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A New Version of the Central Limit Theorem

Enrique Villamor*
Florida International University, Department of Mathematics and Statistics,
Miami FL 33199 USA

Pablo Olivares†
Ryerson University, Department of Mathematics,
Toronto CANADA

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Abstract. In this short note, we present a new version of the Central Limit Theorem whose proof is based on Levy's characterization of Brownian motion. The method in the proof may allow to extend the result to a more general context, e.g. to averaged sums of properly compensated dependent random variables.

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1 Introduction

The traditional proof of the Central Limit Theorem (CLT), i.e. the weak limit of a sum of independent and identically distributed random variables is normally distributed, relies on the convergence of its characteristic function to a Gaussian one.

In this paper we present a new version of the CLT whose proof is based on the characterization of a Brownian Motion via its quadratic variation. This approach may be useful to extend the CLT to a more general context, e.g. results for non-identically distributed random variables or in presence of a sum of dependent random variables, properly normalized. See, for example, Hall and Heyde [1].

*e-mail address: villamor@fiu.edu
†e-mail address: pablo.olivares@ryerson.ca

We recall that a standard Brownian motion, $\{W_t\}_{t\geq 0}$, is defined to be a real-valued stochastic process satisfying the following properties:

1. $W_0 = 0$.

2. $W_t - W_s$ is normally distributed with mean 0 and variance $(t - s)$ independent of $\{W_u : u \leq s\}$, for any $t > s \geq 0$.

3. W_t has continuous sample paths, for any $t \geq 0$.

As always, it only really matters that these properties hold almost surely. Now, to apply the techniques of stochastic calculus, it is assumed that there is an underlying filtered probability space $(\Omega, \{F_t\}_{t\geq 0}, \mathbb{P})$, which necessitates a further definition.

An stochastic process $\{W_t\}_{t\geq 0}$ is a Brownian motion on a filtered probability space $(\Omega, \{F_t\}_{t\geq 0}, \mathbb{P})$ if in addition to the properties 1-3 above it is also adapted, that is, W_t is F_t-measurable, and $W_t - W_s$ is independent of F_s for each $t > s \geq 0$. The filtration $\{F_t\}_{t\geq 0}$ is called the *natural filtration*. It is assumed to verify the usual conditions, i.e. it is right-continuous and contains all the probability null measurable sets.

Note that the above condition that $W_t - W_s$ is independent of $\{W_u : u \leq s\}$ is not explicitly required, as it also follows from the independence from F_s for each $t > s \geq 0$. According to these definitions, an stochastic process is a Brownian motion if and only if it is a Brownian motion with respect to its natural filtration.

The property that $W_t - W_s$ has zero mean independently of F_s for each $t > s \geq 0$ means that Brownian motion is a martingale. Furthermore, its quadratic variation is $[W_t, W_t] = [W]_t = t$. An incredibly useful result is that the converse statement holds. That is, Brownian motion is the only local martingale with this quadratic variation. This is known as Levy characterization, and shows that Brownian motion is a particularly general stochastic process, justifying its ubiquitous influence on the study of continuous-time stochastic processes. We state the theorem below, for a proof see for example Pascucci [2].

Theorem 1.1. *(Levy Characterization of Brownian Motion) Let $\{X_t\}_{t\geq 0}$ be a local martingale with $X_0 = 0$. Then, the following are equivalent.*

1) $\{X_t\}_{t\geq 0}$ is the standard Brownian motion on the underlying filtered probability space.

2) $\{X_t\}_{t\geq 0}$ is continuous and $\{X_t^2 - t\}_{t\geq 0}$ is a local martingale.

3) $\{X_t\}_{t\geq 0}$ has quadratic variation $\{[X]_t = t\}_{t\geq 0}$.

