mathcal{G}(t,u). \end{aligned} \tag{2.32}$$

Note that

$$\mathcal{G}_u \text{ is measurable and } \mathcal{G}_u \in L^\infty(\tau, T; H_1). \tag{2.33}$$

Now we define the mapping $\tilde{\mathcal{G}}$ by

$$\begin{aligned} \tilde{\mathcal{G}} :\quad & C^0(\tau - h, T; H_1) \longrightarrow L^2(\tau, T; H_1) \\ & u \longmapsto \mathcal{G}_u, \end{aligned} \tag{2.34}$$

which has a unique extension to a uniformly continuous mapping $\tilde{\mathcal{G}}$ defined from $L^2(\tau - h, T; H_1)$ into $L^2(\tau, T; H_1)$.
Hereafter we will denote $\mathcal{G}(t, u_t) = \tilde{\mathcal{G}}(u)(t)$ for $u \in L^2(\tau - h, T; H_1)$. Therefore, $\forall t \in [\tau, T], u, v \in L^2(\tau - h, T; H_1)$, we have

$$\int_\tau^t |\mathcal{G}(s,u_s) - \mathcal{G}(s,v_s)|_{L^2}^2 ds \le C_g^2 \int_{\tau-h}^t |u(s) - v(s)|_{L^2}^2 ds. \tag{2.35}$$

We assume that

$$u_0 = (v_0, \phi_0) \in \mathbb{Y},\ \vartheta \in L^2_{\mathbb{Y}},\ g \in L^2_{loc}(\Re; V_1^*)$$

and $\mathcal{G} : \Re \times C_{H_1} \to H_1$ satisfies (2.27)-(2.31).
Using the notations above, we rewrite (2.1), (2.4), (2.7)-(2.5) as (see [8] for the details)

$$\begin{cases} \dfrac{dv}{dt} + \nu A_0 v + B_0(v,v) - \mathcal{K} R_0(\epsilon A_N \phi, \phi) = g(t) + \mathcal{G}(t, v_t), \\ \\ \dfrac{d\phi}{dt} + A_N \mu + B_1(v, \phi) = 0,\ \mu = \epsilon A_N \phi + \alpha f(\phi), \\ \\ (v,\phi)(\tau) = (v_0, \phi_0)\ (v,\phi)(t) = \vartheta(t) = (\vartheta_1, \vartheta_2)(t),\ t \in (\tau - h, \tau). \end{cases} \tag{2.36}$$

Remark 2.1. In the weak formulation (2.36), the term $\mu\nabla\phi$ is replaced by $\epsilon A_N\nabla\phi$. This is justified since $f^{'}(\phi)\nabla\phi$ is the gradient $F(\phi)$ and can be incorporated into the pressure gradient, see [8] for details. For the sake of convenience, as in [8] we will replace μ in $(2.36)_3$ by $\bar{\mu} = \mu - \langle\mu\rangle$, that is $\bar{\mu} = \epsilon A_N\phi + \alpha f(\phi) - \alpha\langle f(\phi)\rangle$, a.e. in $\mathcal{M}\times(0,+\infty)$. Obviously we have $\langle\bar{\mu}(t)\rangle = 0\ \forall t > 0$.

Definition 2.2. A pair (v,ϕ) is called a weak solution to (2.36) if

$$(v,\phi) \in C([\tau,T];\mathbb{Y}) \cap L^2([\tau,T];\mathbb{V}),\ \frac{dv}{dt} \in L^1([\tau,T];V_1^*),\ \frac{d\phi}{dt},\ \mu \in L^1([\tau,T];V_2^*) \tag{2.37}$$

and (v,ϕ) satisfies $(2.36)_1$ and $(2.36)_3$ in V_1^* and V_2^* respectively.
For $(v_0,\phi_0) \in \mathbb{V}$, a weak solution (v,ϕ) is called a strong solution on the time interval $[0,T]$ if in addition to (3.51), it satisfies

$$v \in C([\tau,T];V_1) \cap L^2(\tau,T;D(A_0)),\ \phi \in C([\tau,T];D(A_N)) \cap L^2(\tau,T;D(A_N)\cap H^3(\mathcal{M})). \tag{2.38}$$

In the case when the delay terms τ and $\mathcal{G}$ are zero, the weak formulation of (2.36) was proposed and studied in [9, 8], and the existence and uniqueness of solution was proved. The CH-NS with delay is studied in [15], where the author proved the existence and uniqueness of weak and strong solutions. In particular, the following result can be proved as Theorem 3.1 in [15].

Theorem 2.3. *Let* $u_0 = (v_0,\phi_0) \in \mathbb{Y}$, $\vartheta \in L^2_{\mathbb{Y}}$, $g \in L^2_{loc}(\mathfrak{R};V_1^*)$ *and assume that* $\mathcal{G} : \mathfrak{R}\times C_{H_1} \to H_1$ *satisfies (2.27)-(2.31). Then, for each* $\tau \in \mathfrak{R}$,
(1) there exists a unique solution $u = (v,\phi)$ *to (2.36) that belongs to the space* $C([\tau,+\infty);\mathbb{Y})$.
(2) If $g \in L^2_{loc}(\mathfrak{R};H_1)$ *and* $u_0 = (v_0,\phi_0) \in \mathbb{V}$, *then the solution* $u = (v,\phi)$ *is a strong solution. In particular, if* $\vartheta \in C_{\mathbb{V}}$ *and* $u_0 = (v_0,\phi_0) = \vartheta(0)$, *then* $u = (v,\phi) \in C([\tau-h,+\infty);\mathbb{V})$.

Proof. See [15]. □

Hereafter, to simplify the notation, we set $\mathcal{K} = 1$.

3 Existence of an attractor

3.1 Preliminaries on pullback attractors

We first recall some results on the theory of pullback attractors as developed in [12, 13]. For more details, the reader is referred to [6, 3, 12, 13].
It is well known that for non-autonomous differential equations, the initial time is as important as the final one. Therefore the classical semigroup property of autonomous dynamical systems is no longer available. Instead of a one time-dependent map $S(t)$, we need to use a two-parameter process $U(t,\tau)$ on a complete metric space X. Hereafter $U(t,\tau)\psi$ denotes the value of the solution at the time t which was equal to the initial value ψ at the time τ.

Definition 3.1. Let X be a complete metric space. A family of mappings $\{U(t,\tau), t,\tau \in \mathfrak{R}, t \geq \tau\} \subset C^0(X,X)$ is said to be a process (or a two-parameter semigroup) in X if

$$\begin{aligned} &U(t,\tau)U(\tau,r) = U(t,r) \text{ for all } t \geq \tau \geq r,\\ &U(\tau,\tau) = Id \text{ for all } \tau. \end{aligned} \tag{3.1}$$

The process $U(\cdot,\cdot)$ is said to be continuous if the mapping $(t,\tau) \mapsto U(t,\tau)x$ is continuous for all $x \in X$.

As in the standard theory of attractors, we are interested to invariant sets. Since the system is non-autonomous, these sets also depend on the time. Hereafter, if $A, B \subset X$, $\mathrm{dist}(A, B)$ denotes the Hausdorf semi-distance between the subsets A and B defined by:

$$\mathrm{dist}(A, B) = \sup_{a\in A} \inf_{a\in B} d(a, b). \tag{3.2}$$

Definition 3.2. The family of subsets $\{\mathcal{B}(t)\}_{t\in\Re} \subset X$ is said to be (pullback) absorbing with respect to the process U if, for all $t \in \Re$ and all $D \subset X$ bounded, there exists $T_D(t) > 0$ such that

$$U(t, t-s)D \subset \mathcal{B}(t) \text{ for all } s \geq T_D(s). \tag{3.3}$$

The absorption is said to be uniform if $T_D(t)$ does not depend on the variable t.

