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A Note on Relations Between Hom-Malcev Algebras and Hom-Lie-Yamaguti Algebras

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Abstract. A Hom-Lie-Yamaguti algebra, whose ternary operation expresses through its binary one in a specific way, is a multiplicative Hom-Malcev algebra. Any multiplicative Hom-Malcev algebra over a field of characteristic zero has a natural Hom-Lie-Yamaguti structure.

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1 Introduction and Results

1.1 Introduction

Motivated by quasi-deformations of Lie algebras of vector fields, including q-deformations of Witt and Virasoro algebras, Hom-Lie algebras were introduced in [5]. The intensive development of the theory of Hom-algebras started by the introduction of Hom-associative algebras in [12], where it is shown that the commutator algebra of a given Hom-associative algebra is a Hom-Lie algebra. Since then, various types of Hom-algebras were introduced and investigated (see, e.g., [1], [2], [3], [6], [11]-[13], [22]-[24] and references therein). We refer to [5], [11], [12], [20], [21] for fundamentals on Hom-algebras.

A rough description of a Hom-type algebra is that it is a generalization of a given type of algebras by twisting its defining identity (identities) by a linear self-map in such a way that, when

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the twisting map is the identity map, one recovers the original type of algebras (in [21] a general strategy to twist a given algebraic structure into its corresponding Hom-type via an endomorphism is given). Following this pattern, a number of binary, n-ary or binary-ternary algebras are twisted into their Hom-version. For instance, by twisting alternative algebras, Hom-alternative algebras are introduced in [11], and the notion of a Hom-Malcev algebra is introduced in [22], where it is shown that the commutator Hom-algebra of a Hom-alternative algebra is a Hom-Malcev algebra (this is the Hom-analogue of the Malcev's result stating that alternative algebras are Malcev-admissible). In fact, Malcev algebras were introduced by A.I. Mal'tsev in [14] (calling them Moufang-Lie algebras) as commutator algebras of alternative algebras and also as tangent algebras to local smooth Moufang loops. The terminology "Malcev algebra" is introduced in [16], where a systematic study of such algebras is undertaken.

All vector spaces and algebras throughout will be over a ground field $\mathbb{K}$ of characteristic 0.

1.2 Definitions and Results

A *Malcev algebra* [14] is a nonassociative algebra A with an anticommutative binary operation "$[,]$" satisfying the *Malcev identity*

$$J(x,y,[x,z]) = [J(x,y,z),x] \tag{1.1}$$

for all x,y,z in A, where $J(x,y,z) = \circlearrowleft_{(x,y,z)}[[x,y],z]$ denotes the *Jacobian* and $\circlearrowleft_{(x,y,z)}$ means the sum over cyclic permutation of x,y,z. A *Hom-Malcev algebra* [22] is a *Hom-algebra* $(A,[,],\alpha)$ such that its binary operation "$[,]$" is anticommutative and that the *Hom-Malcev identity*

$$J_\alpha(\alpha(x),\alpha(y),[x,z]) = [J_\alpha(x,y,z),\alpha^2(x)] \tag{1.2}$$

holds for all x,y,z in A, where $J_\alpha(x,y,z) = \circlearrowleft_{(x,y,z)}[[x,y],\alpha(z)]$ is the *Hom-Jacobian*. Observe that $J_\alpha(x,y,z)$ is completely skew-symmetric in its three variables and when the *twisting map* α is the identity map, $\alpha = id$, then the Hom-Malcev algebra $(A,[,],\alpha)$ reduces to the Malcev algebra $(A,[,])$. The Hom-Malcev algebra $(A,[,],\alpha)$ is said to be *multiplicative* if $\alpha([x,y]) = [\alpha(x),\alpha(y)]$, for all x,y in A. We assume in this paper that all Hom-algebras are multiplicative. In [7] an identity, equivalent to the Hom-Malcev identity (1.2), is pointed out (see also section 2).