Let $X_1, ..., X_n$ be a random sample of size n extracted from a sequence of independent and identically distributed random variables drawn from distributions of expected value given by ν and finite variance given by σ^2. Suppose we are interested in the sample average

$$S_n := \frac{X_1 + \cdots + X_n}{n}$$

of these random variables. By the Law of Large Numbers, the sample average converges in probability and almost surely to the expected value ν as $n \to \infty$. The Classical Central Limit Theorem describes the size and the distributional form of the stochastic fluctuations around the deterministic number ν during this convergence. More precisely, it states that as n gets larger, the distribution of the difference between the sample average S_n and its limit ν, when multiplied by the factor $\sqrt{n}$ (that is $\sqrt{n}(S_n - \nu)$), approximates the normal distribution with mean 0 and variance σ^2 in the sense of distribution or weak convergence. For large enough n, the distribution of S_n is close to the normal distribution with mean ν and variance $\frac{\sigma^2}{n}$. The usefulness of the theorem is that the distribution of $\sqrt{n}(S_n - \nu)$ approaches normality regardless of the shape of the distribution of the individual X_is. Formally our theorem can be stated as follows:

Theorem 1.2. (New version of the Central Limit Theorem) *Let $\{X_t\}_{t\geq 0}$ a continuum sequence of independent and identically distributed real valued random variables defined on the same filtered probability space $(\Omega, \mathbb{P}, \{F_t\}_{t\geq 0})$, where $\{F_t\}_{t\geq 0}$ is an increasing family of σ-algebras such that the process $\{X_t\}_{t\geq 0}$ is F_t-adapted. Moreover, suppose that for any $t \geq 0$ we have that $\mathbb{E}[X_t] = \nu$, $Var[X_t] = \sigma^2 < \infty$ and $\mathbb{E}[X_t^4] < \infty$.*

Then, for any infinite sequence $\{X_n\}_{n=0}^{\infty}$ in $\{X_t\}_{t\geq 0}$, there exists a subsequence $\{X_{n_j}\}_{j=0}^{\infty}$ such that $\sqrt{n_j}(S_{n_j} - \nu)$ converges in quadratic mean to a normal $N(0, \sigma^2)$ random variable, i.e.

$$\sqrt{n_j}\Big(S_{n_j} - \nu\Big) \rightarrow N(0,\, \sigma^2) \text{ in } L^2(\Omega).$$

2 Proof of the new version of the Central Limit Theorem

Proof. Our proof is based on Levy's characterization of Brownian motion (Theorem 1.1). Let $\{X_t\}_{t=0}^{\infty}$ a continuum sequence of i.i.d, independent and identically distributed real valued random variables defined on the same filtered probability space $(\Omega, \mathbb{P}, \{F_t\}_{t\geq 0})$, where $\{F_t\}_{t\geq 0}$ is an increasing family of σ-algebras such that the process $\{X_t\}_{t=0}^{\infty}$ is F_t-adapted. That is, for each $t \geq 0$, $X_t(\omega)$ is F_t-measurable.

Without loss of generality we can assume they have mean equal to zero and variance equal to one, the general result will follow by applying this result to the normalized random variables $\{\frac{X_n - \nu}{\sigma}\}_{n=0}^{\infty}$.

We define the stochastic process Y_t in terms of $\{X_t\}_{t=0}^{\infty}$ as follows

$$Y_t := \int_0^t X_s\,(ds)^{1/2},$$

Remark. We will build first Y_t for $0 \leq t \leq 1$ and extend it to any $t \geq 1$ as in the standard construction of the Brownian motion process.

The above integral is to be understood in the following sense. Let $t = 1$ we define for any $n = 1, 2, \ldots$, the dyadic partition $\mathbb{P}_n$ as $0 = t_0 < t_1 < t_2 < \ldots < t_{2^n-1} < t_{2^n} = 1$, where for any $j = 0, 1, 2, \ldots, 2^n - 1, 2^n$, the corresponding node in the dyadic partition is given by $t_j = \frac{j}{2^n}$.