Definition 3.3. A family of compact sets $\{\mathcal{A}(t)\}_{t\in\Re}$ is said to be a (global) pullback attractor for the process U if, for all $\tau \in \Re$, it satisfies:

$$\begin{aligned} &U(t, \tau)\mathcal{A}(t) = \mathcal{A}(t) \text{ for all } t \geq \tau,\\ &\lim_{s\to\infty} \mathrm{dist}(U(t, t-s)D, \mathcal{A}(t)) = 0 \text{ for all bounded subsets } D \subset X. \end{aligned} \tag{3.4}$$

The pullback attractor $\{\mathcal{A}(t)\}_{t\in\Re}$ is said to be uniform if the attraction property is uniform in time, i.e;

$$\lim_{s\to\infty} \sup_{t\in\Re} \mathrm{dist}(U(t, t-s)D, \mathcal{A}(t)) = 0 \text{ for all bounded subsets } D \subset X. \tag{3.5}$$

Definition 3.4. A family of compact sets $\{\mathcal{A}(t)\}_{t\in\Re}$ is said to be a (global) forward attractor for the process U if, for all $\tau \in \Re$, it satisfies:

$$\begin{aligned} &U(t, \tau)\mathcal{A}(t) = \mathcal{A}(t) \text{ for all } t \geq \tau,\\ &\lim_{s\to\infty} \mathrm{dist}(U(t, \tau)D, \mathcal{A}(t)) = 0 \text{ for all bounded subsets } D \subset X. \end{aligned} \tag{3.6}$$

The forward attractor $\{\mathcal{A}(t)\}_{t\in\Re}$ is said to be uniform if the attraction property is uniform in time, i.e;

$$\lim_{s\to\infty} \sup_{\tau\in\Re} \mathrm{dist}(U(t+\tau, \tau)D, \mathcal{A}(t)) = 0 \text{ for all bounded subsets } D \subset X. \tag{3.7}$$

The following result is given in [6, 3].

Theorem 3.5. *Let $U(t, \tau)$ be a two-parameter process such that $U(t, \tau) : X \to X$ is continuous for all $t \geq \tau$. If there exists a family of compact (pullback) absorbing sets $\{\mathcal{B}(t)\}_{t\in\Re}$, then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in\Re}$, and $\mathcal{A}(t) \subset \mathcal{B}(t)$ for all $t \in \Re$. Furthermore,*

$$\mathcal{A}(t) = \overline{\bigcup_{\substack{D \subset X \\ \textit{bounded}}} \bigwedge(D, t)}, \tag{3.8}$$

where

$$\bigwedge(D, t) = \bigcap_{n\in\aleph} \overline{\bigcup_{s\geq n} U(t, t-s)D}. \tag{3.9}$$

3.2 Construction of the associated processes

For $\tau \in \Re$, $u_0 = (v_0, \phi_0) \in \mathbb{Y}$ and $\vartheta \in L^2_{\mathbb{Y}}$, hereafter we will denote by $u(.;\tau,(u_0,\vartheta)) \equiv (v,\phi)(.;\tau,(u_0,\vartheta))$ the unique solution to (2.36), which belongs to

$$L^2(\tau,T;\mathbb{V}) \cap L^2(\tau-h,T,\mathbb{Y}) \cap C^0(\tau,T;\mathbb{Y}),\ \forall T > \tau. \tag{3.10}$$

To study the long-time behavior of (2.36), we define the processes $U = U(t,\tau)$ and $S = S(t,\tau)$ as follows.
For $\tau \le t$, $U(t,\tau)$ and $S(t,\tau)$ are given by:

$$\begin{aligned} U(t,\tau): \quad & C_{\mathbb{Y}} \longrightarrow C_{\mathbb{Y}} \\ & \vartheta \longmapsto U(t,\tau) = u_t(.;\tau,(\vartheta(0),\vartheta)). \end{aligned} \tag{3.11}$$

$$\begin{aligned} S(t,\tau): \quad & \mathcal{M}_{\mathbb{Y}} \longrightarrow \mathcal{M}_{\mathbb{Y}} \\ & (u_0,\vartheta) \longmapsto S(t,\tau)(u_0,\vartheta) = (u(t;\tau,(u_0,\vartheta)), u_t(.;\tau,(u_0,\vartheta))). \end{aligned} \tag{3.12}$$

For $\tau \le t$, we define the mappings $\tilde{U}(t,\tau)$ by:

$$\begin{aligned} \tilde{U}(t,\tau): \quad & \mathcal{M}_{\mathbb{Y}} \longrightarrow L^2_{\mathbb{Y}} \\ & (u_0,\vartheta) \longmapsto \tilde{U}(t,\tau)(u_0,\vartheta) = u_t(.;\tau,(u_0,\vartheta)). \end{aligned} \tag{3.13}$$

Then it is clear that for $\tau \le t$, we have

$$\begin{aligned} & U(t,\tau)\phi = \tilde{U}(t,\tau)(\vartheta(0),\vartheta),\ \forall \vartheta \in C_{\mathbb{Y}}, \\ & S(t,\tau)(u_0,\vartheta) = (u(t;\tau,(u_0,\vartheta)), \tilde{U}(t,\tau)(u_0,\vartheta)),\ \forall (u_0,\vartheta) \in \mathcal{M}_{\mathbb{Y}}. \end{aligned} \tag{3.14}$$

Remark 3.6. If we define the mapping j by

$$\begin{aligned} j: \quad & C_{\mathbb{Y}} \longrightarrow \mathbb{Y} \times C_{\mathbb{Y}} \\ & \vartheta \longmapsto j(\vartheta) = (\vartheta(0),\vartheta), \end{aligned} \tag{3.15}$$

then

$$S(t,\tau)(u_0,\vartheta) = j(\tilde{U}(t,\tau)(u_0,\vartheta)),\ \forall (u_0,\vartheta) \in \mathcal{M}_{\mathbb{Y}},\ t \ge \tau + h. \tag{3.16}$$

Note that for $(u_0,\vartheta) \in \mathcal{M}_{\mathbb{Y}}$, $\tilde{U}(t,\tau)(u_0,\vartheta) \in C_{\mathbb{Y}}$ provided that $t \ge \tau + h$.

Lemma 3.7. *Let $(u_0,\vartheta), (\tilde{u}_0,\tilde{\vartheta}) \in \mathcal{M}_{\mathbb{Y}}$ be two initial data for (2.36), where $u_0 = (v_0,\phi_0)$, $\tilde{u}_0 = (\tilde{v}_0,\tilde{\phi}_0)$. Let $(v,\phi)(\cdot) = u(\cdot) = u(\cdot,\tau,(u_0,\vartheta))$, $(\tilde{v},\tilde{\phi})(\cdot) = \tilde{u}(\cdot) = \tilde{u}(\cdot;\tau,(\tilde{u}_0,\tilde{\vartheta}))$ be the corresponding solutions given by Theorem 2.3. Then for all $t \ge \tau$, we have*

$$|u(t) - \tilde{u}(t)|^2_{\mathbb{Y}} \le \left(C_g^2 \|\vartheta - \tilde{\vartheta}\|^2_{L^2_{\mathbb{Y}}} + |u_0 - \tilde{u}_0|^2_{\mathbb{Y}} \right) \times \exp\left(\int_\tau^t (C_g^2 + \Upsilon(s)) ds \right), \tag{3.17}$$

where

$$\Upsilon(t) = c\|v\|^2 + c(1 + \|\phi\|^2)|A_N\phi|^2_{L^2} + c|\tilde{v}|^2_{L^2}\|\tilde{v}\|^2 + Q_1(|\phi|_{H^1}, |\tilde{\phi}|_{H^1})(1 + |A_N\phi|^2_{L^2} + |A_N\tilde{\phi}|^2_{L^2}),$$

$c > 0$ is a constant and $Q_1 \equiv Q_1(\cdot,\cdot)$ monotone non decreasing function independent of the time and the initial data. Moreover for all $t \ge \tau + h$, we have

$$\|u_t - \tilde{u}_t\|^2_{C_{\mathbb{Y}}} \le \left(C_g^2 \|\vartheta - \tilde{\vartheta}\|^2_{L^2_{\mathbb{Y}}} + |u_0 - \tilde{u}_0|^2_{\mathbb{Y}} \right) \times \exp\left(\int_\tau^t (C_g^2 + \Upsilon(s)) ds \right). \tag{3.18}$$

Proof. Let us set $(w,\psi) = (v,\phi) - (\tilde{v},\tilde{\phi})$, $\tilde{\mu} = \epsilon A_N\psi + \alpha(f(\phi_1) - f(\phi_2))$. Then (w,ψ) satisfies

$$\frac{dw}{dt} + \nu A_0 w + B_0(v,w) + B_0(w,\tilde{v}) - R_0(\epsilon A_N\psi,\phi) - R_0(\epsilon A_N\tilde{\phi},\psi)$$

$$= \mathcal{G}(t,v_t) - \mathcal{G}(t,\tilde{v}_t), \tag{3.19}$$

$$\frac{d\psi}{dt} + A_N\bar{\mu} + B_1(w,\phi) + B_1(\tilde{v},\psi) = 0,\ \ \bar{\mu} = \tilde{\mu} - \langle\tilde{\mu}\rangle.$$