As for binary algebras, n-ary algebras can be twisted into their Hom-version (see, e.g., [1], [23], [24]). Other interesting types of algebras are binary-ternary algebras, i.e. algebras with one (or more) binary operation and one (or more) ternary operation. A well-known class of such algebras is the one of Lie-Yamaguti algebras which are introduced by K. Yamaguti [17] as a generalization of Lie triple systems (this motivates the name "generalized Lie triple systems" used in [17] for these algebras; in [8] they are called "Lie triple algebras" and the terminology "Lie-Yamaguti algebras" is introduced in [9] to call these algebras). It turns out that the operations of a Lie-Yamaguti algebra characterize the torsion and curvature tensors of the Nomizu's canonical connection on reductive homogeneous spaces [15].

A *Lie-Yamaguti algebra* (*LYA*) $(A,*,\{,,\})$ is a vector space A together with a binary operation $*: A\times A \to A$ and a ternary operation $\{,,\}: A\times A\times A \to A$ such that

(LY1) $x*y = -y*x$,
(LY2) $\{x,y,z\} = -\{y,x,z\}$,
(LY3) $\circlearrowleft_{(x,y,z)}[(x*y)*z + \{x,y,z\}] = 0$,

(LY4) $\circlearrowleft_{(x,y,z)}\{x*y,z,u\}=0$,
(LY5) $\{x,y,u*v\}=\{x,y,u\}*v+u*\{x,y,v\}$,
(LY6) $\{x,y,\{u,v,w\}\}=\{\{x,y,u\},v,w\}+\{u,\{x,y,v\},w\}$
$+\{u,v,\{x,y,w\}\}$,

for all u,v,w,x,y,z in A.

One observes that if $x*y=0$, for all x,y in A, then $(A,*,\{,,\})$ reduces to a *Lie triple system* $(A,\{,,\})$ and if $\{x,y,z\}=0$ for all x,y,z in A then one gets a Lie algebra $(A,*)$. Motivated by recent developments of the theory of Hom-algebras, Theorem 2.3 in [21] is extended to the study of a twisted deformation of *Akivis algebras* (see references in [6]) which constitutes a very general class of binary-ternary algebras. It is proved ([6], Corollary 4.5) that every Akivis algebra A can be twisted into a *Hom-Akivis algebra* via an endomorphism of A (this is the first extension of Theorem 2.3 in [21] to the category of binary-ternary Hom-algebras). Following this line, Hom-Lie-Yamaguti algebras are introduced in [3] as a twisted generalization of Lie-Yamaguti algebras (the cohomology theory and representation theory of Hom-Lie-Yamaguti algebras are recently developed in [10] and [25]).

A *Hom-Lie-Yamaguti algebra* (Hom-LYA for short) [3] is a quadruple $(A,*,\{,,\},\alpha)$ in which A is a $\mathbb{K}$-vector space, "$*$" a binary operation and "$\{,,\}$" a ternary operation on A, and $\alpha: A\to A$ a linear map such that

(HLY1) $\alpha(x*y)=\alpha(x)*\alpha(y)$,
(HLY2) $\alpha(\{x,y,z\})=\{\alpha(x),\alpha(y),\alpha(z)\}$,
(HLY3) $x*y=-y*x$,
(HLY4) $\{x,y,z\}=-\{y,x,z\}$,
(HLY5) $\circlearrowleft_{(x,y,z)}[(x*y)*\alpha(z)+\{x,y,z\}]=0$,
(HLY6) $\circlearrowleft_{(x,y,z)}\{x*y,\alpha(z),\alpha(u)\}=0$,
(HLY7) $\{\alpha(x),\alpha(y),u*v\}=\{x,y,u\}*\alpha^2(v)+\alpha^2(u)*\{x,y,v\}$,
(HLY8) $\{\alpha^2(x),\alpha^2(y),\{u,v,w\}\}=\{\{x,y,u\},\alpha^2(v),\alpha^2(w)\}$
$+\{\alpha^2(u),\{x,y,v\},\alpha^2(w)\}$
$+\{\alpha^2(u),\alpha^2(v),\{x,y,w\}\}$,

for all u,v,w,x,y,z in A.