Define the sums $\sum_{j=0}^{2^n-1} X_{t_j}(t_{j+1} - t_j)^{1/2}$. It follows that

$$\mathbb{E}[(\sum_{j=0}^{2^n-1} X_{t_j}(t_{j+1} - t_j)^{1/2})^2] = \mathbb{E}[[\sum_{j=0}^{2^n-1}\sum_{k=0}^{2^n-1} X_{t_j}(t_{j+1} - t_j)^{1/2} X_{t_k}(t_{k+1} - t_k)^{1/2}]],$$

for any $j \neq k$ and since the corresponding X_{t_j} and X_{t_k} are independent with mean equal to 0, the corresponding term in the above sum is equal to 0. Thus,

$$\mathbb{E}[(\sum_{j=0}^{2^n-1} X_{t_j}(t_{j+1} - t_j)^{1/2})^2] = \mathbb{E}[\sum_{j=0}^{2^n-1} X_{t_j}^2(t_{j+1} - t_j)] =$$

$$= \sum_{j=0}^{2^n-1} \mathbb{E}[X_{t_j}^2](t_{j+1} - t_j) = \sum_{j=0}^{2^n-1}(t_{j+1} - t_j) = 1,$$

hence, by the Banch-Alaouglu theorem (see [3]), there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $\{n_k\}_{k=1}^{\infty}$ is strictly increasing, and

$$\lim_{k\to\infty}\sum_{j=0}^{2^{n_k}-1} X_{t_j}(t_{j+1}-t_j)^{1/2} = Y_1 := \int_0^1 X_s(ds)^{1/2}$$

converges weakly in L^2 to Y_1. Observe that requiring that the norm of the partition $||\mathbb{P}_{n_k}|| \to 0$ is equivalent to say that $k\to\infty$. It then follows that $Y_1 \in L^2$ and $||Y_1||_2 = 1$.

We trivially have that $\lim_{k\to\infty} ||\sum_{j=0}^{2^{n_k}-1} X_{t_j}(t_{j+1}-t_j)^{1/2}||_2 = 1 = ||Y_1||$, this implies that the sequence $\sum_{j=0}^{2^{n_k}-1} X_{t_j}(t_{j+1}-t_j)^{1/2}$ converges strongly in L^2 to Y_1 as k approaches infinity. Let us prove this; we clearly have

$$\begin{aligned} ||\sum_{j=0}^{2^{n_k}-1} X_{t_j}(t_{j+1}-t_j)^{1/2} - Y_1||_2^2 &= ||\sum_{j=0}^{2^{n_k}-1} X_{t_j}(t_{j+1}-t_j)^{1/2}||_2^2 \\ &-2 < \sum_{j=0}^{2^{n_k}-1} X_{t_j}(t_{j+1}-t_j)^{1/2}, Y_1 > + ||Y_1||_2^2 \\ &= 2 - 2 < \sum_{j=0}^{2^{n_k}-1} X_{t_j}(t_{j+1}-t_j)^{1/2}, Y_1 >, \end{aligned}$$

since $\sum_{j=0}^{2^{n_k}-1} X_{t_j}(t_{j+1}-t_j)^{1/2}$ converges weakly to Y_1 it follows that $\lim_{k\to\infty} < \sum_{j=0}^{2^{n_k}-1} X_{t_j}(t_{j+1}-t_j)^{1/2}, Y_1 >= ||Y_1||_2^2 = 1$, thus the sequence $\sum_{j=0}^{2^{n_k}-1} X_{t_j}(t_{j+1}-t_j)^{1/2}$ converges to Y_1 strongly in L^2.