Multiplying $(3.19)_1$ by w, $(3.19)_3$ and $(3.19)_2$ respectively by $A_N\bar{\mu} + \epsilon\xi A_N\psi$ (with $\xi > 0$ sufficiently small to be selected in the sequel) and $\epsilon A_N\psi$, respectively, we derive as in [14, 9] that

$$\frac{dy}{dt} + \nu\|w\|^2 + (1-c\xi)\|\bar{\mu}\|^2 + \frac{\epsilon^2\xi}{2}|A_N\psi|^2_{L^2} \leq \Upsilon(t)y(t) + c|\mathcal{G}(t,v_t) - \mathcal{G}(t,\tilde{v}_t)|^2_{L^2}, \tag{3.20}$$

where $c = c_{\mathcal{M}}$ is a constant that depends only on $\mathcal{M}$ and

$$y(t) = |(w,\psi)|^2_{\mathbb{Y}},$$

$$\Upsilon(t) = c\|v\|^2 + c(1+\|\phi\|^2)|A_N\phi|^2_{L^2} + c|\tilde{v}|^2_{L^2}\|\tilde{v}\|^2 \tag{3.21}$$

$$+Q_1(|\phi|_{H^1},|\tilde{\phi}|_{H^1})(1+|A_N\phi|^2_{L^2} + |A_N\tilde{\phi}|^2_{L^2}).$$

It follows from (2.31) that (for $t \geq \tau$) (with ξ small enough such that $1 - c\xi > 0$)

$$y(t) \leq \left(C_g^2\|\vartheta - \tilde{\vartheta}\|^2_{L^2_{\mathbb{Y}}} + |u_0 - \tilde{u}_0|^2_{\mathbb{Y}}\right)\times\exp\left(\int_\tau^t (C_g^2 + \Upsilon(s))ds\right), \tag{3.22}$$

and (3.17) is proved. Note that $\displaystyle\int_\tau^t \Upsilon(s)ds < \infty$.

Now we assume that $t \geq \tau + h$. Then for $\zeta \in [-h,0]$, we have

$$\begin{aligned} y(t+\zeta) &\leq \left(C_g^2\|\vartheta - \tilde{\vartheta}\|^2_{L^2_{\mathbb{Y}}} + |u_0 - \tilde{u}_0|^2_{\mathbb{Y}}\right)\times\exp\left(\int_\tau^{t+\zeta} (C_g^2 + \Upsilon(s))ds\right) \\ &\leq \left(C_g^2\|\vartheta - \tilde{\vartheta}\|^2_{L^2_{\mathbb{Y}}} + |u_0 - \tilde{u}_0|^2_{\mathbb{Y}}\right)\times\exp\left(\int_\tau^{t} (C_g^2 + \Upsilon(s))ds\right). \end{aligned} \tag{3.23}$$

Therefore

$$\|u_t\|^2_{C_{\mathbb{Y}}} \leq \left(C_g^2\|\vartheta - \tilde{\vartheta}\|^2_{L^2_{\mathbb{Y}}} + |u_0 - \tilde{u}_0|^2_{\mathbb{Y}}\right)\times\exp\left(\int_\tau^t (C_g^2 + \Upsilon(s))ds\right), \tag{3.24}$$

and (3.18) is proved. □

We now prove that $U(.,.)$ and $S(.,.)$ are continuous processes.

Lemma 3.8. *The mappings U and S are processes. Moreover, $U(t,\tau): C_{\mathbb{Y}} \longrightarrow C_{\mathbb{Y}}$ and $S(t,\tau): \mathcal{M}_{\mathbb{Y}} \longrightarrow \mathcal{M}_{\mathbb{Y}}$ are continuous for $t \geq \tau$.*

Proof. We proceed as in [6, 3]. From the uniqueness of solutions to (2.36), we conclude that U and S are processes. To prove the second part of the lemma, we consider two solutions $u(\cdot)$ and $\tilde{u}(\cdot)$ to (2.36) corresponding to the initial data $(\vartheta(0),\vartheta)$ and $(\tilde{\vartheta}(0),\tilde{\vartheta})$ respectively. Then from (3.17), we have

$$|u(t) - \tilde{u}(t)|^2_{\mathbb{Y}} \leq (C_g^2 h + 1)\|\vartheta - \tilde{\vartheta}\|^2_{C_{\mathbb{Y}}}\times\exp\left(\int_\tau^t (C_g^2 + \Upsilon(s))ds\right), \tag{3.25}$$

for all $t \geq \tau$.
For $\tau - h \leq t \leq \tau$, we also have

$$u(t) - \tilde{u}(t) = \vartheta(t-\tau) - \tilde{\vartheta}(t-\tau), \tag{3.26}$$

which gives

$$|u(t) - \tilde{u}(t)|_{\mathbb{Y}}^2 \leq (C_g^2 h + 1)\|\vartheta - \tilde{\vartheta}\|_{C_{\mathbb{Y}}}^2 \times \exp\left(\int_{\tau-h}^t (C_g^2 + \Upsilon(s))ds\right), \tag{3.27}$$

for all $t \geq \tau - h$.
Finally we obtain

$$\|u_t - \tilde{u}_t\|_{C_{\mathbb{Y}}}^2 \leq (C_g^2 h + 1)\|\vartheta - \tilde{\vartheta}\|_{C_{\mathbb{Y}}}^2 \times \exp\left(\int_{\tau-h}^t (C_g^2 + \Upsilon(s))ds\right), \tag{3.28}$$

for all $t \geq \tau$, which shows that $U(t,\tau)$ is continuous.
For the continuity of S we note that for $t \geq \tau + h$, if (u_0, ϑ), $(\tilde{u}_0, \tilde{\vartheta}) \in \mathcal{M}_{\mathbb{Y}}$, and $u(.)$ and $\tilde{u}(.)$ are the corresponding solutions, then for $t \geq \tau + h$, we have (see (3.18))

$$\begin{aligned}
\|u_t - \tilde{u}_t\|_{L_{\mathbb{Y}}^2} &= \int_{-h}^0 |u(t+\zeta) - \tilde{u}(t+\zeta)|_{\mathbb{Y}}^2 d\zeta \\
&\leq \int_{-h}^0 \sup_{s\in[-h,0]} |u(t+s) - \tilde{u}(t+s)|_{\mathbb{Y}}^2 d\zeta \\
&\leq h\left(C_g^2\|\vartheta - \tilde{\vartheta}\|_{L_{\mathbb{Y}}^2}^2 + |u_0 - \tilde{u}_0|_{\mathbb{Y}}^2\right) \times \exp\left(\int_\tau^t (C_g^2 + \Upsilon(s))ds\right).
\end{aligned} \tag{3.29}$$

For $\tau \leq t \leq \tau + h$, we also have

$$\begin{aligned}
\|u_t - \tilde{u}_t\|_{L_{\mathbb{Y}}^2}^2 &= \int_{-h}^0 |u(t+\zeta) - \tilde{u}(t+\zeta)|_{\mathbb{Y}}^2 d\zeta \\
&\leq \left((C_g^2 h + 1)\|\vartheta - \tilde{\vartheta}\|_{L_{\mathbb{Y}}^2}^2 + h|u_0 - \tilde{u}_0|_{\mathbb{Y}}^2\right) \times \exp\left(\int_\tau^t (C_g^2 + \Upsilon(s))ds\right).
\end{aligned} \tag{3.30}$$

Then for $t \geq \tau$ we derive

$$\|u_t - \tilde{u}_t\|_{L_{\mathbb{Y}}^2}^2 \leq \left((C_g^2 h + 1)\|\vartheta - \tilde{\vartheta}\|_{L_{\mathbb{Y}}^2}^2 + h|u_0 - \tilde{u}_0|_{\mathbb{Y}}^2)\right) \times \exp\int_\tau^t (C_g^2 + \Upsilon(s))ds \tag{3.31}$$

and the continuity of S follows from (3.17). □

Lemma 3.9. *The mappings $U(t,\tau) : C_{\mathbb{V}} \longrightarrow C_{\mathbb{V}}$ and $S(t,\tau) : \mathcal{M}_{\mathbb{V}} \longrightarrow \mathcal{M}_{\mathbb{V}}$ are continuous for $t \geq \tau$.*

Proof. The proof is similar to that of Lemma 3.3. □

3.3 Absorbing set in $C_{\mathbb{Y}}$ and $\mathcal{M}_{\mathbb{Y}}$

In this part, we prove that for ν large enough, there exists a family of absorbing sets in $C_{\mathbb{Y}}$ and $\mathcal{M}_{\mathbb{Y}}$ for the processes $U(.,.)$ and $S(.,.)$.
We first recall from [6, 3] the following result.