Note that the conditions **(HLY1)** and **(HLY2)** mean the multiplicativity of $(A,*,\{,,\},\alpha)$. Examples of Hom-LYA could be found in [3], [4].

Remark. (i) If $\alpha=Id$, then the Hom-LYA $(A,*,\{,,\},\alpha)$ reduces to a LYA $(A,*,\{,,\})$ (see **(LY1)**-**(LY6)**).

(ii) If $x*y=0$, for all $x,y\in A$, then $(A,*,\{,,\},\alpha)$ is a multiplicative Hom-Lie triple system $(A,\{,,\},\alpha^2)$ and, subsequently, a multiplicative ternary Hom-Nambu algebra since any Hom-Lie triple system is automatically a ternary Hom-Nambu algebra (see [23] for Hom-Lie triple systems and [1] for Hom-Nambu algebras).

(iii) If $\{x,y,z\}=0$ for all $x,y,z\in A$, then the Hom-LYA $(A,*,\{,,\},\alpha)$ becomes a Hom-Lie algebra $(A,*,\alpha)$.

It is shown ([3], Corollary 3.2) that every LYA $(A,*,\{,,\})$ can be twisted into a Hom-LYA via an endomorphism of $(A,*,\{,,\})$.

The relationships between LYA and Malcev algebras are investigated by K. Yamaguti in [18], [19]. In [18] (Theorem 1.1), relying on a result in [16] (Proposition 8.3), it is proved that the Malcev identity is equivalent to **(LY5)** in an anticommutative algebra over a field of characteristic not 2 or

3 with "$\{,,\}$" defined in a specific way. Moreover, any Malcev algebra over a field of characteristic not 2 has a natural LYA structure ([18], proof of Theorem 2.1). Besides, when the ternary operation of a given LYA expresses in a specific way through its binary one, then such a LYA reduces to a Malcev algebra ([19], Theorem 1.1).

The purpose of this note is the study of the twisted version of K. Yamaguti's results relating Malcev algebras and LYA ([18], [19]) that is, in a similar way, we shall relate Hom-Malcev algebras and Hom-LYA. We stress that although the analogue of Yamaguti's results are shown below to hold in the Hom-algebra setting, the methods used in the proofs of these results still cannot be reported in the case of Hom-algebras. So we proceed otherwise as it could be seen in what follows.

Our investigations are based on the trilinear composition

$$\{x,y,z\} := xy * \alpha(z) - yz * \alpha(x) - zx * \alpha(y), \tag{1.3}$$

where "$*$" will denote the binary operation of either the given Hom-Malcev algebra or the Hom-LYA and juxtaposition is used in order to reduce the number of braces i.e., e.g., $xy * \alpha(z)$ means $(x * y) * \alpha(z)$. We shall prove:

Theorem 1.1. *Let $(A, *, \{,,\}, \alpha)$ be a Hom-LYA. If its ternary operation "$\{,,\}$" expresses through its binary one "$*$" as in (1.3) for all x, y, z in A, then $(A, *, \alpha)$ is a multiplicative Hom-Malcev algebra.*

Theorem 1.2. *Let $(A, *, \alpha)$ be a multiplicative Hom-Malcev algebra. If define on $(A, *, \alpha)$ a ternary operation by (1.3), then A has a Hom-LYA structure.*

The next section is devoted to the proofs of Theorems 1.1 and 1.2. Some other results are also mentioned.