Since the subsequence$\{\{\frac{j}{2^{n_k}}\}_{j=0}^{2^{n_k}}\}_{k=1}^{\infty}$ is dense on the interval $[0,1]$, it is clear that for any $\frac{j}{2^{n_k}}$, for $j = 0,1,2,\ldots,2^{n_k}$ and $k = 1,2,\ldots$ the corresponding $Y_{\frac{j}{2^{n_k}}}$ is well defined using the same argument as for Y_1 with the same subsequence $\{n_k\}_{k=1}^{\infty}$ restricted to the interval $[0,\frac{j}{2^{n_k}}]$. Hence we have defined $\{\{Y_{\frac{j}{2^{n_k}}}\}_{j=0}^{2^{n_k}}\}_{k=1}^{\infty}$ with our desired properties, namely

$$\mathbb{E}[(Y_{\frac{i}{2^{n_k}}} - Y_{\frac{j}{2^{n_m}}})^2] = |\frac{i}{2^{n_k}} - \frac{j}{2^{n_m}}|,$$

for any $i = 0,1,\ldots,2^{n_k}$, $j = 0,1,\ldots,2^{n_m}$ and any $k,m = 1,2,\ldots$. This clearly implies that the sequence $\{\{Y_{\frac{j}{2^{n_k}}}\}_{j=0}^{2^{n_k}}\}_{k=1}^{\infty}$ is a Cauchy sequence in L^2. More precisely, we have that for any $0 < t < 1$, there exists a dyadic sequence $\frac{j}{2^{n_k}}$ such that $\lim_{k\to\infty}\frac{j}{2^{n_k}} = t$, and the corresponding sequence $\{Y_{\frac{j}{2^{n_k}}}\}$ is Cauchy in L^2, thus converges to a Y_t in L^2. This completely defines the stochastic process $\{Y_t\}_{0\le t\le 1}$ with our desired properties.

Our argument shows that the X_ts need only to be i.i.d. and defined only for the dyadic numbers on the interval $[0,1]$ since for any other $0 \le t \le 1$, the corresponding Y_ts are defined as L^2 limits, see above. The extension from the interval $0 \le t \le 1$ to any $t \ge 0$ can be accomplished reindexing the dyadic family $\{\{X_{\frac{j}{2^n}}\}_{j=0}^{2^n}\}_{n=1}^{\infty}$ and building as above countably many $\{\{Y_t^{(n)}\}_{0\le t\le 1}\}_{n=1}^{\infty}$ from each reindexing. It clearly follows from our previous construction that for each $\{\{Y_t^{(n)}\}_{0\le t\le 1}\}_{n=1}^{\infty}$, the same subsequence $\{n_k\}_{k=1}^{\infty}$ can be used. Then, define an inductive assembling of them by setting

$$Y_t = Y_{n-1} + Y_{t-(n-1)}^{(n)},$$

for any $n-1 \le t \le n$. It follows from our construction that the stochastic process $\{Y_t\}_{t\ge 0}$ has the desired properties needed in the rest of our proof.

Before we continue, let's make the following observation: It is clear form our previous construction of the process $\{Y_t\}_{t\geq 0}$, that we only need the process X_t to be defined for $0 \leq t \leq 1$.

We need to prove that $\{Y_t\}_{t\geq 0}$ is a standard Brownian motion. According to Levy's characterization of Brownian motion (Theorem 1.1), we need to show that $\{Y_t\}_{t\geq 0}$ is a martingale and that its quadratic variation $[Y_t, Y_t] = t$ since it is already clear by definition that $Y_0 = 0$.

We will show next that $\{Y_t\}_{t\geq 0}$ is a martingale. Let $s < t$, we have that:

$$\mathbb{E}[Y_t/F_s] = \mathbb{E}[\int_0^t X_u (du)^{1/2}/F_s] = \mathbb{E}[\int_0^s X_u (du)^{1/2} + \int_s^t X_u (du)^{1/2}/F_s]$$

$$= \mathbb{E}[\int_0^s X_u (du)^{1/2}/F_s] + \mathbb{E}[\int_s^t X_u (du)^{1/2}/F_s],$$