Lemma 3.10. *Assume that the family of bounded sets $\{B(t)\}_{t\in\Re}$ in $C_{\mathbb{Y}}$ is absorbing (resp. attracting) for the family of mappings $\{U(t,\tau), t \geq \tau\}$. Then*
1) the family $\{B(t)\}_{t\in\Re}$ is absorbing (resp. attracting) for the process $U(.,.)$,
2) the family of bounded sets $j(B(t))_{t\in\Re}$ in $\mathbb{Y}\times C_{\mathbb{Y}}$ is absorbing (resp. attracting) for the process $S(.,.)$.

Proof. See [6, 3]. □

Theorem 3.11. *We assume that (2.27)-(2.34) are satisfied. For ν large enough such $\lambda_1\alpha_1 > C_g$, where α_1 is defined in (3.39) below and $\lambda_1 > 0$ is the constant defined in (2.11). Then, there exists a family $\{B(t)\}_{t\in\Re}$ of bounded absorbing sets in $C_{\mathbb{Y}}$ for the family of mappings $\{\tilde{U}(t,\tau), t \geq \tau\}$. Moreover, $B(t) = B_0\ \forall t \in \Re$, where $B_0 \subset C_{\mathbb{Y}}$ is bounded.*

Proof. We set $\bar{\mu} = \mu - \langle\mu\rangle$. By multiplying $(2.36)_1$ by v, $(2.36)_3$ by $2\xi\phi$, $\xi > 0$ and adding the resulting equations, we derive as in [9] that

$$\frac{dE}{dt} + \kappa E(t) = \wedge_1(t), \tag{3.32}$$

where

$$E(t) = |(v,\phi)(t)|^2_{\mathbb{Y}} + 2\alpha(F(\phi(t)),1)_{L^2} + C_e, \tag{3.33}$$

and

$$\begin{aligned}\wedge_1(t) &= -2\nu\|v\|^2 + \kappa|v|^2_{L^2} - 2|\nabla\mu|^2_{L^2} - (2\xi-\kappa)\epsilon|\nabla\phi)|^2_{L^2}\\ &+2\alpha\left[\kappa(F(\phi)-f(\phi)\phi,1)_{L^2} - (\xi-\kappa)(f(\phi)\phi,1)_{L^2}\right] + 2\xi(\mu,\phi)_{L^2}\\ &+2\langle v, g(t)+\mathcal{G}(t,v_t)\rangle + \kappa|\phi(t)|^2_{L^2} + \kappa C_e.\end{aligned} \tag{3.34}$$

Here $C_e = 2\alpha\alpha C_F|\mathcal{M}| > 0$, where C_F is a constant large enough in order to ensure that$E$ is nonnegative (note that F is bounded from below by a constant independent of α and ϵ) From (3.61), we have

$$\begin{aligned}&c_*|f(y)|(1+|y|) \leq 2f(y)y + c_f,\\ &F(y) - f(y)y \leq c'_f y^2 + c''_f,\\ &2\xi(\bar{\mu},\phi)_{L^2} \leq |\nabla\bar{\mu}|^2_{L^2} + \xi^2 C_{\mathcal{M}}|\mathcal{M}||\nabla\phi|^2_{L^2},\end{aligned} \tag{3.35}$$

for any $y \in \Re$, where c_f, c_*, c'_f and c''_f are positive, sufficiently large constants that depend only on f.
From [9], we also note that

$$\begin{aligned}\wedge_1(t) &\leq -(\nu - \kappa C_m|\mathcal{M}|)\,\|v(t)\|^2 - |\nabla\mu(t)|^2_{L^2} - c_*\alpha(\xi-\kappa)\,(|f(\phi(t))|, 1+|\phi(t)|)_{L^2}\\ &-[\xi(2-\xi C_m|\mathcal{M}|\epsilon^{-1}) - \kappa(1+2\alpha\epsilon^{-1}c'_f|\mathcal{M}|)]\epsilon|\nabla\phi|^2_{L^2}\\ &+\sigma^{-1}|g|^2_{L^2} + \sigma|v|^2_{L^2} + C_g^{-1}|\mathcal{G}(t,v_t)|^2_{L^2} + C_g|v|^2_{L^2} + c_1,\end{aligned} \tag{3.36}$$

where C_m depends on the shape of $\mathcal{M}$, but not its size and c_1 is given by

$$c_1 = 2\kappa\alpha C_F|\mathcal{M}| + 2\alpha c''_f|\mathcal{M}| + c_f\alpha(\xi-\kappa)|\mathcal{M}|. \tag{3.37}$$

Let us choose $\kappa \in (0,1)$ as

$$\kappa = \min\left\{\nu(2C_m|\mathcal{M}|)^{-1}, \epsilon(2C_{\mathcal{M}}|\mathcal{M}|)^{-1}, \xi(1+2\alpha\epsilon^{-1}C_m|\mathcal{M}|c'_f)^{-1}\right\}. \tag{3.38}$$

From now on, c_i will denote a positive constant independent on the initial data and on time. Let us set

$$\alpha_1 = \nu - \kappa C_m|\mathcal{M}|,\ \alpha_2 = [\xi(2-\xi C_m|\mathcal{M}|\epsilon^{-1}) - \kappa(1+2\alpha\epsilon^{-1}c_f^{'}|\mathcal{M}|)]\epsilon. \tag{3.39}$$

As

$$\alpha_1\lambda_1 > 2C_g,$$

we can choose $\sigma > 0$ such that

$$\alpha_1\lambda_1 > 2C_g + \sigma.$$

Then from (3.33)-(3.38), we derive that

$$\begin{aligned}&\frac{dE}{dt} + \kappa E(t) + \alpha_1\|v(t)\|^2 + \alpha_2\|\phi(t)\|^2 + c_3(|f(\phi(t)|, 1+|\phi(t)|)_{L^2}\\ &+2|\nabla\mu(t)|_{L^2}^2 \le \sigma^{-1}|g|_{L^2}^2 + \sigma|v|_{L^2}^2 + C_g^{-1}|\mathcal{G}(t,v_t)|_{L^2}^2 + C_g|v|_{L^2}^2 + c_1.\end{aligned} \tag{3.40}$$

Let $\tilde{D} \subset \mathcal{M}_{\mathbb{Y}}$ be bounded and let $\tilde{d} > 0$ such that

$$|(v_0,\phi_0)|_{\mathbb{Y}}^2 + \|\vartheta\|_{L^2_{\mathbb{Y}}}^2 \le \tilde{d},\ \forall((v_0,\phi_0),\vartheta) \in \tilde{D}.$$

Let $((v_0,\phi_0),\vartheta) \in \tilde{D}$ and $\tau \in \Re$. Let us set $u(.) = u(.;\tau,(u_0,\vartheta)), u_0 = (v_0,\phi_0)$. Then from (3.40), we have

$$\frac{dE}{dt} \le -(\alpha_1\lambda_1 - (\sigma + C_g))|v|_{L^2}^2 + \sigma^{-1}|g|_{L^2}^2 + C_g^{-1}|\mathcal{G}(t,v_t)|_{L^2}^2 + c_1. \tag{3.41}$$

Now let $m \in [0,m_0]$ such that

$$\alpha_1\lambda_1 > 2C_g + \sigma + m. \tag{3.42}$$

Then

$$\begin{aligned}&\frac{d}{dt}(e^{mt}E(t)) = me^{mt}E(t) + e^{mt}\frac{dE(t)}{dt} \le e^{mt}\sigma^{-1}|g|_{L^2}^2\\ &+e^{mt}\Big((m-(\alpha_1\lambda_1 - (\sigma + C_g)))|v|_{L^2}^2 + C_g^{-1}|\mathcal{G}(t,v_t)|_{L^2}^2\Big) + e^{mt}c_1.\end{aligned} \tag{3.43}$$