2 Proofs

In [7] it is proved that, in an anticommutative Hom-algebra $(A, [,], \alpha)$, the Hom-Malcev identity (1.2) is equivalent to the identity

$$\begin{aligned} J_\alpha(\alpha(x), \alpha(y), [u,v]) = {} & [J_\alpha(x,y,u), \alpha^2(v)] + [\alpha^2(u), J_\alpha(x,y,v)] \\ & -2J_\alpha(\alpha(u), \alpha(v), [x,y]) \end{aligned} \tag{2.1}$$

so that (2.1) can also be taken as a defining identity of Hom-Malcev algebras (a proof of the equivalence between (1.2) and (2.1) is the gist of [7]). Now we write (1.3) in an equivalent suitable form as

$$\{x,y,z\} = -J_\alpha(x,y,z) + 2xy * \alpha(z). \tag{2.2}$$

Proof of Theorem 1.1. Observe that (2.2) and multiplicativity imply

$$\{\alpha(x), \alpha(y), z\} = -J_\alpha(\alpha(x), \alpha(y), z) + 2\alpha(xy * z). \tag{2.3}$$

Then, putting (2.3) in **(HLY7)**, we get

$$\begin{aligned} -J_\alpha(\alpha(x), \alpha(y), u * v) = {} & -J_\alpha(x,y,u) * \alpha^2(v) + \alpha^2(u) * (-J_\alpha(x,y,v)) \\ & +(2xy * \alpha(u)) * \alpha^2(v) + \alpha^2(u) * (2xy * \alpha(v)) \end{aligned}$$

$$-2\alpha(xy * uv)$$

and this last equality is written as

$$J_\alpha(\alpha(x),\alpha(y),u*v) = J_\alpha(x,y,u)*\alpha^2(v)+\alpha^2(u)*J_\alpha(x,y,v)$$
$$-2J_\alpha(\alpha(u),\alpha(v),x*y),$$

which is (2.1). Therefore $(A,*,\alpha)$ is a Hom-Malcev algebra. □

Remark. For $\alpha = Id$, the ternary operation (2.2) reduces to the ternary operation, defined by the relation (1.4) in [18], that is considered in Malcev algebras. Thus Theorem 1.1 above is the Hom-analogue of the result of K. Yamaguti [18], which is the converse of a result of A.A. Sagle ([16], Proposition 8.3). The Hom-version of the Sagle's result is the following

Proposition 2.1. *Let $(A,*,\alpha)$ be a multiplicative Hom-Malcev algebra and define on $(A,*,\alpha)$ a ternary operation by (2.2). Then*

$$\{\alpha(x),\alpha(y),u*v\} = \{x,y,u\}*\alpha^2(v)+\alpha^2(u)*\{x,y,v\} \tag{2.4}$$

for all u,v,x,y in A.

Proof. We write the identity (2.1) as

$$-J_\alpha(\alpha(x),\alpha(y),u*v) = -J_\alpha(x,y,u)*\alpha^2(v)+\alpha^2(u)*(-J_\alpha(x,y,v))$$
$$+2J_\alpha(\alpha(u),\alpha(v),x*y)$$

i.e.

$$-J_\alpha(\alpha(x),\alpha(y),u*v) = -J_\alpha(x,y,u)*\alpha^2(v)+\alpha^2(u)*(-J_\alpha(x,y,v))$$
$$+2\alpha(u*v)*\alpha(x*y)+2(\alpha(v)*xy)*\alpha^2(u)$$
$$+2(xy*\alpha(u))*\alpha^2(v)$$

or

$$-J_\alpha(\alpha(x),\alpha(y),u*v)+2\alpha(x*y)*\alpha(u*v)$$
$$=(-J_\alpha(x,y,u)+2(xy*\alpha(u)))*\alpha^2(v)$$
$$+\alpha^2(u)*(-J_\alpha(x,y,v)+2(xy*\alpha(v))).$$

This last equality (according to (2.2) and using multiplicativity) means that

$$\{\alpha(x),\alpha(y),u*v\} = \{x,y,u\}*\alpha^2(v)+\alpha^2(u)*\{x,y,v\}$$

and therefore the proposition is proved. □

Observe that (2.4) is just **(HLY7)** in the definition of a Hom-LYA. Combining Theorem 1.1 and Proposition 2.1, we get the following

Corollary 2.2. *In an anticommutative Hom-algebra $(A,*,\alpha)$, the Hom-Malcev identity (1.2) is equivalent to (2.4), with "$\{,,\}$" defined by (2.2).* □

The untwisted counterpart of Corollary 2.2 is Theorem 1.1 in [18].