clearly $\int_0^s X_u (du)^{1/2}$ is F_s-measurable and $\int_s^t X_u (du)^{1/2}$ is independent of the σ-algebra F_s, thus by the properties of the conditional expectation we have that:

$$\mathbb{E}[Y_t/F_s] = \int_0^s X_u (du)^{1/2} + \mathbb{E}[\int_s^t X_u (du)^{1/2}] = Y_s,$$

since by our definition the $\int_s^t X_u (du)^{1/2}$ is the limit of certain Riemann-Stieltjes sums over some dyadic partitions, it clearly follows from the fact that the expectation of the sum is the sum of the expectations that

$$\mathbb{E}[\int_s^t X_u (du)^{1/2}] = \int_s^t \mathbb{E}[X_u] (du)^{1/2} = 0$$

since each X_t is i.i.d. with the same mean equal to zero and variance equal to one. This leads to

$$\mathbb{E}[Y_t/F_s] = \int_0^s X_u (du)^{1/2} = Y_s,$$

which finishes the proof that $\{Y_t\}_{t\geq 0}$ is a martingale process.

It remains to show that the quadratic variation of $\{Y_t\}_{t\geq 0}$ is equal to $[Y_t, Y_t] = t$, then by Levy's characterization of Brownian motion we'll have that $\{Y_t\}_{t\geq 0}$ is a Brownian motion which will complete the proof of this new version of the Central Limit Theorem.

We need to show that, for a partition $\mathbb{P}$ of the interval $[0, t]$ into m subintervals, $0 = t_0 < t_1 < \ldots, t_{m-1} < t_m = t$:

$$t = [Y_t, Y_t] = \lim_{\|\mathbb{P}\|\to 0} \sum_{j=0}^{m-1} (Y_{t_{j+1}} - Y_{t_j})^2$$

$$= \lim_{\|\mathbb{P}\|\to 0} \sum_{j=0}^{m-1} \left(\int_0^{t_{j+1}} X_u (du)^{1/2} - \int_0^{t_j} X_u (du)^{1/2} \right)^2$$

$$= \lim_{\|\mathbb{P}\|\to 0} \sum_{j=0}^{m-1} \left(\int_{t_j}^{t_{j+1}} X_u (du)^{1/2} \right)^2,$$

notice that this limit is understood in the sense of convergence in probability.

Let us pass now to study each of the terms $(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2})^2$ in the above sum. Without loss of generality we can assume that both t_j and t_{j+1} are nodes on the dyadic subsequence found at the beginning of our proof to define the $\{Y_t\}_{t\geq 0}$ process. Hence,

$$\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2} = \lim_{\|\mathbb{P}_{n_k}\|\to 0} \sum_{i=0}^{2^{n_k}-1} X_{s_i}\,(s_{i+1}-s_i)^{1/2},$$

where the partitions $\mathbb{P}_{n_k}$ above are the dyadic partitions of the interval $[t_j, t_{j+1}]$ into 2^{n_k} subintervals in the definition of the $\{Y_t\}_{t\geq 0}$ process that we constructed at the beginning of the proof, where $t_j = s_0 < s_1 < \ldots, s_{2^{n_k}-1} < s_{2^{n_k}} = t_{j+1}$. Squaring both sides of the above equality, we have that

$$(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2})^2 = \lim_{k\to\infty}(\sum_{i=0}^{2^{n_k}-1} X_{s_i}\,(s_{i+1}-s_i)^{1/2})^2,$$

it is clear now that on the square of the sum on the right hand side of the above equality, after taking expectations and since the X_{s_i}s are independent, the only terms that do not vanish are when the indices are equal and then we obtain

$$\mathbb{E}[\left(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\right)^2] = \lim_{k\to\infty} \sum_{j=0}^{2^{n_k}-1} \mathbb{E}[(X_{s_i})^2]\,(s_{i+1}-s_i),$$

and since we have that $\mathbb{E}[(X_{s_i})^2] = 1$, for all s_i, then we have that

$$\mathbb{E}[\left(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\right)^2] = \lim_{k\to\infty} \sum_{i=0}^{2^{n_k}-1} (s_{i+1}-s_i) = (t_{j+1}-t_j). \qquad (2.1)$$