Therefore (using (2.31) and (3.42))

$$\begin{aligned}&e^{mt}E(t) - e^{m\tau}E(\tau) \le \int_\tau^t e^{ms}(\sigma^{-1}|g|_{L^2}^2 + c_1)ds\\ &+\int_\tau^t e^{ms}\Big((m-(\alpha_1\lambda_1 - (\sigma + C_g)))|v|_{L^2}^2 + C_g^{-1}|\mathcal{G}(t,v_t)|_{L^2}^2\Big)ds\\ &\le \frac{e^{mt}}{m}(\sigma^{-1}|g|_{L^2}^2 + c_1) + C_g\int_{\tau-h}^{\tau} e^{ms}|\vartheta(s-\tau)|_{\mathbb{Y}}^2 ds\\ &+\int_\tau^t e^{ms}\Big(m + C_g - (\alpha_1\lambda_1 - (\sigma + C_g))|v|_{L^2}^2\Big)ds\\ &\le \frac{e^{mt}}{m}(\sigma^{-1}|g|_{L^2}^2 + c_1) + C_g e^{m\tau}\int_{-h}^{\tau}|\vartheta(s)|_{\mathbb{Y}}^2 ds,\end{aligned} \tag{3.44}$$

which gives

$$E(t) \le \frac{e^{mt}}{m}(\sigma^{-1}|g|_{L^2}^2 + c_1) + (Q_0(\tilde{d}^2) + C_g\tilde{d}^2)e^{-mt}e^{m\tau},\ \forall t \ge \tau. \tag{3.45}$$

Note that $E(\tau) \le Q_0(|(v_0,\phi_0)|_{\mathbb{Y}}) \le Q_0(\tilde{d}^2)$.
Now, for $t \ge \tau + h$ and $s \in [-h,0]$ we have

$$\begin{aligned} E(t+s) &\le \frac{e^{mt}}{m}(\sigma^{-1}|g|_{L^2}^2 + c_1) + (Q_0(\tilde{d}^2) + C_g\tilde{d}^2)e^{-mt-ms}e^{m\tau} \\ &\le \frac{e^{mt}}{m}(\sigma^{-1}|g|_{L^2}^2 + c_1) + (Q_0(\tilde{d}^2) + C_g\tilde{d}^2)e^{-mt}e^{m\tau}e^{mh}, \end{aligned} \tag{3.46}$$

which gives

$$\|u_t\|_{C_{\mathbb{Y}}}^2 \le \frac{e^{mt}}{m}(\sigma^{-1}|g|_{L^2}^2 + c_1) + (Q_0(\tilde{d}^2) + C_g\tilde{d}^2)e^{-mt}e^{m\tau}e^{mh},\ \forall t \ge \tau + h. \tag{3.47}$$

Finally we derive that

$$\|\tilde{U}(t,t-s)(u_0,\vartheta)\|_{C_{\mathbb{Y}}}^2 = \|u_t\|_{C_{\mathbb{Y}}}^2 \le \sigma^{-1}m^{-1}|g|_{L^2}^2 + c_1m^{-1} + (Q_0(\tilde{d}^2) + C_g\tilde{d}^2)e^{-mt}e^{m\tau}e^{mh},\ \forall t,s \ge h. \tag{3.48}$$

Let

$$\tilde{\rho}_{\mathbb{Y}}^2 = 2\frac{e^{mt}}{m}(\sigma^{-1}|g|_{L^2}^2 + c_1). \tag{3.49}$$

Then, there exists $\tilde{T}_{\tilde{D}}(t)\ (= \tilde{T}_D) \ge h$ such that for $s \ge \tilde{T}_{\tilde{D}}(t)$ and $(u_0,\vartheta) \in \mathcal{M}_{\mathbb{Y}}$, we have

$$\|\tilde{U}(t,t-s)(u_0,\vartheta)\|_{C_{\mathbb{Y}}} \le \tilde{\rho}_{\mathbb{Y}}. \tag{3.50}$$

Therefore, the balls $B(t) = B_{C_{\mathbb{Y}}}(0,\tilde{\rho}_{\mathbb{Y}})$ form an absorbing family for the mappings $\tilde{U}(t,\tau)$. □

Corollary 3.12. *The assumptions are the same as in Theorem 3.11. There exists a family $\{B(t)\}_{t\in\mathfrak{R}}$ of bounded absorbing sets in $C_{\mathbb{Y}}$ for the process $U(.,.)$, which is given by $B(t) = B_0 = B_{C_{\mathbb{Y}}}(0,\tilde{\rho}_{\mathbb{Y}})$ for all $t \in \mathfrak{R}$. Moreover, if $\mathbf{B}(t) = B_{\mathbb{Y}}(0,\tilde{\rho}_{\mathbb{Y}}) \times B_{L^2_{\mathbb{Y}}}(0,h^{1/2}\tilde{\rho}_{\mathbb{Y}}) \subset \mathcal{M}_{\mathbb{Y}}$ for all $t \in \mathfrak{R}$, then $\{\mathbf{B}(t)\}_{t\in\mathfrak{R}}$ forms a family of bounded absorbing sets for the process $S(.,.)$.*

Proof. The first part follows from Theorem 3.11 and Lemma 3.10. For the second part, we proceed as in [6, 3]. Since

$$\|w\|_{L^2_{\mathbb{Y}}}^2 \le h\|w\|_{C_{\mathbb{Y}}}^2 \tag{3.51}$$

and

$$j(B(t)) = \{(\vartheta(0),\vartheta),\ \phi \in B_{C_{\mathbb{Y}}}(0,\tilde{\rho}_{\mathbb{Y}})\}, \tag{3.52}$$

we have

$$j(B(t)) \subset B_{C_{\mathbb{Y}}}(0,\tilde{\rho}_{\mathbb{Y}}) \times B_{L^2_{\mathbb{Y}}}(0,h^{1/2}\tilde{\rho}_{\mathbb{Y}}) \equiv \mathbf{B}(t). \tag{3.53}$$

This implies that $\{\mathbf{B}(t)\}_{t\in\mathfrak{R}}$ is absorbing for the process $S(.,.)$. □

3.4 Absorbing set in $C_{\mathbb{V}}$

In this part, we prove that for ν large enough, there exists a family of absorbing sets in $C_{\mathbb{V}}$ for the family of mappings $\{\tilde{U}(t,\tau),\ t \ge \tau\}$.

Theorem 3.13. *The assumptions are the same as in Theorem 3.11. For $D \subset C_{\mathbb{Y}}$, let $T_D = \tilde{T}_{j(D)}$, where $\tilde{T}_{j(D)}$ is the absorbing time corresponding to the set B_0 in Theorem 3.6. There exist $\tilde{\rho}_{\mathbb{V}}^2, \tilde{\beta}_1, \tilde{\beta}_2$ such that for any bounded set $\tilde{D} \subset \mathcal{M}_{\mathbb{Y}}$, we have*

$$\begin{aligned} &\|\tilde{U}(t,t-s)(u_0,\vartheta)\|_{C_{\mathbb{V}}}^2 = \max_{\zeta\in[-h,0]} \|u(t+\zeta,t-s,(u_0,\vartheta))\|_{\mathbb{V}}^2 \le \tilde{\rho}_{\mathbb{V}}^2, \\ &\int_{t+t_1}^{t+t_2} (|A_0v(\sigma,t-s,(u_0,\vartheta)|_{L^2}^2 + |A_N^2\phi(\sigma,t-s,(u_0,\vartheta)|_{L^2}^2 + |A_N^{5/2}\phi(\sigma,t-s,(u_0,\vartheta)|_{L^2}^2)d\sigma \\ &\le \tilde{\beta}_1|t_2-t_1| + \tilde{\beta}_2, \end{aligned} \tag{3.54}$$

for all $s \geq \tilde{T}_{\tilde{D}} + 1 + h$, $t \in \Re$, $(u_0, \vartheta) \in \tilde{D}$, $t_1, t_2 \in [-h, 0]$.