Proof of Theorem 1.2. We must prove the validity in $(A,*,\alpha)$ of the set of identities **(HLY1)**-**(HLY8)**. In the transformations below, we shall use the complete skew-symmetry of the Hom-Jacobian $J_\alpha(x,y,z)$ in $(A,*,\alpha)$.

The multiplicativity of $(A,*,\alpha)$ implies **(HLY1)** and **(HLY2)**. The skew-symmetry of "$*$" is

(HLY3) and it implies $\{x,y,z\} = -\{y,x,z\}$ which is **(HLY4)**. Next,

$$\begin{aligned} J_\alpha(x,y,z) &+ \circlearrowleft_{(x,y,z)}\{x,y,z\} = J_\alpha(x,y,z) - J_\alpha(x,y,z) + 2xy * \alpha(z) \\ &- J_\alpha(y,z,x) + 2yz * \alpha(x) - J_\alpha(z,x,y) + 2zx * \alpha(y) \\ &= -J_\alpha(y,z,x) - J_\alpha(z,x,y) + 2xy * \alpha(z) + 2yz * \alpha(x) \\ &+ 2zx * \alpha(y) \\ &= -2J_\alpha(x,y,z) + 2J_\alpha(x,y,z) \\ &= 0 \end{aligned}$$

so we get **(HLY5)**. Now consider $\circlearrowleft_{(x,y,z)}\{x*y,\alpha(z),\alpha(u)\}$ and note that, by (2.2), we have

$$\begin{aligned} \{x*y,\alpha(z),\alpha(u)\} &= -J_\alpha(x*y,\alpha(z),\alpha(u)) + 2(xy*\alpha(z))*\alpha^2(u), \\ \{y*z,\alpha(x),\alpha(u)\} &= -J_\alpha(y*z,\alpha(x),\alpha(u)) + 2(yz*\alpha(x))*\alpha^2(u), \\ \{z*x,\alpha(y),\alpha(u)\} &= -J_\alpha(z*x,\alpha(y),\alpha(u)) + 2(zx*\alpha(y))*\alpha^2(u). \end{aligned}$$

Then

$$\begin{aligned} \circlearrowleft_{(x,y,z)}\{x*y,\alpha(z),\alpha(u)\} &= -J_\alpha(x*y,\alpha(z),\alpha(u)) - J_\alpha(y*z,\alpha(x),\alpha(u)) \\ &\quad -J_\alpha(z*x,\alpha(y),\alpha(u)) + 2J_\alpha(x,y,z)*\alpha^2(u). \end{aligned}$$

We know [7] that the identity (2.1) is equivalent to the identity

$$J_\alpha(\alpha(x),\alpha(y),x*z) = J_\alpha(x,y,z)*\alpha^2(x)$$

(see (1.2)) defining Hom-Malcev algebras [22]. Then, by (2.1),

$$\begin{aligned} J_\alpha(x*y,\alpha(z),\alpha(u)) &= J_\alpha(x,z,u)*\alpha^2(y) + \alpha^2(x)*J_\alpha(y,z,u) \\ &\quad -2J_\alpha(z*u,\alpha(x),\alpha(y)), \\ J_\alpha(y*z,\alpha(x),\alpha(u)) &= J_\alpha(y,x,u)*\alpha^2(z) + \alpha^2(y)*J_\alpha(z,x,u) \\ &\quad -2J_\alpha(x*u,\alpha(y),\alpha(z)), \\ J_\alpha(z*x,\alpha(y),\alpha(u)) &= J_\alpha(z,y,u)*\alpha^2(x) + \alpha^2(z)*J_\alpha(x,y,u) \\ &\quad -2J_\alpha(y*u,\alpha(z),\alpha(x)). \end{aligned}$$