Next, we will show the convergence in quadratic mean, therefore in probability, of the quadratic variation $[Y_t, Y_t]$ toward the limit t. We need to show, under the above notation, that

$$\lim_{\|\mathbb{P}\|\to 0} \mathbb{E}\Big[(\sum_{j=0}^{m-1}\left(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\right)^2 - t)^2\Big]$$

$$= \lim_{\|\mathbb{P}\|\to 0} \mathbb{E}\Big[(\sum_{j=0}^{m-1}\left(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\right)^2 - (t_{j+1}-t_j))^2\Big]$$

$$= \lim_{\|\mathbb{P}\|\to 0} \mathbb{E}\Big[\sum_{j=0}^{m-1}\sum_{k=0}^{m-1}[\left(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\right)^2 - (t_{j+1}-t_j)]\times$$

$$\times[\left(\int_{t_k}^{t_{k+1}} X_u\,(du)^{1/2}\right)^2 - (t_{k+1}-t_k)]\Big] = 0.$$

For $k \neq j$ the term in the double sum above is

$$\mathbb{E}\Big[[\left(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\right)^2 - (t_{j+1}-t_j)]\,[\left(\int_{t_k}^{t_{k+1}} X_u\,(du)^{1/2}\right)^2 - (t_{k+1}-t_k)]\Big] =$$

$$= \mathbb{E}\Big[[\left(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\right)^2 - (t_{j+1}-t_j)]\Big]\mathbb{E}\Big[[\left(\int_{t_k}^{t_{k+1}} X_u\,(du)^{1/2}\right)^2 - (t_{k+1}-t_k)]\Big],$$

according to the independent increments and thus equal to 0 by (2.1) above. Hence,

$$\mathbb{E}\Big[\sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\Big[\Big(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\Big)^2-(t_{j+1}-t_j)\Big]\times$$

$$\times\Big[\Big(\int_{t_k}^{t_{k+1}} X_u\,(du)^{1/2}\Big)^2-(t_{k+1}-t_k)\Big]\Big]=$$

$$=\mathbb{E}\Big[\sum_{j=0}^{m-1}\Big[\Big(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\Big)^2-(t_{j+1}-t_j)\Big]^2\Big]=$$

$$=\sum_{j=0}^{m-1}\mathbb{E}\Big[\Big[\frac{\Big(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\Big)^2}{(t_{j+1}-t_j)}-1\Big]^2\Big](t_{j+1}-t_j)^2.$$

Next, we will show that

$$\mathbb{E}\Big[\Big[\frac{\Big(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\Big)^2}{(t_{j+1}-t_j)}-1\Big]^2\Big]<C<\infty,$$

where C is a finite positive constant independent of the partitions. Clearly

$$\mathbb{E}\Big[\Big[\frac{\Big(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\Big)^2}{(t_{j+1}-t_j)}-1\Big]^2\Big]=\mathbb{E}\Big[\frac{\Big(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\Big)^4}{(t_{j+1}-t_j)^2}-2\,\frac{\Big(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\Big)^2}{(t_{j+1}-t_j)}+1\Big].$$

The expectations of the second and third terms above are obviously bounded by a constant independent of the partitions. Let us examine then $\mathbb{E}\Big[\frac{\left(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\right)^4}{(t_{j+1}-t_j)^2}\Big]$. We have that

$$\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}=\lim_{\|\mathbb{P}_{n_k}\|\to 0}\sum_{i=0}^{2^{n_k}-1} X_{s_i}\,(s_{i+1}-s_i)^{1/2},$$

where the partitions $\mathbb{P}_{n_k}$ above are the dyadic partitions of the interval $[t_j,t_{j+1}]$ into 2^{n_k} subintervals in the definition of the $\{Y_t\}_{t\geq 0}$ process that we constructed at the beginning of the proof, where $t_j=s_0<s_1<\ldots,s_{2^{n_k}-1}<s_{2^{n_k}}=t_{j+1}$, and $s_i=t_j+i\,\frac{(t_{j+1}-t_j)}{2^{n_k}}$, for $i=0,1,2,\ldots,2^{n_k}$.