Proof Let $\tilde{D} \subset \mathcal{M}_{\mathbb{Y}}$ be a bounded set and let $\tilde{d} > 0$ such that

$$\|(u_0, \vartheta)\|_{\mathcal{M}_{\mathbb{Y}}} \leq \tilde{d}, \ \forall (u_0, \vartheta) \in \tilde{D}. \tag{3.55}$$

For $(u_0, \vartheta) \in \tilde{D}$, let $u(\cdot) = u(.; t_0 - s, (u_0, \vartheta))$ where $t_0 \in \Re$ is fixed and $s \geq \tilde{T}_{\tilde{D}}$.
We derive from (3.40) that (by integrating between t and $t+1$, for $t \geq 0$ and $s \geq \tilde{T}_D$)

$$\begin{aligned} & E(t+1) - E(t) + \kappa \int_t^{t+1} E(r) dr + (\alpha_1 - (\sigma + C_g)\lambda_1^{-1}) \int_t^{t+1} \|v(r)\|^2 dr \\ & + \int_t^{t+1} |\nabla \mu(r)|_{L^2}^2 dr \leq \sigma^{-1} |g|_{L^2}^2 + c_1 + C_g^{-1} \int_t^{t+1} |\mathcal{G}(t, v_t)|_{L^2}^2 dr \\ & \leq \sigma^{-1} |g|_{L^2}^2 + c_1 + \frac{1}{C_g} C_g^2 \int_{t-h}^{t+1} |v(r)|_{L^2}^2 dr \\ & \leq \sigma^{-1} |g|_{L^2}^2 + c_1 + C_g \int_{t-h}^{t} |v(r)|_{L^2}^2 dr + C_g \int_t^{t+1} |v(r)|_{L^2}^2 dr, \end{aligned} \tag{3.56}$$

which gives

$$\begin{aligned} & (\alpha_1 - (\sigma + 2C_g)\lambda_1^{-1}) \int_t^{t+1} |v(r)|_{L^2}^2 dr + \int_t^{t+1} |\nabla \bar{\mu}(r)|_{L^2}^2 dr \\ & \leq \sigma^{-1} |g|_{L^2}^2 + c_1 + C_g \int_{t-h}^{t} |(v(r)|_{L^2}^2 dr + E(t) \\ & \leq \sigma^{-1} |g|_{L^2}^2 + c_1 + C_g \int_{t-h}^{t} \|u_r\|_{C_{\mathbb{Y}}}^2 dr + E(t) \\ & \leq \sigma^{-1} |g|_{L^2}^2 + c_1 + C_g h \tilde{\rho}_{\mathbb{Y}}^2 + Q_0(\tilde{\rho}_{\mathbb{Y}}^2). \end{aligned} \tag{3.57}$$

Therefore if ν is large enough such that

$$\alpha_1 - (\sigma + 2C_g)\lambda_1^{-1} > 0, \tag{3.58}$$

then

$$\int_t^{t+1} (\|v(r)\|^2 + |\nabla \bar{\mu}(r)|_{L^2}^2) dr \leq \tilde{J}_{\mathbb{V}}, \ \forall t \geq t_0, \tag{3.59}$$

where

$$\tilde{J}_{\mathbb{V}} = \kappa_1^{-1} \left(\sigma^{-1} |g|_{L^2}^2 + c_1 + C_g h \tilde{\rho}_{\mathbb{Y}}^2 + Q_0(\tilde{\rho}_{\mathbb{Y}}^2) \right), \tag{3.60}$$

and

$$\kappa_1 = \min(1, \alpha_1 - (\sigma + 2C_g)\lambda_1^{-1}).$$

From $\mu = \epsilon A_N \phi + \alpha f(\phi)$, we also have (for $t \geq t_0$)

$$\begin{aligned} & \epsilon^2 \int_t^{t+1} |A_N^{3/2} \phi(r)|_{L^2}^2 dr \leq c \int_t^{t+1} (|\nabla \mu(r)|_{L^2}^2 + \alpha |A_N^{1/2} f(\phi)(r)|_{L^2}^2) dr \\ & \leq c \int_t^{t+1} (|\nabla \bar{\mu}(r)|_{L^2}^2 + Q_0(\tilde{\rho}_{\mathbb{Y}}^2)) dr, \end{aligned} \tag{3.61}$$

which gives

$$\int_t^{t+1} |A_N^{3/2}\phi(r)|_{L^2}^2 dr \le c\tilde{J}_{\mathbb{V}} + Q_0(\tilde{\rho}_{\mathbb{Y}}^2). \tag{3.62}$$

It follows from (3.59) and (3.62) that

$$\int_t^{t+1} (\|v(r)\|^2 + |A_N^{3/2}\phi(r)|_{L^2}^2) dr \le \tilde{I}_{\mathbb{V}},\ \forall t \ge t_0, \tag{3.63}$$

where $\tilde{I}_{\mathbb{V}} = c\tilde{I}_{\mathbb{V}} + Q_0(\tilde{\rho}_{\mathbb{Y}}^2)$.

Note that (3.58) implies that we can choose $m \in [0, m_0]$ such that (3.42) is satisfied.

For $r \ge t_0$, taking the inner product in H_1 of $(2.36)_1$ with $2A_0 v$, the inner product in $L^2(\mathcal{M})$ of $(2.36)_2$ and $(2.36)_3$ with $2A_N^2\bar{\mu} + 2\zeta B_N^3\phi$ ($\zeta > 0$ small enough) and $2\epsilon A_N^2\phi$ respectively. Adding the resulting equalities gives (see [8] for the details)

$$\begin{aligned}
&\frac{d\mathcal{Y}}{dt} + 2\nu|A_0 v|_{L^2}^2 + 2\zeta\epsilon|A_N^2\phi|_{L^2}^2 + 2|B_N\bar{\mu}|_{L^2}^2 = 2\mathcal{K}(R_0(\epsilon A_N\phi, \phi), A_0 v)_{L^2} \\
&-2\alpha\langle A_N(f(\phi) - \langle f(\phi)\rangle), B_N\bar{\mu}\rangle - 2\langle B_1(u, \phi), A_N^2\phi\rangle \\
&-2(B_0(v, v), A_0 v)_{L^2} + (g + \mathcal{G}(t, u_t), A_0 v)_{L^2} + 2\zeta\langle B_N\bar{\mu}, A_N^2\phi\rangle \\
&+2\alpha\zeta(B_N(f(\phi) - \langle f(\phi)\rangle), A_N^2\phi)_{L^2},
\end{aligned} \tag{3.64}$$

where

$$\mathcal{Y}(t) = \|v(t)\|^2 + \epsilon|A_N\phi(t)|_{L^2}^2,\ \forall t \ge t_0.$$

Using the estimates of [8] for the nonlinear terms appearing in (2.36), we can check that $\mathcal{Y}(t)$ satisfies

$$\begin{aligned}
&\frac{d\mathcal{Y}}{dt} + \nu|A_0 v|_{L^2}^2 + \tfrac{\zeta\epsilon}{4}|B_N^2\phi|_{L^2}^2 + [2 - (4\epsilon^{-1} + \alpha)\zeta]|B_N\bar{\mu}|_{L^2}^2 \\
&\le \Psi(t)\mathcal{Y}(t) + \Pi(t),
\end{aligned} \tag{3.65}$$

where

$$\begin{aligned}
\Psi(t) &= c(|\nabla\phi|_{L^2}^2|A_N\phi|_{L^2}^2 + |v|_{L^2}^2\|v\|^2 + Q_1(|\phi|_{H^1})(1 + |A_N\phi|_{L^2}^2), \\
\Upsilon(t) &= c(1 + |v|_{L^2}^2 + |\nabla\phi|_{L^2}^2 + |g|_{L^2}^2 + |\mathcal{G}(t, v_t)|_{L^2}^2).
\end{aligned} \tag{3.66}$$

We note that

$$\begin{aligned}
&\int_t^{t+1} \Psi(s) ds \equiv a_1 < \infty,\ \int_t^{t+1} \Upsilon(s) ds \equiv a_2^{'} < \infty, \\
&\int_t^{t+1} \mathcal{Y}(s) ds \equiv a_3 = \tilde{I}_{\mathbb{V}}, < \infty,
\end{aligned} \tag{3.67}$$

and

$$\begin{aligned}
\int_t^{t+1} |\mathcal{G}(s, v_s)|_{L^2}^2 ds &\le L_g^2 \int_{t-h}^{t+1} |v(r)|_{L^2}^2 dr \\
&\le L_g^2 \int_{t-h}^{t} |v(r)|_{L^2}^2 dr + L_g^2 \int_t^{t+1} |v(r)|_{L^2}^2 dr \\
&\le L_g^2 \int_{t-h}^{t} \|u_r\|_{C_{\mathbb{Y}}}^2 dr + L_g^2 \int_t^{t+1} |v(r)|_{L^2}^2 dr \\
&\le hL_g^2\tilde{\rho}_{\mathbb{Y}}^2 + L_g^2\tilde{\rho}_{\mathbb{Y}}^2 \equiv a_2^{''}.
\end{aligned} \tag{3.68}$$