Therefore

$$\begin{aligned} \circlearrowleft_{(x,y,z)}\{x*y,\alpha(z),\alpha(u)\} &= -2J_\alpha(x,z,u)*\alpha^2(y) - 2\alpha^2(x)*J_\alpha(y,z,u) \\ &\quad -2J_\alpha(y,x,u)*\alpha^2(z) + 2J_\alpha(z*u,\alpha(x),\alpha(y)) \\ &\quad +2J_\alpha(x*u,\alpha(y),\alpha(z)) + 2J_\alpha(y*u,\alpha(z),\alpha(x)) \\ &\quad +2J_\alpha(x,y,z)*\alpha^2(u). \end{aligned}$$

Now, observe that by the identity

$$\begin{aligned} &J_\alpha(\alpha(w),\alpha(y),x*z) + J_\alpha(\alpha(x),\alpha(y),w*z) = \\ &J_\alpha(w,y,z)*\alpha^2(x) + J_\alpha(x,y,z)*\alpha^2(w) \end{aligned} \tag{2.5}$$

(see (8) in [22]) which is shown [22] to be equivalent to (1.2), we have

$$\begin{aligned} &-2J_\alpha(y,x,u)*\alpha^2(z) + 2J_\alpha(y*u,\alpha(z),\alpha(x)) + 2J_\alpha(x,y,z)*\alpha^2(u) = \\ &2J_\alpha(\alpha(u),\alpha(x),z*y) \end{aligned}$$

so that

$$\begin{aligned} \circlearrowleft_{(x,y,z)}\{x*y,\alpha(z),\alpha(u)\} &= -2J_\alpha(x,z,u)*\alpha^2(y) - 2\alpha^2(x)*J_\alpha(y,z,u) \\ &\quad +2J_\alpha(\alpha(u),\alpha(x),z*y) + 2J_\alpha(z*u,\alpha(x),\alpha(y)) \end{aligned}$$

$$+2J_{\alpha}(x*u,\alpha(y),\alpha(z)).$$

Next, by (2.1), we have

$$2J_{\alpha}(z*y,\alpha(u),\alpha(x)) = -J_{\alpha}(x*u,\alpha(y),\alpha(z)) + \alpha^2(x)*J_{\alpha}(y,z,u) + \alpha^2(u)*J_{\alpha}(x,z,y),$$

$$2J_{\alpha}(z*u,\alpha(x),\alpha(y)) = -J_{\alpha}(x*y,\alpha(z),\alpha(u)) + J_{\alpha}(x,z,u)*\alpha^2(y) + \alpha^2(x)*J_{\alpha}(y,z,u)$$

so that

$$\begin{aligned}\circlearrowleft_{(x,y,z)}\{x*y,\alpha(z),\alpha(u)\} &= -2J_{\alpha}(x,z,u)*\alpha^2(y) - 2\alpha^2(x)*J_{\alpha}(y,z,u)\\ &\quad -J_{\alpha}(x*u,\alpha(y),\alpha(z)) + \alpha^2(x)*J_{\alpha}(y,z,u)\\ &\quad +\alpha^2(u)*J_{\alpha}(x,z,y) - J_{\alpha}(x*y,\alpha(z),\alpha(u))\\ &\quad +J_{\alpha}(x,z,u)*\alpha^2(y) + \alpha^2(x)*J_{\alpha}(y,z,u)\\ &\quad +2J_{\alpha}(x*u,\alpha(y),\alpha(z))\\ &= -J_{\alpha}(x,z,u)*\alpha^2(y) + \alpha^2(u)*J_{\alpha}(x,z,y)\\ &\quad +J_{\alpha}(x*u,\alpha(y),\alpha(z)) - J_{\alpha}(x*y,\alpha(z),\alpha(u)).\end{aligned}$$

By (2.5), we note that

$$J_{\alpha}(x*u,\alpha(y),\alpha(z)) - J_{\alpha}(x*y,\alpha(z),\alpha(u)) = -J_{\alpha}(y,z,x)*\alpha^2(u) - J_{\alpha}(u,z,x)*\alpha^2(y).$$

Therefore, from the last expression of $\circlearrowleft_{(x,y,z)}\{x*y,\alpha(z),\alpha(u)\}$ above, we get

$$\begin{aligned}\circlearrowleft_{(x,y,z)}\{x*y,\alpha(z),\alpha(u)\} &= -J_{\alpha}(x,z,u)*\alpha^2(y) + \alpha^2(u)*J_{\alpha}(x,z,y)\\ &\quad -J_{\alpha}(y,z,x)*\alpha^2(u) - J_{\alpha}(u,z,x)*\alpha^2(y)\\ &= 0\end{aligned}$$

and thus **(HLY6)** holds.