Hence, $\mathbb{E}\Big[\frac{\left(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\right)^4}{(t_{j+1}-t_j)^2}\Big]$ is approximated by $\mathbb{E}\Big[\frac{\left(\sum_{i=0}^{2^{n_k}-1} X_{s_i}\,(s_{i+1}-s_i)^{1/2}\right)^4}{(t_{j+1}-t_j)^2}\Big]$. It is now clear that since the X_is are i.i.d.

$$\mathbb{E}\Big[\frac{(\sum_{i=0}^{2^{n_k}-1} X_{s_i}\,(s_{i+1}-s_i)^{1/2})^4}{(t_{j+1}-t_j)^2}\Big]=\frac{\sum_{i=0}^{2^{n_k}-1}\mathbb{E}[X_{s_i}^4]\,(s_{i+1}-s_i)^2}{(t_{j+1}-t_j)^2}+$$

$$+\frac{\sum_{i=0}^{2^{n_k}-1}\sum_{l=0,l\neq i}^{2^{n_k}-1}\mathbb{E}[X_{s_i}^2]\,\mathbb{E}[X_{s_l}^2]\,(s_{i+1}-s_i)\,(s_{l+1}-s_l)}{(t_{j+1}-t_j)^2},$$

since any other term in the above sum carries a factor of the form $\mathbb{E}[X_{s_i}]$ which is equal to zero and another one of the form $\mathbb{E}[X_{s_i}^3]$ which is finite since $\mathbb{E}[X_{s_i}^4]$ is. Thus all those terms vanish.

The first sum above is of the order of a big $O(\frac{1}{2^{n_k}})$ since by assumption the common fourth moment $\mathbb{E}[X_{s_i}^4]$ is finite, and the second double sum is of the order of $O(1)$. Thus $\mathbb{E}\Big[\frac{\left(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\right)^4}{(t_{j+1}-t_j)^2}\Big]\leq$ C a constant independent of the partition, therefore

$$\mathbb{E}\Big[\Big[\frac{\Big(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\Big)^2}{(t_{j+1}-t_j)} - 1\Big]^2\Big] < C < \infty,$$

as stated. This leads to

$$\lim_{\|\mathbb{P}\|\to 0}\mathbb{E}\Big[\sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\Big[\Big(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\Big)^2 - (t_{j+1}-t_j)\Big]\times$$

$$\times\Big[\Big(\int_{t_k}^{t_{k+1}} X_u\,(du)^{1/2}\Big)^2 - (t_{k+1}-t_k)\Big]\Big] \le C \lim_{\|\mathbb{P}\|\to 0}\sum_{j=0}^{m-1}(t_{j+1}-t_j)^2$$

$$\le C\,(\lim_{\|\mathbb{P}\|\to 0}\|\mathbb{P}\|)\,t = 0,$$

as we wanted to show. Thus,

$$[Y_t, Y_t] = \lim_{\|\mathbb{P}\|\to 0}\sum_{j=0}^{m-1}\Big(\int_{t_j}^{t_{j+1}} X_u\,(du)^{1/2}\Big)^2 \to t,$$

in L^2, and thus in probability. Thus, by Levy's characterization theorem of Brownian motion, we have that

$$Y_t = \int_0^t X_s\,(ds)^{1/2}$$

is normally distributed with mean 0 and variance t, $N(0,t)$. That is, $Y_t = W_t$ is Brownian motion, and thus

$$\frac{1}{\sqrt{t}}\,Y_t = \frac{1}{\sqrt{t}}\int_0^t X_s\,(ds)^{1/2}$$

is normally distributed with mean 0 and variance 1, $N(0,1)$.