Therefore

$$\mathcal{Y}(t) \le (a_3 + a_2)e^{a_1} \equiv \tilde{\rho}_V^2, \ \forall t \ge t_0 + 1, \tag{3.69}$$

where

$$a_2 = a_2^{'} + a_2^{''} + c|g|_{L^2}^2. \tag{3.70}$$

Letting

$$u(\cdot) = u(\cdot; t-s, (u_0, \vartheta)), \tag{3.71}$$

we obtain

$$\|u_t\|_{C_{\mathbb{V}}}^2 \le \tilde{\rho}_{\mathbb{V}}^2, \ \forall s \ge \tilde{T}_{\tilde{D}} + 1 + h. \tag{3.72}$$

Now, multiplying $(2.36)_2$ by $A_N^3\phi$ and using $(2.36)_3$, we derive that

$$\frac{d\mathcal{Y}_1}{dt} + 2\epsilon|A_N^{5/2}\phi|_{L^2}^2 \le 2\alpha|\langle A_N^{3/2} f(\phi), A_N^{5/2}\phi\rangle| + 2|\langle A_N^{1/2} B_1(v,\phi), A_N^{5/2}\phi\rangle|, \tag{3.73}$$

where

$$\mathcal{Y}_1(t) = |A_N^{3/2}\phi(t)|_{L^2}^2.$$

Note that

$$\begin{aligned} &2|\langle A_N^{1/2} B_1(v,\phi), A_N^{5/2}\phi\rangle| \le 2|A_N^{1/2} B_1(v,\phi)|_{L^2}|A_N^{5/2}\phi|_{L^2} \\ &\le \tfrac{\epsilon}{4}|A_N^{5/2}|_{L^2}^2 + c\|v\||A_0 v|_{L^2}\|\phi\||A_N\phi|_{L^2} + |v|_{L^2}\|v\||A_N\phi|_{L^2}|A_N^{3/2}\phi|_{L^2}, \end{aligned} \tag{3.74}$$

and

$$\begin{aligned} &2\alpha|\langle A_N^{3/2} f(\phi), A_N^{5/2}\phi\rangle| \le \tfrac{\epsilon}{4}|A_N^{5/2}\phi|_{L^2}^2 + c|f^{'''}(\phi)(A_N^{1/2}\phi)^3|_{L^2}^2 \\ &+c|f^{''}(\phi)(A_N^{1/2}\phi)A_N\phi|_{L^2}^2 + |f^{'}(\phi)A_N^{3/2}\phi|_{L^2}^2 \\ &\le \tfrac{\epsilon}{4}|A_N^{5/2}|_{L^2}^2 + Q_2(|A_N\phi|)|A_N^{3/2}|_{L^2}^2. \end{aligned} \tag{3.75}$$

It follows from (3.73)-(3.75) that

$$\frac{d\mathcal{Y}_1}{dt} + \epsilon|A_N^{5/2}\phi|_{L^2}^2 \le \Pi_1(t), \tag{3.76}$$

where

$$\Pi_1(t) = Q_2(|A_N\phi|)|A_N^{3/2}\phi|_{L^2}^2 + c\|v\||A_0 v|_{L^2}\|\phi\||A_N\phi|_{L^2} + |v|_{L^2}\|v\||A_N\phi|_{L^2}|A_N^{3/2}\phi|_{L^2}. \tag{3.77}$$

It follows from the Gronwall lemma that

$$|A_N^{3/2}\phi(t)|_{L^2}^2 \le c_1, \ \forall t \ge t_0. \tag{3.78}$$

For the bound on $\int_{t+t_1}^{t+t_2}(|A_0 v|_{L^2}^2 + |A_N^2\phi|_{L^2}^2 + |A_N^{5/2}\phi|_{L^2}^2)dr$, we proceed as follows. From (3.65) and (3.76), we have

$$\nu|A_0 v|_{L^2}^2 + \epsilon|A_N^2\phi|_{L^2}^2 + \epsilon|A_N^{5/2}\phi|_{L^2}^2 \le -\frac{d\mathcal{Y}}{dt} - \frac{d\mathcal{Y}_1}{dt} + \Psi(t)\mathcal{Y}(t) + \Pi(t) + \Pi_1(t). \tag{3.79}$$

Then for $s \geq \tilde{T}_D + 1 + h$ and $t_1, t_2 \in [-h, 0]$ with $t_2 > t_1$, we derive from (3.69) and (3.79) that

$$\begin{aligned}
&\int_{t+t_1}^{t+t_2} (\nu|A_0 v|_{L^2}^2 + \epsilon|A_N^2\phi|_{L^2}^2 + \epsilon|A_N^{5/2}\phi|_{L^2}^2)dr \leq \mathcal{Y}(t+t_1) \\
&+\mathcal{Y}_1(t+1) + \int_{t+t_1}^{t+t_2} (\Psi(r)\mathcal{Y}(r) + \Pi(r) + \Pi_1(r))dr \\
&\leq \tilde{\rho}_{\mathbb{V}}^2 + \tilde{\alpha}_1|t_2 - t_1| + \tilde{\alpha}_2 + c_1,
\end{aligned} \tag{3.80}$$

which gives $(3.54)_2$. □

Corollary 3.14. *The assumptions and the notations are the same as in Theorems 3.11 and 3.13. There exist $\rho_{\mathbb{V}}^2, \beta_1, \beta_2$ such that for any bounded set $D \subset C_{\mathbb{Y}}$ we have*

$$\begin{aligned}
&||\tilde{U}(t, t-s)(u_0, \vartheta)||_{C_{\mathbb{V}}}^2 = \max_{\zeta\in[-h,0]} |u(t+\zeta, t-s, j(\vartheta))|_{L^2}^2 \leq \rho_{\mathbb{V}}^2, \\
&\int_{t+t_1}^{t+t_2} \Big|A_0 v(\sigma, t-s, j(\vartheta))|_{L^2}^2 + |A_\gamma^{3/2}\phi(\sigma, t-s, j(\vartheta))|_{L^2}^2\Big) d\sigma \\
&\leq \beta_1|t_2 - t_1| + \beta_2,
\end{aligned} \tag{3.81}$$

for all $s \geq T_D + 1 + h$, $t \in \Re$, $(u_0, \vartheta) \in D$, $t_1, t_2 \in [-h, 0]$.
Moreover, the family $\{B_s(t)\}_{t\in\Re}$, where $B_s(t) = B_s = B_{C_{\mathbb{V}}}(0, \rho_{\mathbb{V}}) \times B_{L_{\mathbb{V}}}(0, h^{1/2}\rho_{\mathbb{V}})$ is absorbing for $S(.,.)$.

Proof. The proof is similar to that of Corollary 3.12. □

3.5 Existence of the pullback attractor

In this part, we prove that $U(.,.)$ and $S(.,.)$ have a unique uniformly bounded pullback attractor in $C_{\mathbb{Y}}$ and $M_{\mathbb{Y}}$ respectively.

Theorem 3.15. *There exists a unique uniformly bounded attractor $\{\mathcal{A}_{C\mathbb{Y}}(t)\}_{t\in\Re}$ for the process $\Theta(.,.)$ in $C_{\mathbb{Y}}$ and a unique uniformly bounded attractor $\{\mathcal{A}_{\mathcal{M}_{\mathbb{Y}}}(t)\}_{t\in\Re}$ for the process $S(.,.)$ in $\mathcal{M}_{\mathbb{Y}}$. Furthermore these attractors satisfy*

$$\mathcal{A}_{\mathcal{M}_{\mathbb{Y}}}(t) \subset \mathbb{Y} \times C_{\mathbb{Y}},\ \mathcal{A}_{\mathcal{M}_{\mathbb{Y}}}(t) = j(\mathcal{A}_{C_{\mathbb{Y}}}(t)),\ \forall t \in \Re. \tag{3.82}$$

Proof. We proceed as in [6, 3]. Let us set $B_2(t) = B_2 = B_{C_{\mathbb{V}}}(0, \rho_{\mathbb{V}})$ for all $t \in \Re$. Then $\{B_2(t)\}_{t\in\Re}$ is a family of bounded set in $C_{\mathbb{V}}$ and uniformly absorbing for $\tilde{U}(.,.)$. Let us set $\tilde{B}_2 = j(B_2)$. Then, there exists $\tilde{T}_{\tilde{B}_2} = T_{B_2} + 1 + h > 0$ such that

$$\tilde{U}(t, t-s)\tilde{B}_2 \subset B_2,\ \forall t \in \Re, s \geq \tilde{T}_{\tilde{B}_2}. \tag{3.83}$$

Now let us set

$$B_3(t) = \bigcup_{s\geq\tilde{T}_{\tilde{B}_2}} \tilde{U}(t, t-s)\tilde{B}_2 \subset B_2 \subset C_{\mathbb{V}}. \tag{3.84}$$

Then $\{B_3(t)\}$ is a family of uniformly bounded sets in $C_{\mathbb{V}}$ which is (uniformly) absorbing for $\tilde{U}(.,.)$. Let us prove that $(B_3(t))$ is relatively compact in $C_{\mathbb{Y}}$.