The checking of **(HLY7)** is given by the proof of Proposition 2.1 above.

Finally we check the validity of **(HLY8)** for $(A,*,\alpha)$. We have

$$\{\{x,y,u\},\alpha^2(v),\alpha^2(w)\} + \{\alpha^2(u),\{x,y,v\},\alpha^2(w)\} + \{\alpha^2(u),\alpha^2(v),\{x,y,w\}\}$$

$$\begin{aligned}&= \{x,y,u\}\alpha^2(v)*\alpha^3(w) - \alpha^2(v)\alpha^2(w)*\alpha(\{x,y,u\}) - \alpha^2(w)\{x,y,u\}*\alpha^3(v)\\ &+\alpha^2(u)\{x,y,v\}*\alpha^3(w) - \{x,y,v\}\alpha^2(w)*\alpha^3(u) - \alpha^2(w)\alpha^2(u)*\alpha(\{x,y,v\})\\ &+\alpha^2(u)\alpha^2(v)*\alpha(\{x,y,w\}) - \alpha^2(v)\{x,y,w\}*\alpha^3(u) - \{x,y,w\}\alpha^2(u)*\alpha^3(v) \text{ (by (1.3))}\end{aligned}$$

$$\begin{aligned}&= (\{x,y,u\}*\alpha^2(v) + \alpha^2(u)*\{x,y,v\})*\alpha^3(w) + \alpha^2(u)\alpha^2(v)*\alpha(\{x,y,w\})\\ &-(\{x,y,v\}*\alpha^2(w) + \alpha^2(v)*\{x,y,w\})*\alpha^3(u) - \alpha^2(v)\alpha^2(w)*\alpha(\{x,y,u\})\\ &-(\{x,y,w\}*\alpha^2(u) + \alpha^2(w)*\{x,y,u\})*\alpha^3(v) - \alpha^2(w)\alpha^2(u)*\alpha(\{x,y,v\})\end{aligned}$$

$$\begin{aligned}&= \{\alpha(x),\alpha(y),u*v\}*\alpha^3(w) + \alpha^2(u*v)*\{\alpha(x),\alpha(y),\alpha(w)\}\\ &-\{\alpha(x),\alpha(y),v*w\}*\alpha^3(u) - \alpha^2(v*w)*\{\alpha(x),\alpha(y),\alpha(u)\}\\ &-\{\alpha(x),\alpha(y),w*u\}*\alpha^3(v) - \alpha^2(w*u)*\{\alpha(x),\alpha(y),\alpha(v)\}\end{aligned}$$

(by (2.4) and multiplicativity)

$$\begin{aligned}&= \{\alpha^2(x),\alpha^2(y),uv*\alpha(w)\} - \{\alpha^2(x),\alpha^2(y),vw*\alpha(u)\}\\ &-\{\alpha^2(x),\alpha^2(y),wu*\alpha(v)\} \text{ (by (2.4))}\end{aligned}$$

$$= \{\alpha^2(x),\alpha^2(y),\{u,v,w\}\} \text{ (by (1.3))}$$

and thus we get **(HLY8)**.

This completes the proof. □

The untwisted version of Theorems 1.1 and 1.2 above is proved by K. Yamaguti in [18], [19]. One observes that Yamaguti's proofs are quite different of our proofs above in the situation of Hom-algebras. It could be of some interest to know in which extent the Yamaguti's approach can be applied in the Hom-algebra setting.

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