Letting $t = 1$, we have that $\int_0^1 X_s\,(ds)^{1/2}$ is a standard normal random variable with mean 0 and variance 1. By our construction of the continuous time stochastic process $\{Y_t\}_{t\ge 0}$ above, the integral above is approximated in quadratic mean by

$$\delta^{1/2}\sum_{j=0}^{2^{n_k}-1} X_j = \frac{1}{\sqrt{2^{n_k}}}\sum_{j=0}^{2^{n_k}-1} X_j = \frac{X_0 + X_1 + \ldots + X_{2^{n_k}-1}}{\sqrt{2^{n_k}}},$$

for an equally spaced dyadic partition $\mathbb{P}_{n_k}$ of the interval $[0,1]$ into 2^{n_k} equally spaced subintervals of length $\delta = \frac{1}{2^{n_k}}$, where for each $j = 0,1,2,\ldots,2^{n_k}$, $X_j = X_{t_j} = X_{\frac{j}{2^{n_k}}}$. This means that $\frac{X_0+X_1+\ldots+X_{2^{n_k}-1}}{\sqrt{2^{n_k}}}$ converges in quadratic mean to a $N(0,1)$ random variable.

Finally, for any infinite sequence $\{X_n\}_{n=0}^{\infty}$ in $\{X_t\}_{0\le t\le 1}$, we place it in the dyadic nodes of the interval $[0,1]$ as follows. Place X_0 at $t = 0$ and X_1 at $t = 1$, X_2 at $t = \frac{1}{2}$, X_3 at $t = \frac{1}{4}$, X_4 at $t = \frac{3}{4}$, X_5 at $t = \frac{1}{8}$, X_6 at $t = \frac{3}{8}$, X_t at $t = \frac{5}{8}$, X_8 at $t = \frac{7}{8}$; the next eight X_is at the new eight nodes of the dyadic partition with 2^4 nodes, so on and so forth.

Generally, place X_0 at $t = 0$ and X_1 at $t = 1$. Henceforth, for each $n = 1,2,3,\ldots$ place

$$X_{2^{n-1}+1}, X_{2^{n-1}+2}, X_{2^{n-1}+3}, \ldots, X_{2^n-1}, X_{2^n}$$

at the dyadic nodes $t = \frac{2k-1}{2^n}$ for each $k = 1, 2, 3, \ldots, 2^{n-1} - 1, 2^{n-1}$. Thus, by this construction, it is clear that the

$$\lim_{k\to\infty} \frac{X_1 + X_2 + \ldots + X_{2^{n_k}}}{\sqrt{2^{n_k}}} = \lim_{j\to\infty} \frac{X_1 + X_2 + \ldots + X_{n_j}}{\sqrt{n_j}}$$

in quadratic mean (and thus in distribution) is equal to a $N(0, 1)$ random variable. □

It follows from our Theorem 1.2 that the extra assumption that the common fourth moment of the i.i.d $\{X_t\}_{t\geq 0}$ process is finite allows us to improve the convergence in our new version of the Central Limit Theorem in the sense that we can prove that the convergence is in quadratic mean rather than in just distribution. The payoff is that this convergence is for a subsequence $\{X_{n_j}\}_{j=0}^{\infty}$ of the original sequence $\{X_n\}_{n=0}^{\infty}$.

References

[1] Hall, P.; Heyde, C. C. Martingale limit theory and its application. Probability and Mathematical Statistics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980. xii+308 pp. ISBN: 0-12-319350-8

[2] Pascucci, A. PDE and Martingale Methods in Option Pricing. Springer-Verlag Italia, 2011.

[3] Rudin, W. Functional Analysis. MacGraw-Hill, New York, 1991.