Step 1. First for $t_1 \in [-h, 0]$, the set

$$\bigcup_{s \geq \tilde{T}_{\tilde{B}_2}} \bigcup_{\vartheta \in B_2} \tilde{U}(t-s)(j(\vartheta))(t_1) \tag{3.85}$$

is relatively compact in $\mathbb{Y}$. In fact, for $t_1 \in [-h, 0]$, $t \in \mathfrak{R}$, the set

$$\{u(t+t_1, t-s, j(\vartheta)),\ s \geq \tilde{T}_{\tilde{B}_2}, \vartheta \in B_2\} \tag{3.86}$$

is relatively compact in $\mathbb{Y}$ since it is bounded in $\mathbb{V}$ and the injection of $\mathbb{V}$ in $\mathbb{Y}$ is compact.

Step 2. Let us now prove that the set

$$\bigcup_{s \geq \tilde{T}_{\tilde{\mathcal{B}}_2}} \tilde{U}(t, t-s)\tilde{\mathcal{B}}_2 \tag{3.87}$$

is equi-continuous. We proceed as in [6, 3] and we note that for $t \in \mathfrak{R}$, $t_1, t_2 \in [-h, 0]$, $s \geq \tilde{T}_{\tilde{B}_2}$ and $\vartheta \in \tilde{B}_2$, we have

$$|\tilde{U}(t, t-s)(j(\vartheta))(t_1) - \tilde{U}(t, t-s)(j(\vartheta))(t_2)|_{\mathbb{Y}} = |u(t+t_1, t-s, j(\vartheta) - u(t+t_2, t-s, j(\vartheta))|_{\mathbb{Y}}. \tag{3.88}$$

Now let us denote $u(.) = u(.; t-s, j(\vartheta))$. From $(2.36)_1$, we have

$$\begin{aligned}
&|v(t+t_1) - v(t+t_2)|_{L^2} = \left| \int_{t+t_1}^{t+t_2} v^{'}(r) dr \right|_{L^2} \leq \int_{t+t_1}^{t+t_2} |v^{'}(r)|_{L^2} dr \\
&\leq \int_{t+t_1}^{t+t_2} (\nu |A_0 v(r)|_{L^2} + |B_0(v,v)|_{L^2} + \epsilon |R_0(A_N\phi, \phi)|_{L^2} + |g|_{L^2} + |\mathcal{G}(t, v_r)|_{L^2}) dr \\
&\leq |g|_{L^2}|t_2 - t_1| + \int_{t+t_1}^{t+t_2} \left(\nu |A_0 v(r)|_{L^2} + c|A_0 v(r)|_{L^2} \|v\| + c\epsilon |A_N\phi|_{L^2} |A_N^{3/2}\phi|_{L^2} + L_g \|v_r\|_{C_{H_1}} \right) dr \\
&\leq |g|_{L^2}|t_2 - t_1| + (\nu + c\rho_{\mathbb{V}})|t_2 - t_1|^{1/2} \int_{t+t_1}^{t+t_2} |A_0 v(r)|^2_{L^2} dr + c\epsilon |t_2 - t_1|^{1/2} \rho_{\mathbb{V}} \int_{t+t_1}^{t+t_2} |A_N^{3/2}\phi|^2_{L^2} dr \\
&\leq |g|_{L^2}|t_2 - t_1| + (\nu + \epsilon + c\rho_{\mathbb{V}})|t_2 - t_1|^{1/2}(\beta_1 |t_2 - t_1| + \beta_2) + c\tilde{\rho}_{\mathbb{Y}} |t_2 - t_1|.
\end{aligned} \tag{3.89}$$

From $(2.36)_2$, we also have

$$\begin{aligned}
&|A_N^{1/2}(\phi(t+t_1) - \phi(t+t_2))|_{L^2} = \left| \int_{t+t_1}^{t+t_2} A_N^{1/2}\phi^{'}(r) dr \right|_{L^2} \leq \int_{t+t_1}^{t+t_2} |A_N^{1/2}\phi^{'}(r)|_{L^2} dr \\
&\leq \int_{t+t_1}^{t+t_2} \left(\epsilon |A_N^{5/2}\phi(r)|_{L^2} + |A_N^{1/2} B_1(v,\phi)|_{L^2} + |A_N^{3/2} f(\phi)|_{L^2} \right) dr \\
&\leq c|t_2 - t_1|^{1/2} \int_{t+t_1}^{t+t_2} |A_N^{5/2}\phi(r)|^2_{L^2} dr + c \int_{t+t_1}^{t+t_2} (\|v\|^{1/2} |A_0 v|^{1/2}_{L^2} \|\phi\|^{1/2} |A_N\phi|^{1/2}_{L^2}) dr \\
&+ c \int_{t+t_1}^{t+t_2} \left(|v|^{1/2}_{L^2} \|v\|^{1/2} |A_N\phi|^{1/2}_{L^2} |A_N^{3/2}\phi(r)|^{1/2}_{L^2} + Q_2(|A_N\phi|_{L^2}) |A_N^{3/2}\phi|_{L^2} \right) dr \\
&\leq c|t_2 - t_1|^{1/2}(\beta_1 |t_2 - t_1| + \beta_2)\rho_{\mathbb{V}} + c_1 Q_2(\rho_{\mathbb{V}})|t_2 - t_1|.
\end{aligned} \tag{3.90}$$

It follows from (3.89) and (3.90) that the set

$$\bigcup_{s\geq \tilde{T}_{\tilde{\mathcal{B}}_2}} \tilde{U}(t,t-s)\tilde{\mathcal{B}}_2 \tag{3.91}$$

is equi-continuous.
It follows from Step 1, Step 2 and the Ascoli-Arzela theorem that $B_3(t)$ is relatively compact in $\mathbb{Y}$. The proves that $\{\overline{B_3(t)}\}_{t\in\mathfrak{R}}$ (where the closure is taken in $C_{\mathbb{Y}}$) is a family of compact absorbing set in $C_{\mathbb{Y}}$ for the process $\tilde{U}(\cdot,\cdot)$. Consequently, it is also a family of compact (uniformly) absorbing sets for the process $U(\cdot,\cdot)$ in $C_{\mathbb{Y}}$. Moreover, $\{j(\overline{B_3(t)})\}_{t\in\mathfrak{R}}$ is also a family of compact (uniformly) absorbing sets for the process $S(\cdot,\cdot)$ in $\mathcal{M}_{\mathbb{Y}}$, which ensures the existence of the pullback attractors the processes. □

3.6 Example of a forcing term with variable delays

In this part, we give an example of a delay term $\mathcal{G}$ that satisfies (2.27)-(2.31). We assume that the delay term is given by

$$\mathcal{G}(t,v_t) = \mathcal{G}_0(v(t-\rho(t))), \tag{3.92}$$

where $\mathcal{G}_0 : \mathfrak{R}^2 \to \mathfrak{R}^2$ satisfies

$$\mathcal{G}_0(0) = 0,\ |\mathcal{G}_0(u)-\mathcal{G}_0(v)|_{\mathfrak{R}^2} \leq L_1|u-v|_{\mathfrak{R}^2}, \tag{3.93}$$

for some fixed constant $L_1 > 0$. We assume that $\rho \in C^1[0,+\infty)$, $\rho(t) \geq 0,\ \forall t \geq 0$, $h = \sup_{t\geq 0}\rho(t) \in (0,+\infty)$, $\rho^* = \sup_{t\geq 0}\rho^{'}(t) < 1$. We can prove as in [4] that this situation is within our framework and (2.27)-(2.31) are all satisfied.

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